Exercise 10.3

Question 1:

Reduce the following equations into slope-intercept form and find their slopes and the *y*-intercepts.

(i) $x + 7y = 0$ (ii) $6x + 3y - 5 = 0$ (iii) $y = 0$

Answer

(i) The given equation is $x + 7y = 0$.

It can be written as

$$y = -\frac{1}{7}x + 0 \qquad \ldots(1)$$

This equation is of the form $y = mx + c$, where $m = -\frac{1}{7}$ and $c = 0$.

Therefore, equation (1) is in the slope-intercept form, where the slope and the *y*-intercept are $-\frac{1}{7}$ and 0 respectively.

(ii) The given equation is $6x + 3y - 5 = 0$.

It can be written as

$$y = \frac{1}{3}(-6x + 5)$$

$$y = -2x + \frac{5}{3} \qquad \ldots(2)$$

This equation is of the form $y = mx + c$, where $m = -2$ and $c = \frac{5}{3}$.

Therefore, equation (2) is in the slope-intercept form, where the slope and the *y*-intercept are–2 and $\frac{5}{3}$ respectively.

(iii) The given equation is $y = 0$.

It can be written as

$y = 0.x + 0$ … (3)

This equation is of the form $y = mx + c$, where $m = 0$ and $c = 0$.

Therefore, equation (3) is in the slope-intercept form, where the slope and the y-intercept are 0 and 0 respectively.

Question 2:

Reduce the following equations into intercept form and find their intercepts on the axes.

(i) $3x + 2y - 12 = 0$ (ii) $4x - 3y = 6$ (iii) $3y + 2 = 0$.

Answer

(i) The given equation is $3x + 2y - 12 = 0$.

It can be written as

$$3x+2y=12$$

$$\frac{3x}{12}+\frac{2y}{12}=1$$

$$\text{i.e., } \frac{x}{4}+\frac{y}{6}=1 \qquad \ldots(1)$$

This equation is of the form $\frac{x}{a}+\frac{y}{b}=1$, where $a = 4$ and $b = 6$.

Therefore, equation (1) is in the intercept form, where the intercepts on the x and y axes are 4 and 6 respectively.

(ii) The given equation is $4x - 3y = 6$.

It can be written as

$$\frac{4x}{6}-\frac{3y}{6}=1$$

$$\frac{2x}{3}-\frac{y}{2}=1$$

$$\text{i.e., } \frac{x}{\left(\frac{3}{2}\right)}+\frac{y}{(-2)}=1 \qquad \ldots(2)$$

This equation is of the form $\frac{x}{a}+\frac{y}{b}=1$, where $a = \frac{3}{2}$ and $b = -2$.

Therefore, equation (2) is in the intercept form, where the intercepts on the *x* and *y* axes are $\frac{3}{2}$ and –2 respectively.

(iii) The given equation is 3*y* + 2 = 0.

It can be written as

$$3y = -2$$

$$\text{i.e., } \frac{y}{\left(-\frac{2}{3}\right)} = 1 \qquad \ldots(3)$$

This equation is of the form $\frac{x}{a} + \frac{y}{b} = 1$, where *a* = 0 and *b* = $-\frac{2}{3}$.

Therefore, equation (3) is in the intercept form, where the intercept on the *y*-axis is $-\frac{2}{3}$ and it has no intercept on the *x*-axis.

Question 3:

Reduce the following equations into normal form. Find their perpendicular distances from the origin and angle between perpendicular and the positive *x*-axis.

(i) $x - \sqrt{3}y + 8 = 0$ (ii) *y* – 2 = 0 (iii) *x* – *y* = 4

Answer

(i) The given equation is $x - \sqrt{3}y + 8 = 0$.

It can be reduced as:

$$x - \sqrt{3}y = -8$$
$$\Rightarrow -x + \sqrt{3}y = 8$$

On dividing both sides by $\sqrt{(-1)^2 + \left(\sqrt{3}\right)^2} = \sqrt{4} = 2$, we obtain

$$-\frac{x}{2}+\frac{\sqrt{3}}{2}y=\frac{8}{2}$$

$$\Rightarrow\left(-\frac{1}{2}\right)x+\left(\frac{\sqrt{3}}{2}\right)y=4$$

$$\Rightarrow x\cos 120^\circ + y\sin 120^\circ = 4 \qquad ...(1)$$

Equation (1) is in the normal form.

