Exercise 13.2

Question 1:

Find the derivative of $x^2 - 2$ at $x = 10$.

Answer

Let $f(x) = x^2 - 2$. Accordingly,

$$f'(10) = \lim_{h \to 0} \frac{f(10+h) - f(10)}{h}$$
$$= \lim_{h \to 0} \frac{\left[(10+h)^2 - 2\right] - (10^2 - 2)}{h}$$
$$= \lim_{h \to 0} \frac{10^2 + 2.10.h + h^2 - 2 - 10^2 + 2}{h}$$
$$= \lim_{h \to 0} \frac{20h + h^2}{h}$$
$$= \lim_{h \to 0} (20 + h) = (20 + 0) = 20$$

Thus, the derivative of $x^2 - 2$ at $x = 10$ is 20.

Question 2:

Find the derivative of $99x$ at $x = 100$.

Answer

Let $f(x) = 99x$. Accordingly,

$$f'(100) = \lim_{h \to 0} \frac{f(100+h) - f(100)}{h}$$
$$= \lim_{h \to 0} \frac{99(100+h) - 99(100)}{h}$$
$$= \lim_{h \to 0} \frac{99 \times 100 + 99h - 99 \times 100}{h}$$
$$= \lim_{h \to 0} \frac{99h}{h}$$
$$= \lim_{h \to 0} (99) = 99$$

Thus, the derivative of $99x$ at $x = 100$ is 99.

Question 3:

Find the derivative of x at $x = 1$.

Answer

Let $f(x) = x$. Accordingly,

$$f'(1) = \lim_{h \to 0} \frac{f(1+h) - f(1)}{h}$$
$$= \lim_{h \to 0} \frac{(1+h) - 1}{h}$$
$$= \lim_{h \to 0} \frac{h}{h}$$
$$= \lim_{h \to 0} (1)$$
$$= 1$$

Thus, the derivative of x at $x = 1$ is 1.

Question 4:

Find the derivative of the following functions from first principle.

(i) $x^3 - 27$ (ii) $(x - 1)(x - 2)$

(ii) $\frac{1}{x^2}$ (iv) $\frac{x+1}{x-1}$

Answer

(i) Let $f(x) = x^3 - 27$. Accordingly, from the first principle,

$$f'(x) = \lim_{h \to 0} \frac{f(x+h) - f(x)}{h}$$
$$= \lim_{h \to 0} \frac{\left[(x+h)^3 - 27\right] - (x^3 - 27)}{h}$$
$$= \lim_{h \to 0} \frac{x^3 + h^3 + 3x^2h + 3xh^2 - x^3}{h}$$
$$= \lim_{h \to 0} \frac{h^3 + 3x^2h + 3xh^2}{h}$$
$$= \lim_{h \to 0} \left(h^2 + 3x^2 + 3xh\right)$$
$$= 0 + 3x^2 + 0 = 3x^2$$

(ii) Let $f(x) = (x - 1)(x - 2)$. Accordingly, from the first principle,

$$f'(x) = \lim_{h\to 0}\frac{f(x+h)-f(x)}{h}$$

$$= \lim_{h\to 0}\frac{(x+h-1)(x+h-2)-(x-1)(x-2)}{h}$$

$$= \lim_{h\to 0}\frac{\left(x^2+hx-2x+hx+h^2-2h-x-h+2\right)-\left(x^2-2x-x+2\right)}{h}$$

