Exercise 10.3

Question 1:

Find the angle between two vectors $\vec{a}$ and $\vec{b}$ with magnitudes $\sqrt{3}$ and 2, respectively having $\vec{a}\cdot\vec{b}=\sqrt{6}$.

Answer

It is given that,

$$\left|\vec{a}\right|=\sqrt{3},\ \left|\vec{b}\right|=2 \text{ and, } \vec{a}\cdot\vec{b}=\sqrt{6}$$

Now, we know that $\vec{a}\cdot\vec{b}=\left|\vec{a}\right|\left|\vec{b}\right|\cos\theta$.

$$\therefore \sqrt{6}=\sqrt{3}\times 2\times\cos\theta$$

$$\Rightarrow \cos\theta=\frac{\sqrt{6}}{\sqrt{3}\times 2}$$

$$\Rightarrow \cos\theta=\frac{1}{\sqrt{2}}$$

$$\Rightarrow \theta=\frac{\pi}{4}$$

Hence, the angle between the given vectors $\vec{a}$ and $\vec{b}$ is $\frac{\pi}{4}$.

Question 2:

Find the angle between the vectors $\hat{i}-2\hat{j}+3\hat{k}$ and $3\hat{i}-2\hat{j}+\hat{k}$

Answer

The given vectors are $\vec{a}=\hat{i}-2\hat{j}+3\hat{k}$ and $\vec{b}=3\hat{i}-2\hat{j}+\hat{k}$.

$$|\vec{a}| = \sqrt{1^2 + (-2)^2 + 3^2} = \sqrt{1+4+9} = \sqrt{14}$$

$$|\vec{b}| = \sqrt{3^2 + (-2)^2 + 1^2} = \sqrt{9+4+1} = \sqrt{14}$$

$$\begin{aligned}\text{Now, } \vec{a}\cdot\vec{b} &= (\hat{i} - 2\hat{j} + 3\hat{k})(3\hat{i} - 2\hat{j} + \hat{k}) \\ &= 1.3 + (-2)(-2) + 3.1 \\ &= 3 + 4 + 3 \\ &= 10\end{aligned}$$

Also, we know that $\vec{a}\cdot\vec{b} = |\vec{a}||\vec{b}|\cos\theta$.

$$\therefore 10 = \sqrt{14}\sqrt{14}\cos\theta$$

$$\Rightarrow \cos\theta = \frac{10}{14}$$

$$\Rightarrow \theta = \cos^{-1}\left(\frac{5}{7}\right)$$

Question 3:

Find the projection of the vector $\hat{i} - \hat{j}$ on the vector $\hat{i} + \hat{j}$.

Answer

Let $\vec{a} = \hat{i} - \hat{j}$ and $\vec{b} = \hat{i} + \hat{j}$.

Now, projection of vector $\vec{a}$ on $\vec{b}$ is given by,

$$\frac{1}{|\vec{b}|}(\vec{a}.\vec{b}) = \frac{1}{\sqrt{1+1}}\{1.1 + (-1)(1)\} = \frac{1}{\sqrt{2}}(1-1) = 0$$

Hence, the projection of vector $\vec{a}$ on $\vec{b}$ is 0.

Question 4:

Find the projection of the vector $\hat{i} + 3\hat{j} + 7\hat{k}$ on the vector $7\hat{i} - \hat{j} + 8\hat{k}$.

Answer

Let $\vec{a} = \hat{i} + 3\hat{j} + 7\hat{k}$ and $\hat{b} = 7\hat{i} - \hat{j} + 8\hat{k}$.

Now, projection of vector $\vec{a}$ on $\vec{b}$ is given by,

$$\frac{1}{|\vec{b}|}\left(\vec{a}\cdot\vec{b}\right)=\frac{1}{\sqrt{7^2+(-1)^2+8^2}}\{1(7)+3(-1)+7(8)\}=\frac{7-3+56}{\sqrt{49+1+64}}=\frac{60}{\sqrt{114}}$$

Question 5:

Show that each of the given three vectors is a unit vector:

$$\frac{1}{7}\left(2\hat{i}+3\hat{j}+6\hat{k}\right),\frac{1}{7}\left(3\hat{i}-6\hat{j}+2\hat{k}\right),\frac{1}{7}\left(6\hat{i}+2\hat{j}-3\hat{k}\right)$$

Also, show that they are mutually perpendicular to each other.