On comparing equation (1) with the normal form of equation of line

$x \cos \omega + y \sin \omega = p$, we obtain $\omega = 120°$ and $p = 4$.

Thus, the perpendicular distance of the line from the origin is 4, while the angle between the perpendicular and the positive *x*-axis is 120°.

(ii) The given equation is $y - 2 = 0$.

It can be reduced as $0.x + 1.y = 2$

On dividing both sides by $\sqrt{0^2+1^2}=1$, we obtain $0.x + 1.y = 2$

$\Rightarrow x \cos 90° + y \sin 90° = 2$ … (1)

Equation (1) is in the normal form.

On comparing equation (1) with the normal form of equation of line

$x \cos \omega + y \sin \omega = p$, we obtain $\omega = 90°$ and $p = 2$.

Thus, the perpendicular distance of the line from the origin is 2, while the angle between the perpendicular and the positive *x*-axis is 90°.

(iii) The given equation is $x - y = 4$.

It can be reduced as $1.x + (-1)\ y = 4$

On dividing both sides by $\sqrt{1^2+(-1)^2}=\sqrt{2}$, we obtain

$$\frac{1}{\sqrt{2}}x+\left(-\frac{1}{\sqrt{2}}\right)y=\frac{4}{\sqrt{2}}$$

$$\Rightarrow x\cos\left(2\pi-\frac{\pi}{4}\right)+y\sin\left(2\pi-\frac{\pi}{4}\right)=2\sqrt{2}$$

$$\Rightarrow x\cos 315^\circ + y\sin 315^\circ = 2\sqrt{2} \qquad ...(1)$$

Equation (1) is in the normal form.

On comparing equation (1) with the normal form of equation of line

$x \cos \omega + y \sin \omega = p$, we obtain $\omega = 315°$ and $p=2\sqrt{2}$.

Thus, the perpendicular distance of the line from the origin is $2\sqrt{2}$, while the angle between the perpendicular and the positive *x*-axis is 315°.

Question 4:

Find the distance of the point (−1, 1) from the line 12(*x* + 6) = 5(*y* − 2).

Answer

The given equation of the line is 12(*x* + 6) = 5(*y* − 2).

⇒ 12*x* + 72 = 5*y* − 10

⇒12*x* − 5*y* + 82 = 0 … (1)

On comparing equation (1) with general equation of line *Ax* + *By* + *C* = 0, we obtain *A* = 12, *B* = −5, and *C* = 82.

It is known that the perpendicular distance (*d*) of a line *Ax* + *By* + *C* = 0 from a point (x_1, y_1) is given by $d = \frac{|Ax_1 + By_1 + C|}{\sqrt{A^2 + B^2}}$.

The given point is (x_1, y_1) = (−1, 1).

Therefore, the distance of point (−1, 1) from the given line

$$= \frac{|12(-1)+(-5)(1)+82|}{\sqrt{(12)^2+(-5)^2}} \text{ units} = \frac{|-12-5+82|}{\sqrt{169}} \text{ units} = \frac{|65|}{13} \text{ units} = 5 \text{ units}$$

Question 5:

Find the points on the *x*-axis, whose distances from the line $\frac{x}{3} + \frac{y}{4} = 1$ are 4 units.

Answer

The given equation of line is

$$\frac{x}{3} + \frac{y}{4} = 1$$
$$\text{or, } 4x + 3y - 12 = 0 \qquad \ldots(1)$$

On comparing equation (1) with general equation of line *Ax* + *By* + *C* = 0, we obtain *A* = 4, *B* = 3, and *C* = −12.

Let (*a*, 0) be the point on the *x*-axis whose distance from the given line is 4 units.

It is known that the perpendicular distance (d) of a line $Ax + By + C = 0$ from a point (x_1, y_1) is given by $d = \frac{|Ax_1 + By_1 + C|}{\sqrt{A^2 + B^2}}$.

Therefore,

$$4 = \frac{|4a + 3 \times 0 - 12|}{\sqrt{4^2 + 3^2}}$$

$$\Rightarrow 4 = \frac{|4a - 12|}{5}$$

$$\Rightarrow |4a - 12| = 20$$

$$\Rightarrow \pm(4a - 12) = 20$$

$$\Rightarrow (4a - 12) = 20 \text{ or } -(4a - 12) = 20$$

$$\Rightarrow 4a = 20 + 12 \text{ or } 4a = -20 + 12$$

$$\Rightarrow a = 8 \text{ or } -2$$

Thus, the required points on the x-axis are (−2, 0) and (8, 0).