$$= \lim_{h\to 0}\frac{\left(hx+hx+h^2-2h-h\right)}{h}$$

$$= \lim_{h\to 0}\frac{2hx+h^2-3h}{h}$$

$$= \lim_{h\to 0}(2x+h-3)$$

$$= (2x+0-3)$$

$$= 2x-3$$

(iii) Let $f(x)=\frac{1}{x^2}$. Accordingly, from the first principle,

$$f'(x) = \lim_{h\to 0}\frac{f(x+h)-f(x)}{h}$$

$$= \lim_{h\to 0}\frac{\frac{1}{(x+h)^2}-\frac{1}{x^2}}{h}$$

$$= \lim_{h\to 0}\frac{1}{h}\left[\frac{x^2-(x+h)^2}{x^2(x+h)^2}\right]$$

$$= \lim_{h\to 0}\frac{1}{h}\left[\frac{x^2-x^2-h^2-2hx}{x^2(x+h)^2}\right]$$

$$= \lim_{h\to 0}\frac{1}{h}\left[\frac{-h^2-2hx}{x^2(x+h)^2}\right]$$

$$= \lim_{h\to 0}\left[\frac{-h-2x}{x^2(x+h)^2}\right]$$

$$= \frac{0-2x}{x^2(x+0)^2} = \frac{-2}{x^3}$$

(iv) Let $f(x)=\frac{x+1}{x-1}$. Accordingly, from the first principle,

$$f'(x)=\lim_{h\to 0}\frac{f(x+h)-f(x)}{h}$$

$$=\lim_{h\to 0}\frac{\left(\frac{x+h+1}{x+h-1}-\frac{x+1}{x-1}\right)}{h}$$

$$=\lim_{h\to 0}\frac{1}{h}\left[\frac{(x-1)(x+h+1)-(x+1)(x+h-1)}{(x-1)(x+h-1)}\right]$$

$$=\lim_{h\to 0}\frac{1}{h}\left[\frac{(x^2+hx+x-x-h-1)-(x^2+hx-x+x+h-1)}{(x-1)(x+h-1)}\right]$$

$$=\lim_{h\to 0}\frac{1}{h}\left[\frac{-2h}{(x-1)(x+h-1)}\right]$$

$$=\lim_{h\to 0}\left[\frac{-2}{(x-1)(x+h-1)}\right]$$

$$=\frac{-2}{(x-1)(x-1)}=\frac{-2}{(x-1)^2}$$

Question 5:

For the function

$$f(x)=\frac{x^{100}}{100}+\frac{x^{99}}{99}+\ldots+\frac{x^2}{2}+x+1$$

Prove that $f'(1)=100f'(0)$

Answer

The given function is

$f(x)=\frac{x^{100}}{100}+\frac{x^{99}}{99}+...+\frac{x^2}{2}+x+1$

$\frac{d}{dx}f(x)=\frac{d}{dx}\left[\frac{x^{100}}{100}+\frac{x^{99}}{99}+...+\frac{x^2}{2}+x+1\right]$

$\frac{d}{dx}f(x)=\frac{d}{dx}\left(\frac{x^{100}}{100}\right)+\frac{d}{dx}\left(\frac{x^{99}}{99}\right)+...+\frac{d}{dx}\left(\frac{x^2}{2}\right)+\frac{d}{dx}(x)+\frac{d}{dx}(1)$

On using theorem $\frac{d}{dx}(x^n)=nx^{n-1}$, we obtain

$\frac{d}{dx}f(x)=\frac{100x^{99}}{100}+\frac{99x^{98}}{99}+...+\frac{2x}{2}+1+0$

$=x^{99}+x^{98}+...+x+1$

$\therefore f'(x)=x^{99}+x^{98}+...+x+1$

At $x=0$,

$f'(0)=1$

At $x=1$,

$f'(1)=1^{99}+1^{98}+...+1+1=[1+1+...+1+1]_{100\text{ terms}}=1\times 100=100$

Thus, $f'(1)=100\times f^1(0)$

Question 6:

Find the derivative of $x^n+ax^{n-1}+a^2x^{n-2}+...+a^{n-1}x+a^n$ for some fixed real number *a*.

Answer

Let $f(x)=x^n+ax^{n-1}+a^2x^{n-2}+...+a^{n-1}x+a^n$

$\therefore f'(x)=\frac{d}{dx}\left(x^n+ax^{n-1}+a^2x^{n-2}+...+a^{n-1}x+a^n\right)$

$=\frac{d}{dx}(x^n)+a\frac{d}{dx}(x^{n-1})+a^2\frac{d}{dx}(x^{n-2})+...+a^{n-1}\frac{d}{dx}(x)+a^n\frac{d}{dx}(1)$

On using theorem $\frac{d}{dx}x^n=nx^{n-1}$, we obtain

$f'(x)=nx^{n-1}+a(n-1)x^{n-2}+a^2(n-2)x^{n-3}+...+a^{n-1}+a^n(0)$

$=nx^{n-1}+a(n-1)x^{n-2}+a^2(n-2)x^{n-3}+...+a^{n-1}$

Question 7:

For some constants *a* and *b*, find the derivative of

(i) $(x - a)(x - b)$ (ii) $(ax^2 + b)^2$ (iii) $\frac{x-a}{x-b}$

Answer

(i) Let $f(x) = (x - a)(x - b)$

$$\Rightarrow f(x) = x^2 - (a+b)x + ab$$

$$\therefore f'(x) = \frac{d}{dx}\left(x^2 - (a+b)x + ab\right)$$

$$= \frac{d}{dx}\left(x^2\right) - (a+b)\frac{d}{dx}(x) + \frac{d}{dx}(ab)$$

On using theorem $\frac{d}{dx}\left(x^n\right) = nx^{n-1}$, we obtain

$$f'(x) = 2x - (a+b) + 0 = 2x - a - b$$

(ii) Let $f(x) = \left(ax^2 + b\right)^2$

$$\Rightarrow f(x) = a^2x^4 + 2abx^2 + b^2$$

$$\therefore f'(x) = \frac{d}{dx}\left(a^2x^4 + 2abx^2 + b^2\right) = a^2\frac{d}{dx}\left(x^4\right) + 2ab\frac{d}{dx}\left(x^2\right) + \frac{d}{dx}\left(b^2\right)$$

On using theorem $\frac{d}{dx}x^n = nx^{n-1}$, we obtain

$$f'(x) = a^2\left(4x^3\right) + 2ab(2x) + b^2(0)$$

$$= 4a^2x^3 + 4abx$$

$$= 4ax\left(ax^2 + b\right)$$

(iii) Let $f(x) = \frac{(x-a)}{(x-b)}$

$$\Rightarrow f'(x) = \frac{d}{dx}\left(\frac{x-a}{x-b}\right)$$

By quotient rule,

$$f'(x)=\frac{(x-b)\frac{d}{dx}(x-a)-(x-a)\frac{d}{dx}(x-b)}{(x-b)^2}$$

$$=\frac{(x-b)(1)-(x-a)(1)}{(x-b)^2}$$

$$=\frac{x-b-x+a}{(x-b)^2}$$

$$=\frac{a-b}{(x-b)^2}$$

Question 8:

Find the derivative of $\frac{x^n-a^n}{x-a}$ for some constant *a*.

Answer

$$\text{Let } f(x)=\frac{x^n-a^n}{x-a}$$

$$\Rightarrow f'(x)=\frac{d}{dx}\left(\frac{x^n-a^n}{x-a}\right)$$

By quotient rule,

$$f'(x)=\frac{(x-a)\frac{d}{dx}(x^n-a^n)-(x^n-a^n)\frac{d}{dx}(x-a)}{(x-a)^2}$$

$$=\frac{(x-a)(nx^{n-1}-0)-(x^n-a^n)}{(x-a)^2}$$

$$=\frac{nx^n-anx^{n-1}-x^n+a^n}{(x-a)^2}$$

Question 9:

Find the derivative of

(i) $2x-\frac{3}{4}$ (ii) $(5x^3+3x-1)(x-1)$

(iii) x^{-3} (5 + 3x) (iv) x^5 (3 – 6x^{-9})

(v) x^{-4} (3 – 4x^{-5}) (vi) $\frac{2}{x+1}-\frac{x^2}{3x-1}$

Answer

(i) Let $f(x)=2x-\frac{3}{4}$

$$f'(x)=\frac{d}{dx}\left(2x-\frac{3}{4}\right)$$
$$=2\frac{d}{dx}(x)-\frac{d}{dx}\left(\frac{3}{4}\right)$$
$$=2-0$$
$$=2$$

(ii) Let $f(x)$ = (5x^3 + 3x – 1) (x – 1)

By Leibnitz product rule,

$$f'(x)=\left(5x^3+3x-1\right)\frac{d}{dx}(x-1)+(x-1)\frac{d}{dx}\left(5x^3+3x-1\right)$$
$$=\left(5x^3+3x-1\right)(1)+(x-1)\left(5.3x^2+3-0\right)$$
$$=\left(5x^3+3x-1\right)+(x-1)\left(15x^2+3\right)$$
$$=5x^3+3x-1+15x^3+3x-15x^2-3$$
$$=20x^3-15x^2+6x-4$$

(iii) Let $f(x)$ = x^{-3} (5 + 3x)