Answer

$$\text{Let } \vec{a}=\frac{1}{7}\left(2\hat{i}+3\hat{j}+6\hat{k}\right)=\frac{2}{7}\hat{i}+\frac{3}{7}\hat{j}+\frac{6}{7}\hat{k},$$

$$\vec{b}=\frac{1}{7}\left(3\hat{i}-6\hat{j}+2\hat{k}\right)=\frac{3}{7}\hat{i}-\frac{6}{7}\hat{j}+\frac{2}{7}\hat{k},$$

$$\vec{c}=\frac{1}{7}\left(6\hat{i}+2\hat{j}-3\hat{k}\right)=\frac{6}{7}\hat{i}+\frac{2}{7}\hat{j}-\frac{3}{7}\hat{k}.$$

$$|\vec{a}|=\sqrt{\left(\frac{2}{7}\right)^2+\left(\frac{3}{7}\right)^2+\left(\frac{6}{7}\right)^2}=\sqrt{\frac{4}{49}+\frac{9}{49}+\frac{36}{49}}=1$$

$$|\vec{b}|=\sqrt{\left(\frac{3}{7}\right)^2+\left(-\frac{6}{7}\right)^2+\left(\frac{2}{7}\right)^2}=\sqrt{\frac{9}{49}+\frac{36}{49}+\frac{4}{49}}=1$$

$$|\vec{c}|=\sqrt{\left(\frac{6}{7}\right)^2+\left(\frac{2}{7}\right)^2+\left(-\frac{3}{7}\right)^2}=\sqrt{\frac{36}{49}+\frac{4}{49}+\frac{9}{49}}=1$$

Thus, each of the given three vectors is a unit vector.

$$\vec{a}\cdot\vec{b}=\frac{2}{7}\times\frac{3}{7}+\frac{3}{7}\times\left(\frac{-6}{7}\right)+\frac{6}{7}\times\frac{2}{7}=\frac{6}{49}-\frac{18}{49}+\frac{12}{49}=0$$

$$\vec{b}\cdot\vec{c}=\frac{3}{7}\times\frac{6}{7}+\left(\frac{-6}{7}\right)\times\frac{2}{7}+\frac{2}{7}\times\left(\frac{-3}{7}\right)=\frac{18}{49}-\frac{12}{49}-\frac{6}{49}=0$$

$$\vec{c}\cdot\vec{a}=\frac{6}{7}\times\frac{2}{7}+\frac{2}{7}\times\frac{3}{7}+\left(\frac{-3}{7}\right)\times\frac{6}{7}=\frac{12}{49}+\frac{6}{49}-\frac{18}{49}=0$$

Hence, the given three vectors are mutually perpendicular to each other.

Question 6:

Find $|\vec{a}|$ and $|\vec{b}|$, if $(\vec{a}+\vec{b})\cdot(\vec{a}-\vec{b})=8$ and $|\vec{a}|=8|\vec{b}|$.

Answer

$$(\vec{a}\cdot\vec{b}).(\vec{a}-\vec{b})=8$$

$$\Rightarrow \vec{a}\cdot\vec{a}-\vec{a}.\vec{b}+\vec{b}\cdot\vec{a}-\vec{b}\cdot\vec{b}=8$$

$$\Rightarrow |\vec{a}|^2-|\vec{b}|^2=8$$

$$\Rightarrow (8|\vec{b}|)^2-|\vec{b}|^2=8 \qquad [|\vec{a}|=8|\vec{b}|]$$

$$\Rightarrow 64|\vec{b}|^2-|\vec{b}|^2=8$$

$$\Rightarrow 63|\vec{b}|^2=8$$

$$\Rightarrow |\vec{b}|^2=\frac{8}{63}$$

$$\Rightarrow |\vec{b}|=\sqrt{\frac{8}{63}} \qquad [\text{Magnitude of a vector is non-negative}]$$

$$\Rightarrow |\vec{b}|=\frac{2\sqrt{2}}{3\sqrt{7}}$$

$$|\vec{a}|=8|\vec{b}|=\frac{8\times 2\sqrt{2}}{3\sqrt{7}}=\frac{16\sqrt{2}}{3\sqrt{7}}$$

Question 7:

Evaluate the product $(3\vec{a}-5\vec{b})\cdot(2\vec{a}+7\vec{b})$.

Answer

$$(3\vec{a}-5\vec{b})\cdot(2\vec{a}+7\vec{b})$$

$$=3\vec{a}\cdot 2\vec{a}+3\vec{a}\cdot 7\vec{b}-5\vec{b}\cdot 2\vec{a}-5\vec{b}\cdot 7\vec{b}$$

$$=6\vec{a}\cdot\vec{a}+21\vec{a}\cdot\vec{b}-10\vec{a}\cdot\vec{b}-35\vec{b}\cdot\vec{b}$$

$$=6|\vec{a}|^2+11\vec{a}\cdot\vec{b}-35|\vec{b}|^2$$

Question 8:

Find the magnitude of two vectors $\vec{a}$ and $\vec{b}$, having the same magnitude and such that the angle between them is 60° and their scalar product is $\frac{1}{2}$.