Question 6:

Find the distance between parallel lines

(i) $15x + 8y - 34 = 0$ and $15x + 8y + 31 = 0$

(ii) $l(x + y) + p = 0$ and $l(x + y) - r = 0$

Answer

It is known that the distance (d) between parallel lines $Ax + By + C_1 = 0$ and $Ax + By + C_2 = 0$ is given by $d = \frac{|C_1 - C_2|}{\sqrt{A^2 + B^2}}$.

(i) The given parallel lines are $15x + 8y - 34 = 0$ and $15x + 8y + 31 = 0$.

Here, $A = 15$, $B = 8$, $C_1 = -34$, and $C_2 = 31$.

Therefore, the distance between the parallel lines is

$$d = \frac{|C_1 - C_2|}{\sqrt{A^2 + B^2}} = \frac{|-34 - 31|}{\sqrt{(15)^2 + (8)^2}} \text{ units} = \frac{|-65|}{17} \text{ units} = \frac{65}{17} \text{ units}$$

(ii) The given parallel lines are $l(x + y) + p = 0$ and $l(x + y) - r = 0$.

$lx + ly + p = 0$ and $lx + ly - r = 0$

Here, $A = l$, $B = l$, $C_1 = p$, and $C_2 = -r$.

Therefore, the distance between the parallel lines is

$$d = \frac{|C_1 - C_2|}{\sqrt{A^2 + B^2}} = \frac{|p+r|}{\sqrt{l^2 + l^2}} \text{ units} = \frac{|p+r|}{\sqrt{2l^2}} \text{ units} = \frac{|p+r|}{l\sqrt{2}} \text{ units} = \frac{1}{\sqrt{2}}\left|\frac{p+r}{l}\right| \text{ units}$$

Question 7:

Find equation of the line parallel to the line $3x - 4y + 2 = 0$ and passing through the point (–2, 3).

Answer

The equation of the given line is

$$3x - 4y + 2 = 0$$

$$\text{or } y = \frac{3x}{4} + \frac{2}{4}$$

$\text{or } y = \frac{3}{4}x + \frac{1}{2}$, which is of the form $y = mx + c$

∴ Slope of the given line $= \frac{3}{4}$

It is known that parallel lines have the same slope.

∴ Slope of the other line = $m = \frac{3}{4}$

Now, the equation of the line that has a slope of $\frac{3}{4}$ and passes through the point (–2, 3) is

$$(y-3) = \frac{3}{4}\{x - (-2)\}$$

$$4y - 12 = 3x + 6$$

$$\text{i.e., } 3x - 4y + 18 = 0$$

Question 8:

Find equation of the line perpendicular to the line $x - 7y + 5 = 0$ and having x intercept 3.

Answer

The given equation of line is $x - 7y + 5 = 0$.

Or, $y=\frac{1}{7}x+\frac{5}{7}$, which is of the form $y = mx + c$

∴Slope of the given line $=\frac{1}{7}$

The slope of the line perpendicular to the line having a slope of $\frac{1}{7}$ is $m=-\frac{1}{\left(\frac{1}{7}\right)}=-7$

The equation of the line with slope −7 and x-intercept 3 is given by

$y = m (x - d)$

⇒ $y = -7 (x - 3)$

⇒ $y = -7x + 21$

⇒ $7x + y = 21$

Question 9:

Find angles between the lines $\sqrt{3}x+y=1$ and $x+\sqrt{3}y=1$

Answer

The given lines are $\sqrt{3}x+y=1$ and $x+\sqrt{3}y=1$.

$y=-\sqrt{3}x+1$ ...(1) and $y=-\frac{1}{\sqrt{3}}x+\frac{1}{\sqrt{3}}$ ...(2)

The slope of line (1) is $m_1=-\sqrt{3}$, while the slope of line (2) is $m_2=-\frac{1}{\sqrt{3}}$.