By Leibnitz product rule,

$$f'(x) = x^{-3}\frac{d}{dx}(5+3x)+(5+3x)\frac{d}{dx}(x^{-3})$$
$$= x^{-3}(0+3)+(5+3x)(-3x^{-3-1})$$
$$= x^{-3}(3)+(5+3x)(-3x^{-4})$$
$$= 3x^{-3}-15x^{-4}-9x^{-3}$$
$$= -6x^{-3}-15x^{-4}$$
$$= -3x^{-3}\left(2+\frac{5}{x}\right)$$
$$= \frac{-3x^{-3}}{x}(2x+5)$$
$$= \frac{-3}{x^4}(5+2x)$$

(iv) Let $f(x) = x^5(3 - 6x^{-9})$

By Leibnitz product rule,

$$f'(x) = x^5\frac{d}{dx}(3-6x^{-9})+(3-6x^{-9})\frac{d}{dx}(x^5)$$
$$= x^5\left\{0-6(-9)x^{-9-1}\right\}+(3-6x^{-9})(5x^4)$$
$$= x^5(54x^{-10})+15x^4-30x^{-5}$$
$$= 54x^{-5}+15x^4-30x^{-5}$$
$$= 24x^{-5}+15x^4$$
$$= 15x^4+\frac{24}{x^5}$$

(v) Let $f(x) = x^{-4}(3 - 4x^{-5})$

By Leibnitz product rule,

$$f'(x) = x^{-4}\frac{d}{dx}(3-4x^{-5})+(3-4x^{-5})\frac{d}{dx}(x^{-4})$$
$$= x^{-4}\left\{0-4(-5)x^{-5-1}\right\}+(3-4x^{-5})(-4)x^{-4-1}$$
$$= x^{-4}(20x^{-6})+(3-4x^{-5})(-4x^{-5})$$
$$= 20x^{-10}-12x^{-5}+16x^{-10}$$
$$= 36x^{-10}-12x^{-5}$$
$$= -\frac{12}{x^5}+\frac{36}{x^{10}}$$

(vi) Let $f(x) = \frac{2}{x+1} - \frac{x^2}{3x-1}$

$$f'(x) = \frac{d}{dx}\left(\frac{2}{x+1}\right) - \frac{d}{dx}\left(\frac{x^2}{3x-1}\right)$$

By quotient rule,

$$f'(x) = \left[\frac{(x+1)\frac{d}{dx}(2) - 2\frac{d}{dx}(x+1)}{(x+1)^2}\right] - \left[\frac{(3x-1)\frac{d}{dx}(x^2) - x^2\frac{d}{dx}(3x-1)}{(3x-1)^2}\right]$$

$$= \left[\frac{(x+1)(0) - 2(1)}{(x+1)^2}\right] - \left[\frac{(3x-1)(2x) - (x^2)(3)}{(3x-1)^2}\right]$$

$$= \frac{-2}{(x+1)^2} - \left[\frac{6x^2 - 2x - 3x^2}{(3x-1)^2}\right]$$

$$= \frac{-2}{(x+1)^2} - \left[\frac{3x^2 - 2x^2}{(3x-1)^2}\right]$$

$$= \frac{-2}{(x+1)^2} - \frac{x(3x-2)}{(3x-1)^2}$$

Question 10:

Find the derivative of cos *x* from first principle.

Answer

Let *f* (*x*) = cos *x*. Accordingly, from the first principle,

$$f'(x) = \lim_{h\to 0}\frac{f(x+h) - f(x)}{h}$$

$$= \lim_{h\to 0}\frac{\cos(x+h) - \cos x}{h}$$

$$=\lim_{h\to 0}\left[\frac{\cos x\cos h-\sin x\sin h-\cos x}{h}\right]$$

$$=\lim_{h\to 0}\left[\frac{-\cos x(1-\cos h)-\sin x\sin h}{h}\right]$$

$$=\lim_{h\to 0}\left[\frac{-\cos x(1-\cos h)}{h}-\frac{\sin x\sin h}{h}\right]$$

$$=-\cos x\left(\lim_{h\to 0}\frac{1-\cos h}{h}\right)-\sin x\lim_{h\to 0}\left(\frac{\sin h}{h}\right)$$

$$=-\cos x(0)-\sin x(1) \qquad \left[\lim_{h\to 0}\frac{1-\cos h}{h}=0 \text{ and } \lim_{h\to 0}\frac{\sin h}{h}=1\right]$$

$$=-\sin x$$

$$\therefore f'(x)=-\sin x$$

Question 11:

Find the derivative of the following functions:

(i) sin *x* cos *x* (ii) sec *x* (iii) 5 sec *x* + 4 cos *x*

(iv) cosec *x* (v) 3cot *x* + 5cosec *x*

(vi) 5sin *x* – 6cos *x* + 7 (vii) 2tan *x* – 7sec *x*

Answer

(i) Let *f* (*x*) = sin *x* cos *x*. Accordingly, from the first principle,

$$f'(x) = \lim_{h \to 0} \frac{f(x+h) - f(x)}{h}$$

$$= \lim_{h \to 0} \frac{\sin(x+h)\cos(x+h) - \sin x \cos x}{h}$$

$$= \lim_{h \to 0} \frac{1}{2h}\left[2\sin(x+h)\cos(x+h) - 2\sin x \cos x\right]$$

$$= \lim_{h \to 0} \frac{1}{2h}\left[\sin 2(x+h) - \sin 2x\right]$$

$$= \lim_{h \to 0} \frac{1}{2h}\left[2\cos\frac{2x+2h+2x}{2} \cdot \sin\frac{2x+2h-2x}{2}\right]$$

$$= \lim_{h \to 0} \frac{1}{h}\left[\cos\frac{4x+2h}{2}\sin\frac{2h}{2}\right]$$

$$= \lim_{h \to 0} \frac{1}{h}\left[\cos(2x+h)\sin h\right]$$

$$= \lim_{h \to 0} \cos(2x+h) . \lim_{h \to 0} \frac{\sin h}{h}$$

$$= \cos(2x+0).1$$

$$= \cos 2x$$

(ii) Let $f(x) = \sec x$. Accordingly, from the first principle,

$$f'(x)=\lim_{h\to 0}\frac{f(x+h)-f(x)}{h}$$
$$=\lim_{h\to 0}\frac{\sec(x+h)-\sec x}{h}$$
$$=\lim_{h\to 0}\frac{1}{h}\left[\frac{1}{\cos(x+h)}-\frac{1}{\cos x}\right]$$
$$=\lim_{h\to 0}\frac{1}{h}\left[\frac{\cos x-\cos(x+h)}{\cos x\cos(x+h)}\right]$$
$$=\frac{1}{\cos x}.\lim_{h\to 0}\frac{1}{h}\left[\frac{-2\sin\left(\frac{x+x+h}{2}\right)\sin\left(\frac{x-x-h}{2}\right)}{\cos(x+h)}\right]$$
$$=\frac{1}{\cos x}.\lim_{h\to 0}\frac{1}{h}\left[\frac{-2\sin\left(\frac{2x+h}{2}\right)\sin\left(-\frac{h}{2}\right)}{\cos(x+h)}\right]$$
$$=\frac{1}{\cos x}.\lim_{h\to 0}\frac{\left[\sin\left(\frac{2x+h}{2}\right)\frac{\sin\left(\frac{h}{2}\right)}{\left(\frac{h}{2}\right)}\right]}{\cos(x+h)}$$
$$=\frac{1}{\cos x}.\lim_{\frac{h}{2}\to 0}\frac{\sin\left(\frac{h}{2}\right)}{\left(\frac{h}{2}\right)}.\lim_{h\to 0}\frac{\sin\left(\frac{2x+h}{2}\right)}{\cos(x+h)}$$
$$=\frac{1}{\cos x}.1.\frac{\sin x}{\cos x}$$
$$=\sec x\tan x$$