Answer

Let θ be the angle between the vectors $\vec{a}$ and $\vec{b}$.

It is given that $|\vec{a}| = |\vec{b}|,\ \vec{a}\cdot\vec{b} = \frac{1}{2}$, and $\theta = 60°.$ ...(1)

We know that $\vec{a}\cdot\vec{b} = |\vec{a}||\vec{b}|\cos\theta$.

$$\therefore \frac{1}{2} = |\vec{a}||\vec{a}|\cos 60° \qquad [\text{Using (1)}]$$

$$\Rightarrow \frac{1}{2} = |\vec{a}|^2 \times \frac{1}{2}$$

$$\Rightarrow |\vec{a}|^2 = 1$$

$$\Rightarrow |\vec{a}| = |\vec{b}| = 1$$

Question 9:

Find $|\vec{x}|$, if for a unit vector $\vec{a},\ (\vec{x}-\vec{a})\cdot(\vec{x}+\vec{a}) = 12$.

Answer

$$(\vec{x}-\vec{a})\cdot(\vec{x}+\vec{a}) = 12$$

$$\Rightarrow \vec{x}\cdot\vec{x} + \vec{x}\cdot\vec{a} - \vec{a}\cdot\vec{x} - \vec{a}\cdot\vec{a} = 12$$

$$\Rightarrow |\vec{x}|^2 - |\vec{a}|^2 = 12$$

$$\Rightarrow |\vec{x}|^2 - 1 = 12 \qquad [|\vec{a}| = 1 \text{ as } \vec{a} \text{ is a unit vector}]$$

$$\Rightarrow |\vec{x}|^2 = 13$$

$$\therefore |\vec{x}| = \sqrt{13}$$

Question 10:

If $\vec{a} = 2\hat{i} + 2\hat{j} + 3\hat{k},\ \vec{b} = -\hat{i} + 2\hat{j} + \hat{k}$ and $\vec{c} = 3\hat{i} + \hat{j}$ are such that $\vec{a} + \lambda\vec{b}$ is perpendicular to $\vec{c}$, then find the value of λ.

Answer

The given vectors are $\vec{a} = 2\hat{i} + 2\hat{j} + 3\hat{k},\ \vec{b} = -\hat{i} + 2\hat{j} + \hat{k},$ and $\vec{c} = 3\hat{i} + \hat{j}.$

Now,

$$\vec{a} + \lambda\vec{b} = \left(2\hat{i} + 2\hat{j} + 3\hat{k}\right) + \lambda\left(-\hat{i} + 2\hat{j} + \hat{k}\right) = (2-\lambda)\hat{i} + (2+2\lambda)\hat{j} + (3+\lambda)\hat{k}$$

If $\left(\vec{a} + \lambda\vec{b}\right)$ is perpendicular to $\vec{c}$, then

$$\left(\vec{a} + \lambda\vec{b}\right)\cdot\vec{c} = 0.$$

$$\Rightarrow \left[(2-\lambda)\hat{i} + (2+2\lambda)\hat{j} + (3+\lambda)\hat{k}\right]\cdot\left(3\hat{i} + \hat{j}\right) = 0$$

$$\Rightarrow (2-\lambda)3 + (2+2\lambda)1 + (3+\lambda)0 = 0$$

$$\Rightarrow 6 - 3\lambda + 2 + 2\lambda = 0$$

$$\Rightarrow -\lambda + 8 = 0$$

$$\Rightarrow \lambda = 8$$

Hence, the required value of λ is 8.

Question 11:

Show that $|\vec{a}|\vec{b} + |\vec{b}|\vec{a}$ is perpendicular to $|\vec{a}|\vec{b} - |\vec{b}|\vec{a}$, for any two nonzero vectors $\vec{a}$ and $\vec{b}$

Answer

$$\left(|\vec{a}|\vec{b} + |\vec{b}|\vec{a}\right)\cdot\left(|\vec{a}|\vec{b} - |\vec{b}|\vec{a}\right)$$

$$= |\vec{a}|^2\,\vec{b}\cdot\vec{b} - |\vec{a}||\vec{b}|\vec{b}\cdot\vec{a} + |\vec{b}||\vec{a}|\vec{a}\cdot\vec{b} - |\vec{b}|^2\,\vec{a}\cdot\vec{a}$$

$$= |\vec{a}|^2|\vec{b}|^2 - |\vec{b}|^2|\vec{a}|^2$$

$$= 0$$

Hence, $|\vec{a}|\vec{b} + |\vec{b}|\vec{a}$ and $|\vec{a}|\vec{b} - |\vec{b}|\vec{a}$ are perpendicular to each other.