The acute angle i.e., θ between the two lines is given by

$$\tan\theta = \left|\frac{m_1 - m_2}{1 + m_1 m_2}\right|$$

$$\tan\theta = \left|\frac{-\sqrt{3} + \frac{1}{\sqrt{3}}}{1 + \left(-\sqrt{3}\right)\left(-\frac{1}{\sqrt{3}}\right)}\right|$$

$$\tan\theta = \left|\frac{\frac{-3+1}{\sqrt{3}}}{1+1}\right| = \left|\frac{-2}{2\times\sqrt{3}}\right|$$

$$\tan\theta = \frac{1}{\sqrt{3}}$$

$$\theta = 30^\circ$$

Thus, the angle between the given lines is either 30° or 180° – 30° = 150°.

Question 10:

The line through the points (h, 3) and (4, 1) intersects the line $7x - 9y - 19 = 0$. at right angle. Find the value of h.

Answer

The slope of the line passing through points (h, 3) and (4, 1) is

$$m_1 = \frac{1-3}{4-h} = \frac{-2}{4-h}$$

The slope of line $7x - 9y - 19 = 0$ or $y = \frac{7}{9}x - \frac{19}{9}$ is $m_2 = \frac{7}{9}$.

It is given that the two lines are perpendicular.

$$\therefore m_1 \times m_2 = -1$$

$$\Rightarrow \left(\frac{-2}{4-h}\right) \times \left(\frac{7}{9}\right) = -1$$

$$\Rightarrow \frac{-14}{36-9h} = -1$$

$$\Rightarrow 14 = 36 - 9h$$

$$\Rightarrow 9h = 36 - 14$$

$$\Rightarrow h = \frac{22}{9}$$

Thus, the value of h is $\frac{22}{9}$.

Question 11:

Prove that the line through the point (x_1, y_1) and parallel to the line Ax + By + C = 0 is A (x –x_1) + B (y – y_1) = 0.

Answer

The slope of line Ax + By + C = 0 or $y = \left(\frac{-A}{B}\right)x + \left(\frac{-C}{B}\right)$ is $m = -\frac{A}{B}$

It is known that parallel lines have the same slope.

∴ Slope of the other line = $m = -\frac{A}{B}$

The equation of the line passing through point (x_1, y_1) and having a slope $m = -\frac{A}{B}$ is

$$y - y_1 = m(x - x_1)$$

$$y - y_1 = -\frac{A}{B}(x - x_1)$$

$$B(y - y_1) = -A(x - x_1)$$

$$A(x - x_1) + B(y - y_1) = 0$$

Hence, the line through point (x_1, y_1) and parallel to line Ax + By + C = 0 is

A (x –x_1) + B (y – y_1) = 0

Question 12:

Two lines passing through the point (2, 3) intersects each other at an angle of 60°. If slope of one line is 2, find equation of the other line.

Answer

It is given that the slope of the first line, m_1 = 2.

Let the slope of the other line be m_2.

The angle between the two lines is 60°.

$$\therefore \tan 60° = \left|\frac{m_1 - m_2}{1 + m_1 m_2}\right|$$

$$\Rightarrow \sqrt{3} = \left|\frac{2 - m_2}{1 + 2m_2}\right|$$

$$\Rightarrow \sqrt{3} = \pm\left(\frac{2 - m_2}{1 + 2m_2}\right)$$

$$\Rightarrow \sqrt{3} = \frac{2 - m_2}{1 + 2m_2} \text{ or } \sqrt{3} = -\left(\frac{2 - m_2}{1 + 2m_2}\right)$$

$$\Rightarrow \sqrt{3}\left(1 + 2m_2\right) = 2 - m_2 \text{ or } \sqrt{3}\left(1 + 2m_2\right) = -\left(2 - m_2\right)$$

$$\Rightarrow \sqrt{3} + 2\sqrt{3}m_2 + m_2 = 2 \text{ or } \sqrt{3} + 2\sqrt{3}m_2 - m_2 = -2$$

$$\Rightarrow \sqrt{3} + \left(2\sqrt{3} + 1\right)m_2 = 2 \text{ or } \sqrt{3} + \left(2\sqrt{3} - 1\right)m_2 = -2$$

$$\Rightarrow m_2 = \frac{2 - \sqrt{3}}{\left(2\sqrt{3} + 1\right)} \text{ or } m_2 = \frac{-\left(2 + \sqrt{3}\right)}{\left(2\sqrt{3} - 1\right)}$$

