Ex - 1.3

Q1. Prove that $\sqrt{5}$ is irrational.

Sol. Let us assume, to the contrary, that $\sqrt{5}$ is rational.

So, we can find coprime integers a and b ($\neq 0$) such that

$$\sqrt{5} = \frac{a}{b}$$

$\Rightarrow \sqrt{5}\, b = a$

Squaring on both sides, we get

$$5b^2 = a^2$$

Therefore, 5 divides a^2.

Therefore, 5, divides a

So, we can write a = 5c for some integer c.

Substituting for a, we get

$$5b^2 = 25c^2$$

$\Rightarrow b^2 = 5c^2$

This means that 5 divides b^2, and so 5 divides b.

Therefore, a and b have at least 5 as a common factor.

But this contradicts the fact that a and b have no common factor other than 1.

This contradiction arose because of our incorrect assumption that $\sqrt{5}$ is rational.

So, we conclude that $\sqrt{5}$ is irrational.

Q2. Prove that $3 + 2\sqrt{5}$ is irrational.

Sol. Let us assume, to the contrary, that $3 + 2\sqrt{5}$ is rational. That is, we can find coprime integers a and b ($b \neq 0$) such that $3 + 2\sqrt{5} = \frac{a}{b}$

Therefore, $\frac{a}{b} - 3 = 2\sqrt{5}$

$\Rightarrow \frac{a-3b}{b} = 2\sqrt{5}$

$\Rightarrow \frac{a-3b}{2b} = \sqrt{5} \Rightarrow \frac{a}{2b} - \frac{3}{2} = \sqrt{5}$

Since a and b are integers, we get $\frac{a}{2b} - \frac{3}{2}$ is rational, and so $\frac{a-3b}{2b} = \sqrt{5}$ is rational.

But this contradicts the fact that $\sqrt{5}$ is irrational. This contradiction has arisen because of our incorrect assumption that $3 + 2\sqrt{5}$ is rational.

So, we conclude that $3 + 2\sqrt{5}$ is irrational.

Q3. Prove that the following are irrationals :

(i) $\frac{1}{\sqrt{2}}$ (ii) $7\sqrt{5}$ (iii) $6 + \sqrt{2}$

Sol. (i) Let us assume, to the contrary, that $\frac{1}{\sqrt{2}}$ is rational. That is we can find coprime integers a and b (b ≠ 0) such that,

$\frac{1}{\sqrt{2}} = \frac{p}{q}$

Therefore, q = $\sqrt{2}$p

Squaring on both sides, we get

$q^2 = 2p^2$ ...(i)

Therefore, 2 divides q^2

so, 2 divides q

so we can write q = 2r for some integer r

squaring both sides, we get

$q^2 = 4r^2$ ...(ii)

From (i) & (ii), we get

$2p^2 = 4r^2$

$p^2 = 2r^2$

Therefore, 2 divides p^2

So, 2 divides p

So, p & q have atleast 2 as a common factor.

But this contradict the fact that p & q have no common factor other than 1.

This contradict our assumption that $\frac{1}{\sqrt{2}}$ is rational. So, we condude that $\frac{1}{\sqrt{2}}$ is irrational.

(ii) Let us assume, to the contrary, that $7\sqrt{5}$ is rational.

That is, we can find coprime integers a and b (b ≠ 0) such that $7\sqrt{5} = \frac{a}{b}$

Therefore, $\frac{a}{7b} = \sqrt{5}$

Since a and b are integers, we get $\frac{a}{7b}$ is rational, and so $\frac{a}{7b} = \sqrt{5}$ is rational.

But this contradicts the fact that $\sqrt{5}$ is irrational. This contradiction has arisen because of our incorrect assumption that $7\sqrt{5}$ is rational.

So, we conclude that $7\sqrt{5}$ is irrational.

(iii) Let us assume, to the contrary, that $6 + \sqrt{2}$ is rational.

That is, we can find coprime integers a and b (b ≠ 0) such that $6 + \sqrt{2} = \frac{a}{b}$

Therefore, $\frac{a}{b} - 6 = \sqrt{2}$

$\Rightarrow \frac{a-6b}{b} = \sqrt{2}$

Since a and b are integers, we get $\frac{a}{b} - 6$ is rational, and so $\frac{a-6b}{b} = \sqrt{2}$ is rational.

But this contradicts the fact that $\sqrt{2}$ is irrational. This contradiction has arisen because of our incorrect assumption that

$6+\sqrt{2}$ is rational.

So, we conclude that $6+\sqrt{2}$ is irrational.