Question 12:

If $\vec{a}\cdot\vec{a} = 0$ and $\vec{a}\cdot\vec{b} = 0$, then what can be concluded about the vector $\vec{b}$?

Answer

It is given that $\vec{a}\cdot\vec{a}=0$ and $\vec{a}\cdot\vec{b}=0$.

Now,

$\vec{a}\cdot\vec{a}=0 \Rightarrow |\vec{a}|^2=0 \Rightarrow |\vec{a}|=0$

$\therefore \vec{a}$ is a zero vector.

Hence, vector $\vec{b}$ satisfying $\vec{a}\cdot\vec{b}=0$ can be any vector.

Question 14:

If either vector $\vec{a}=\vec{0}$ or $\vec{b}=\vec{0}$, then $\vec{a}\cdot\vec{b}=0$. But the converse need not be true. Justify your answer with an example.

Answer

Consider $\vec{a}=2\hat{i}+4\hat{j}+3\hat{k}$ and $\vec{b}=3\hat{i}+3\hat{j}-6\hat{k}$.

Then,

$\vec{a}\cdot\vec{b}=2.3+4.3+3(-6)=6+12-18=0$

We now observe that:

$|\vec{a}|=\sqrt{2^2+4^2+3^2}=\sqrt{29}$

$\therefore \vec{a}\neq\vec{0}$

$|\vec{b}|=\sqrt{3^2+3^2+(-6)^2}=\sqrt{54}$

$\therefore \vec{b}\neq\vec{0}$

Hence, the converse of the given statement need not be true.

Question 15:

If the vertices A, B, C of a triangle ABC are (1, 2, 3), (–1, 0, 0), (0, 1, 2), respectively, then find ☐ABC. [☐ABC is the angle between the vectors $\overrightarrow{BA}$ and $\overrightarrow{BC}$]

Answer

The vertices of ΔABC are given as A (1, 2, 3), B (–1, 0, 0), and C (0, 1, 2).

Also, it is given that ☐ABC is the angle between the vectors $\overrightarrow{BA}$ and $\overrightarrow{BC}$.

$$\overrightarrow{BA} = \{1-(-1)\}\hat{i} + (2-0)\hat{j} + (3-0)\hat{k} = 2\hat{i} + 2\hat{j} + 3\hat{k}$$

$$\overrightarrow{BC} = \{0-(-1)\}\hat{i} + (1-0)\hat{j} + (2-0)\hat{k} = \hat{i} + \hat{j} + 2\hat{k}$$

$$\therefore \overrightarrow{BA} \cdot \overrightarrow{BC} = \left(2\hat{i} + 2\hat{j} + 3\hat{k}\right) \cdot \left(\hat{i} + \hat{j} + 2\hat{k}\right) = 2\times1 + 2\times1 + 3\times2 = 2+2+6 = 10$$

$$\left|\overrightarrow{BA}\right| = \sqrt{2^2 + 2^2 + 3^2} = \sqrt{4+4+9} = \sqrt{17}$$

$$\left|\overrightarrow{BC}\right| = \sqrt{1+1+2^2} = \sqrt{6}$$

Now, it is known that:

$$\overrightarrow{BA} \cdot \overrightarrow{BC} = \left|\overrightarrow{BA}\right|\left|\overrightarrow{BC}\right|\cos(\angle ABC).$$

$$\therefore 10 = \sqrt{17} \times \sqrt{6}\cos(\angle ABC)$$

$$\Rightarrow \cos(\angle ABC) = \frac{10}{\sqrt{17}\times\sqrt{6}}$$

$$\Rightarrow \angle ABC = \cos^{-1}\left(\frac{10}{\sqrt{102}}\right)$$

Question 16:

Show that the points A (1, 2, 7), B (2, 6, 3) and C (3, 10, –1) are collinear.