(iii) Let $f(x) = 5\sec x + 4\cos x$. Accordingly, from the first principle,

$$f'(x)=\lim_{h\to 0}\frac{f(x+h)-f(x)}{h}$$

$$=\lim_{h\to 0}\frac{5\sec(x+h)+4\cos(x+h)-[5\sec x+4\cos x]}{h}$$

$$=5\lim_{h\to 0}\frac{[\sec(x+h)-\sec x]}{h}+4\lim_{h\to 0}\frac{[\cos(x+h)-\cos x]}{h}$$

$$=5\lim_{h\to 0}\frac{1}{h}\left[\frac{1}{\cos(x+h)}-\frac{1}{\cos x}\right]+4\lim_{h\to 0}\frac{1}{h}[\cos(x+h)-\cos x]$$

$$=5\lim_{h\to 0}\frac{1}{h}\left[\frac{\cos x-\cos(x+h)}{\cos x\cos(x+h)}\right]+4\lim_{h\to 0}\frac{1}{h}[\cos x\cos h-\sin x\sin h-\cos x]$$

$$=\frac{5}{\cos x}\lim_{h\to 0}\frac{1}{h}\left[\frac{-2\sin\left(\frac{x+x+h}{2}\right)\sin\left(\frac{x-x-h}{2}\right)}{\cos(x+h)}\right]+4\lim_{h\to 0}\frac{1}{h}[-\cos x(1-\cos h)-\sin x\sin h]$$

$$=\frac{5}{\cos x}.\lim_{h\to 0}\frac{1}{h}\left[\frac{-2\sin\left(\frac{2x+h}{2}\right)\sin\left(-\frac{h}{2}\right)}{\cos(x+h)}\right]+4\left[-\cos x\lim_{h\to 0}\frac{(1-\cos h)}{h}-\sin x\lim_{h\to 0}\frac{\sin h}{h}\right]$$

$$=\frac{5}{\cos x}.\lim_{h\to 0}\left[\frac{\sin\left(\frac{2x+h}{2}\right).\frac{\sin\left(\frac{h}{2}\right)}{\frac{h}{2}}}{\cos(x+h)}\right]+4[(-\cos x).(0)-(\sin x).1]$$

$$=\frac{5}{\cos x}.\left[\lim_{h\to 0}\frac{\sin\left(\frac{2x+h}{2}\right)}{\cos(x+h)}.\lim_{h\to 0}\frac{\sin\left(\frac{h}{2}\right)}{\frac{h}{2}}\right]-4\sin x$$

$$=\frac{5}{\cos x}.\frac{\sin x}{\cos x}.1-4\sin x$$

$$=5\sec x\tan x.-4\sin x$$

(iv) Let $f(x)=\operatorname{cosec} x$. Accordingly, from the first principle,

$$f'(x)=\lim_{h\to 0}\frac{f(x+h)-f(x)}{h}$$

$$f'(x)=\lim_{h\to 0}\frac{1}{h}\left[\operatorname{cosec}(x+h)-\operatorname{cosec}x\right]$$

$$=\lim_{h\to 0}\frac{1}{h}\left[\frac{1}{\sin(x+h)}-\frac{1}{\sin x}\right]$$

$$=\lim_{h\to 0}\frac{1}{h}\left[\frac{\sin x-\sin(x+h)}{\sin(x+h)\sin x}\right]$$

$$=\lim_{h\to 0}\frac{1}{h}\left[\frac{2\cos\left(\frac{x+x+h}{2}\right)\cdot\sin\left(\frac{x-x-h}{2}\right)}{\sin(x+h)\sin x}\right]$$

$$=\lim_{h\to 0}\frac{1}{h}\left[\frac{2\cos\left(\frac{2x+h}{2}\right)\sin\left(-\frac{h}{2}\right)}{\sin(x+h)\sin x}\right]$$

$$=\lim_{h\to 0}\frac{-\cos\left(\frac{2x+h}{2}\right)\cdot\frac{\sin\left(\frac{h}{2}\right)}{\left(\frac{h}{2}\right)}}{\sin(x+h)\sin x}$$

$$=\lim_{h\to 0}\left(\frac{-\cos\left(\frac{2x+h}{2}\right)}{\sin(x+h)\sin x}\right).\lim_{\frac{h}{2}\to 0}\frac{\sin\left(\frac{h}{2}\right)}{\left(\frac{h}{2}\right)}$$

$$=\left(\frac{-\cos x}{\sin x\sin x}\right).1$$

$$=-\operatorname{cosec}x\cot x$$

(v) Let *f* (*x*) = 3cot *x* + 5cosec *x*. Accordingly, from the first principle,

$$f'(x)=\lim_{h\to 0}\frac{f(x+h)-f(x)}{h}$$

$$=\lim_{h\to 0}\frac{3\cot(x+h)+5\operatorname{cosec}(x+h)-3\cot x-5\operatorname{cosec} x}{h}$$

$$=3\lim_{h\to 0}\frac{1}{h}\left[\cot(x+h)-\cot x\right]+5\lim_{h\to 0}\frac{1}{h}\left[\operatorname{cosec}(x+h)-\operatorname{cosec} x\right] \qquad ...(1)$$