Case I : $m_2 = \left(\frac{2 - \sqrt{3}}{2\sqrt{3} + 1}\right)$

The equation of the line passing through point (2, 3) and having a slope of $\frac{\left(2 - \sqrt{3}\right)}{\left(2\sqrt{3} + 1\right)}$ is

$$\left(y - 3\right) = \frac{2 - \sqrt{3}}{2\sqrt{3} + 1}\left(x - 2\right)$$

$$\left(2\sqrt{3} + 1\right)y - 3\left(2\sqrt{3} + 1\right) = \left(2 - \sqrt{3}\right)x - 2\left(2 - \sqrt{3}\right)$$

$$\left(\sqrt{3} - 2\right)x + \left(2\sqrt{3} + 1\right)y = -4 + 2\sqrt{3} + 6\sqrt{3} + 3$$

$$\left(\sqrt{3} - 2\right)x + \left(2\sqrt{3} + 1\right)y = -1 + 8\sqrt{3}$$

In this case, the equation of the other line is $\left(\sqrt{3}-2\right)x+\left(2\sqrt{3}+1\right)y=-1+8\sqrt{3}$.

Case II : $m_2=\dfrac{-\left(2+\sqrt{3}\right)}{\left(2\sqrt{3}-1\right)}$

The equation of the line passing through point (2, 3) and having a slope of $\dfrac{-\left(2+\sqrt{3}\right)}{\left(2\sqrt{3}-1\right)}$ is

$$(y-3)=\frac{-\left(2+\sqrt{3}\right)}{\left(2\sqrt{3}-1\right)}(x-2)$$
$$\left(2\sqrt{3}-1\right)y-3\left(2\sqrt{3}-1\right)=-\left(2+\sqrt{3}\right)x+2\left(2+\sqrt{3}\right)$$
$$\left(2\sqrt{3}-1\right)y+\left(2+\sqrt{3}\right)x=4+2\sqrt{3}+6\sqrt{3}-3$$
$$\left(2+\sqrt{3}\right)x+\left(2\sqrt{3}-1\right)y=1+8\sqrt{3}$$

In this case, the equation of the other line is $\left(2+\sqrt{3}\right)x+\left(2\sqrt{3}-1\right)y=1+8\sqrt{3}$.

Thus, the required equation of the other line is $\left(\sqrt{3}-2\right)x+\left(2\sqrt{3}+1\right)y=-1+8\sqrt{3}$ or $\left(2+\sqrt{3}\right)x+\left(2\sqrt{3}-1\right)y=1+8\sqrt{3}$.

Question 13:

Find the equation of the right bisector of the line segment joining the points (3, 4) and (–1, 2).

Answer

The right bisector of a line segment bisects the line segment at 90°.

The end-points of the line segment are given as A (3, 4) and B (–1, 2).

Accordingly, mid-point of AB $=\left(\dfrac{3-1}{2},\dfrac{4+2}{2}\right)=(1,3)$

Slope of AB $=\dfrac{2-4}{-1-3}=\dfrac{-2}{-4}=\dfrac{1}{2}$

∴Slope of the line perpendicular to AB = $-\frac{1}{\left(\frac{1}{2}\right)}=-2$

The equation of the line passing through (1, 3) and having a slope of −2 is

$(y - 3) = -2(x - 1)$

$y - 3 = -2x + 2$

$2x + y = 5$

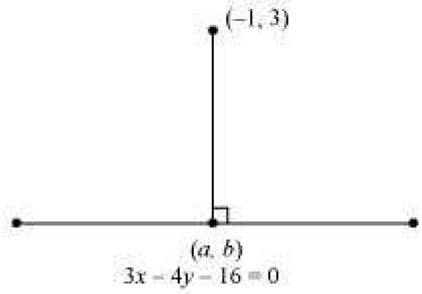
Thus, the required equation of the line is $2x + y = 5$.

Question 14:

Find the coordinates of the foot of perpendicular from the point (−1, 3) to the line $3x - 4y - 16 = 0$.

Answer

Let (a, b) be the coordinates of the foot of the perpendicular from the point (−1, 3) to the line $3x - 4y - 16 = 0$.