Answer

The given points are A (1, 2, 7), B (2, 6, 3), and C (3, 10, –1).

$$\therefore \overrightarrow{AB} = (2-1)\hat{i} + (6-2)\hat{j} + (3-7)\hat{k} = \hat{i} + 4\hat{j} - 4\hat{k}$$

$$\overrightarrow{BC} = (3-2)\hat{i} + (10-6)\hat{j} + (-1-3)\hat{k} = \hat{i} + 4\hat{j} - 4\hat{k}$$

$$\overrightarrow{AC} = (3-1)\hat{i} + (10-2)\hat{j} + (-1-7)\hat{k} = 2\hat{i} + 8\hat{j} - 8\hat{k}$$

$$\left|\overrightarrow{AB}\right| = \sqrt{1^2 + 4^2 + (-4)^2} = \sqrt{1+16+16} = \sqrt{33}$$

$$\left|\overrightarrow{BC}\right| = \sqrt{1^2 + 4^2 + (-4)^2} = \sqrt{1+16+16} = \sqrt{33}$$

$$\left|\overrightarrow{AC}\right| = \sqrt{2^2 + 8^2 + 8^2} = \sqrt{4+64+64} = \sqrt{132} = 2\sqrt{33}$$

$$\therefore \left|\overrightarrow{AC}\right| = \left|\overrightarrow{AB}\right| + \left|\overrightarrow{BC}\right|$$

Hence, the given points A, B, and C are collinear.

Question 17:

Show that the vectors $2\hat{i}-\hat{j}+\hat{k},\ \hat{i}-3\hat{j}-5\hat{k}$ and $3\hat{i}-4\hat{j}-4\hat{k}$ form the vertices of a right angled triangle.

Answer

Let vectors $2\hat{i}-\hat{j}+\hat{k},\ \hat{i}-3\hat{j}-5\hat{k}$ and $3\hat{i}-4\hat{j}-4\hat{k}$ be position vectors of points A, B, and C respectively.

i.e., $\overrightarrow{OA}=2\hat{i}-\hat{j}+\hat{k},\ \overrightarrow{OB}=\hat{i}-3\hat{j}-5\hat{k}$ and $\overrightarrow{OC}=3\hat{i}-4\hat{j}-4\hat{k}$

Now, vectors $\overrightarrow{AB}, \overrightarrow{BC}$, and $\overrightarrow{AC}$ represent the sides of ΔABC.

i.e., $\overrightarrow{OA}=2\hat{i}-\hat{j}+\hat{k},\ \overrightarrow{OB}=\hat{i}-3\hat{j}-5\hat{k}$, and $\overrightarrow{OC}=3\hat{i}-4\hat{j}-4\hat{k}$

$\therefore \overrightarrow{AB}=(1-2)\hat{i}+(-3+1)\hat{j}+(-5-1)\hat{k}=-\hat{i}-2\hat{j}-6\hat{k}$

$\overrightarrow{BC}=(3-1)\hat{i}+(-4+3)\hat{j}+(-4+5)\hat{k}=2\hat{i}-\hat{j}+\hat{k}$

$\overrightarrow{AC}=(2-3)\hat{i}+(-1+4)\hat{j}+(1+4)\hat{k}=-\hat{i}+3\hat{j}+5\hat{k}$

$\left|\overrightarrow{AB}\right|=\sqrt{(-1)^2+(-2)^2+(-6)^2}=\sqrt{1+4+36}=\sqrt{41}$

$\left|\overrightarrow{BC}\right|=\sqrt{2^2+(-1)^2+1^2}=\sqrt{4+1+1}=\sqrt{6}$

$\left|\overrightarrow{AC}\right|=\sqrt{(-1)^2+3^2+5^2}=\sqrt{1+9+25}=\sqrt{35}$

$\therefore \left|\overrightarrow{BC}\right|^2+\left|\overrightarrow{AC}\right|^2=6+35=41=\left|\overrightarrow{AB}\right|^2$

Hence, ΔABC is a right-angled triangle.

Question 18:

If $\vec{a}$ is a nonzero vector of magnitude '*a*' and λ a nonzero scalar, then $\lambda\vec{a}$ is unit vector if

(A) λ = 1 (B) λ = –1 (C) $a=|\lambda|$

(D) $a=\dfrac{1}{|\lambda|}$

Answer

Vector $\lambda\vec{a}$ is a unit vector if $|\lambda\vec{a}|=1$.

Now,

$|\lambda \vec{a}| = 1$

$\Rightarrow |\lambda||\vec{a}| = 1$

$\Rightarrow |\vec{a}| = \dfrac{1}{|\lambda|}$ $\quad [\lambda \neq 0]$

$\Rightarrow a = \dfrac{1}{|\lambda|}$ $\quad [|\vec{a}| = a]$

Hence, vector $\lambda \vec{a}$ is a unit vector if $a = \dfrac{1}{|\lambda|}$.

The correct answer is D.