Now, $\lim_{h\to 0}\frac{1}{h}\left[\cot(x+h)-\cot x\right]$

$$=\lim_{h\to 0}\frac{1}{h}\left[\frac{\cos(x+h)}{\sin(x+h)}-\frac{\cos x}{\sin x}\right]$$

$$=\lim_{h\to 0}\frac{1}{h}\left[\frac{\cos(x+h)\sin x-\cos x\sin(x+h)}{\sin x\sin(x+h)}\right]$$

$$=\lim_{h\to 0}\frac{1}{h}\left[\frac{\sin(x-x-h)}{\sin x\sin(x+h)}\right]$$

$$=\lim_{h\to 0}\frac{1}{h}\left[\frac{\sin(-h)}{\sin x\sin(x+h)}\right]$$

$$=-\left(\lim_{h\to 0}\frac{\sin h}{h}\right).\left(\lim_{h\to 0}\frac{1}{\sin x\cdot\sin(x+h)}\right)$$

$$=-1.\frac{1}{\sin x\cdot\sin(x+0)}=\frac{-1}{\sin^2 x}=-\operatorname{cosec}^2 x \qquad ...(2)$$

$$\lim_{h\to 0}\frac{1}{h}\left[\operatorname{cosec}(x+h)-\operatorname{cosec}x\right]$$

$$=\lim_{h\to 0}\frac{1}{h}\left[\frac{1}{\sin(x+h)}-\frac{1}{\sin x}\right]$$

$$=\lim_{h\to 0}\frac{1}{h}\left[\frac{\sin x-\sin(x+h)}{\sin(x+h)\sin x}\right]$$

$$=\lim_{h\to 0}\frac{1}{h}\left[\frac{2\cos\left(\frac{x+x+h}{2}\right)\cdot\sin\left(\frac{x-x-h}{2}\right)}{\sin(x+h)\sin x}\right]$$

$$=\lim_{h\to 0}\frac{1}{h}\left[\frac{2\cos\left(\frac{2x+h}{2}\right)\sin\left(-\frac{h}{2}\right)}{\sin(x+h)\sin x}\right]$$

$$=\lim_{h\to 0}\frac{-\cos\left(\frac{2x+h}{2}\right)\cdot\frac{\sin\left(\frac{h}{2}\right)}{\left(\frac{h}{2}\right)}}{\sin(x+h)\sin x}$$

$$=\lim_{h\to 0}\left(\frac{-\cos\left(\frac{2x+h}{2}\right)}{\sin(x+h)\sin x}\right).\lim_{\frac{h}{2}\to 0}\frac{\sin\left(\frac{h}{2}\right)}{\left(\frac{h}{2}\right)}$$

$$=\left(\frac{-\cos x}{\sin x\sin x}\right).1$$

$$=-\operatorname{cosec}x\cot x \qquad ...(3)$$

From (1), (2), and (3), we obtain

$$f'(x)=-3\operatorname{cosec}^2 x-5\operatorname{cosec}x\cot x$$

(vi) Let $f(x) = 5\sin x - 6\cos x + 7$. Accordingly, from the first principle,

$$f'(x)=\lim_{h\to 0}\frac{f(x+h)-f(x)}{h}$$

$$=\lim_{h\to 0}\frac{1}{h}\left[5\sin(x+h)-6\cos(x+h)+7-5\sin x+6\cos x-7\right]$$

$$=\lim_{h\to 0}\frac{1}{h}\left[5\{\sin(x+h)-\sin x\}-6\{\cos(x+h)-\cos x\}\right]$$

$$=5\lim_{h\to 0}\frac{1}{h}\left[\sin(x+h)-\sin x\right]-6\lim_{h\to 0}\frac{1}{h}\left[\cos(x+h)-\cos x\right]$$