Slope of the line joining (−1, 3) and (a, b), $m_1 = \frac{b-3}{a+1}$

Slope of the line $3x - 4y - 16 = 0$ or $y=\frac{3}{4}x-4,\ m_2=\frac{3}{4}$

Since these two lines are perpendicular, $m_1m_2 = -1$

$$\therefore \left(\frac{b-3}{a+1}\right)\times\left(\frac{3}{4}\right)=-1$$

$$\Rightarrow \frac{3b-9}{4a+4}=-1$$

$$\Rightarrow 3b-9=-4a-4$$

$$\Rightarrow 4a+3b=5 \qquad \ldots(1)$$

Point (a, b) lies on line $3x - 4y = 16$.

$\therefore 3a - 4b = 16$ … (2)

On solving equations (1) and (2), we obtain

$$a = \frac{68}{25} \text{ and } b = -\frac{49}{25}$$

Thus, the required coordinates of the foot of the perpendicular are $\left(\frac{68}{25}, -\frac{49}{25}\right)$.

Question 15:

The perpendicular from the origin to the line $y = mx + c$ meets it at the point (–1, 2). Find the values of m and c.

Answer

The given equation of line is $y = mx + c$.

It is given that the perpendicular from the origin meets the given line at (–1, 2).

Therefore, the line joining the points (0, 0) and (–1, 2) is perpendicular to the given line.

$\therefore$ Slope of the line joining (0, 0) and (–1, 2) $= \frac{2}{-1} = -2$

The slope of the given line is m.

$$\therefore m \times -2 = -1 \qquad \left[\text{The two lines are perpendicular}\right]$$

$$\Rightarrow m = \frac{1}{2}$$

Since point (–1, 2) lies on the given line, it satisfies the equation $y = mx + c$.

$$\therefore 2 = m(-1) + c$$

$$\Rightarrow 2 = \frac{1}{2}(-1) + c$$

$$\Rightarrow c = 2 + \frac{1}{2} = \frac{5}{2}$$

Thus, the respective values of m and c are $\frac{1}{2}$ and $\frac{5}{2}$.

Question 16:

If p and q are the lengths of perpendiculars from the origin to the lines $x \cos \theta - y \sin \theta = k \cos 2\theta$ and $x \sec \theta + y \operatorname{cosec} \theta = k$, respectively, prove that $p^2 + 4q^2 = k^2$

Answer

The equations of given lines are

$x \cos \theta - y \sin\theta = k \cos 2\theta$ … (1)

$x \sec\theta + y \operatorname{cosec} \theta = k$ … (2)

The perpendicular distance (d) of a line $Ax + By + C = 0$ from a point (x_1, y_1) is given by

$$d = \frac{|Ax_1 + By_1 + C|}{\sqrt{A^2 + B^2}}.$$

On comparing equation (1) to the general equation of line i.e., $Ax + By + C = 0$, we obtain $A = \cos\theta$, $B = -\sin\theta$, and $C = -k \cos 2\theta$.

It is given that p is the length of the perpendicular from (0, 0) to line (1).

$$\therefore p = \frac{|A(0) + B(0) + C|}{\sqrt{A^2 + B^2}} = \frac{|C|}{\sqrt{A^2 + B^2}} = \frac{|-k \cos 2\theta|}{\sqrt{\cos^2 \theta + \sin^2 \theta}} = |-k \cos 2\theta| \qquad \ldots(3)$$

On comparing equation (2) to the general equation of line i.e., $Ax + By + C = 0$, we obtain $A = \sec\theta$, $B = \operatorname{cosec}\theta$, and $C = -k$.

It is given that q is the length of the perpendicular from (0, 0) to line (2).

$$\therefore q = \frac{|A(0) + B(0) + C|}{\sqrt{A^2 + B^2}} = \frac{|C|}{\sqrt{A^2 + B^2}} = \frac{|-k|}{\sqrt{\sec^2 \theta + \operatorname{cosec}^2 \theta}} \qquad \ldots(4)$$

From (3) and (4), we have

$$p^2+4q^2=\left(\left|-k\cos 2\theta\right|\right)^2+4\left(\frac{\left|-k\right|}{\sqrt{\sec^2\theta+\operatorname{cosec}^2\theta}}\right)^2$$

$$=k^2\cos^2 2\theta+\frac{4k^2}{\left(\sec^2\theta+\operatorname{cosec}^2\theta\right)}$$

$$=k^2\cos^2 2\theta+\frac{4k^2}{\left(\frac{1}{\cos^2\theta}+\frac{1}{\sin^2\theta}\right)}$$

$$=k^2\cos^2 2\theta+\frac{4k^2}{\left(\frac{\sin^2\theta+\cos^2\theta}{\sin^2\theta\cos^2\theta}\right)}$$

$$=k^2\cos^2 2\theta+\frac{4k^2}{\left(\frac{1}{\sin^2\theta\cos^2\theta}\right)}$$

$$=k^2\cos^2 2\theta+4k^2\sin^2\theta\cos^2\theta$$

$$=k^2\cos^2 2\theta+k^2\left(2\sin\theta\cos\theta\right)^2$$

$$=k^2\cos^2 2\theta+k^2\sin^2 2\theta$$

$$=k^2\left(\cos^2 2\theta+\sin^2 2\theta\right)$$

$$=k^2$$

Hence, we proved that $p^2 + 4q^2 = k^2$.

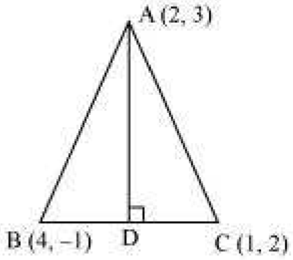
Question 17:

In the triangle ABC with vertices A (2, 3), B (4, –1) and C (1, 2), find the equation and length of altitude from the vertex A.

Answer

Let AD be the altitude of triangle ABC from vertex A.

Accordingly, AD⊥BC



The equation of the line passing through point (2, 3) and having a slope of 1 is

$(y - 3) = 1(x - 2)$

$\Rightarrow x - y + 1 = 0$

$\Rightarrow y - x = 1$

Therefore, equation of the altitude from vertex A = $y - x = 1$.

Length of AD = Length of the perpendicular from A (2, 3) to BC

The equation of BC is

$$(y+1) = \frac{2+1}{1-4}(x-4)$$
$$\Rightarrow (y+1) = -1(x-4)$$
$$\Rightarrow y+1 = -x+4$$
$$\Rightarrow x+y-3 = 0 \qquad ...(1)$$

The perpendicular distance (d) of a line $Ax + By + C = 0$ from a point (x_1, y_1) is given by

$$d = \frac{|Ax_1 + By_1 + C|}{\sqrt{A^2 + B^2}}.$$

On comparing equation (1) to the general equation of line $Ax + By + C = 0$, we obtain $A = 1$, $B = 1$, and $C = -3$.

∴Length of AD $= \dfrac{|1\times 2 + 1\times 3 - 3|}{\sqrt{1^2 + 1^2}}$ units $= \dfrac{|2|}{\sqrt{2}}$ units $= \dfrac{2}{\sqrt{2}}$ units $= \sqrt{2}$ units

Thus, the equation and the length of the altitude from vertex A are $y - x = 1$ and $\sqrt{2}$ units respectively.

Question 18:

If p is the length of perpendicular from the origin to the line whose intercepts on the axes are a and b, then show that $\frac{1}{p^2}=\frac{1}{a^2}+\frac{1}{b^2}$.

Answer

It is known that the equation of a line whose intercepts on the axes are a and b is

$$\frac{x}{a}+\frac{y}{b}=1$$
$$\text{or } bx+ay=ab$$
$$\text{or } bx+ay-ab=0 \qquad ...(1)$$

The perpendicular distance (d) of a line $Ax + By + C = 0$ from a point (x_1, y_1) is given by

$$d=\frac{|Ax_1+By_1+C|}{\sqrt{A^2+B^2}}.$$

On comparing equation (1) to the general equation of line $Ax + By + C = 0$, we obtain $A = b$, $B = a$, and $C = -ab$.

Therefore, if p is the length of the perpendicular from point $(x_1, y_1) = (0, 0)$ to line (1), we obtain

$$p=\frac{|A(0)+B(0)-ab|}{\sqrt{b^2+a^2}}$$
$$\Rightarrow p=\frac{|-ab|}{\sqrt{a^2+b^2}}$$

On squaring both sides, we obtain

$$p^2=\frac{(-ab)^2}{a^2+b^2}$$
$$\Rightarrow p^2\left(a^2+b^2\right)=a^2b^2$$
$$\Rightarrow \frac{a^2+b^2}{a^2b^2}=\frac{1}{p^2}$$
$$\Rightarrow \frac{1}{p^2}=\frac{1}{a^2}+\frac{1}{b^2}$$