$$=5\lim_{h\to 0}\frac{1}{h}\left[2\cos\left(\frac{x+h+x}{2}\right)\sin\left(\frac{x+h-x}{2}\right)\right]-6\lim_{h\to 0}\frac{\cos x\cos h-\sin x\sin h-\cos x}{h}$$

$$=5\lim_{h\to 0}\frac{1}{h}\left[2\cos\left(\frac{2x+h}{2}\right)\sin\frac{h}{2}\right]-6\lim_{h\to 0}\left[\frac{-\cos x(1-\cos h)-\sin x\sin h}{h}\right]$$

$$=5\lim_{h\to 0}\left(\cos\left(\frac{2x+h}{2}\right)\frac{\sin\frac{h}{2}}{\frac{h}{2}}\right)-6\lim_{h\to 0}\left[\frac{-\cos x(1-\cos h)}{h}-\frac{\sin x\sin h}{h}\right]$$

$$=5\left[\lim_{h\to 0}\cos\left(\frac{2x+h}{2}\right)\right]\left[\lim_{\frac{h}{2}\to 0}\frac{\sin\frac{h}{2}}{\frac{h}{2}}\right]-6\left[(-\cos x)\left(\lim_{h\to 0}\frac{1-\cos h}{h}\right)-\sin x\lim_{h\to 0}\left(\frac{\sin h}{h}\right)\right]$$

$$=5\cos x.1-6\left[(-\cos x).(0)-\sin x.1\right]$$

$$=5\cos x+6\sin x$$

(vii) Let $f(x) = 2\tan x - 7\sec x$. Accordingly, from the first principle,

$$f'(x)=\lim_{h\to 0}\frac{f(x+h)-f(x)}{h}$$

$$=\lim_{h\to 0}\frac{1}{h}\left[2\tan(x+h)-7\sec(x+h)-2\tan x+7\sec x\right]$$

$$=\lim_{h\to 0}\frac{1}{h}\left[2\left\{\tan(x+h)-\tan x\right\}-7\left\{\sec(x+h)-\sec x\right\}\right]$$

$$=2\lim_{h\to 0}\frac{1}{h}\left[\tan(x+h)-\tan x\right]-7\lim_{h\to 0}\frac{1}{h}\left[\sec(x+h)-\sec x\right]$$

$$=2\lim_{h\to 0}\frac{1}{h}\left[\frac{\sin(x+h)}{\cos(x+h)}-\frac{\sin x}{\cos x}\right]-7\lim_{h\to 0}\frac{1}{h}\left[\frac{1}{\cos(x+h)}-\frac{1}{\cos x}\right]$$

$$=2\lim_{h\to 0}\frac{1}{h}\left[\frac{\sin(x+h)\cos x-\sin x\cos(x+h)}{\cos x\cos(x+h)}\right]-7\lim_{h\to 0}\frac{1}{h}\left[\frac{\cos x-\cos(x+h)}{\cos x\cos(x+h)}\right]$$

$$=2\lim_{h\to 0}\frac{1}{h}\left[\frac{\sin(x+h-x)}{\cos x\cos(x+h)}\right]-7\lim_{h\to 0}\frac{1}{h}\left[\frac{-2\sin\left(\frac{x+x+h}{2}\right)\sin\left(\frac{x-x-h}{2}\right)}{\cos x\cos(x+h)}\right]$$

$$=2\lim_{h\to 0}\left[\left(\frac{\sin h}{h}\right)\frac{1}{\cos x\cos(x+h)}\right]-7\lim_{h\to 0}\frac{1}{h}\left[\frac{-2\sin\left(\frac{2x+h}{2}\right)\sin\left(-\frac{h}{2}\right)}{\cos x\cos(x+h)}\right]$$

$$=2\left(\lim_{h\to 0}\frac{\sin h}{h}\right)\left(\lim_{h\to 0}\frac{1}{\cos x\cos(x+h)}\right)-7\left(\lim_{\frac{h}{2}\to 0}\frac{\sin\frac{h}{2}}{\frac{h}{2}}\right)\left(\lim_{h\to 0}\frac{\sin\left(\frac{2x+h}{2}\right)}{\cos x\cos(x+h)}\right)$$

$$=2.1.\frac{1}{\cos x\cos x}-7.1\left(\frac{\sin x}{\cos x\cos x}\right)$$

$$=2\sec^2 x-7\sec x\tan x$$