Exercise 13.5

Question 1:

A die is thrown 6 times. If 'getting an odd number' is a success, what is the probability of

(i) 5 successes? (ii) at least 5 successes?

(iii) at most 5 successes?

Answer

The repeated tosses of a die are Bernoulli trials. Let X denote the number of successes of getting odd numbers in an experiment of 6 trials.

Probability of getting an odd number in a single throw of a die is, $p = \frac{3}{6} = \frac{1}{2}$

$\therefore q = 1 - p = \frac{1}{2}$

X has a binomial distribution.

Therefore, P (X = x) = ${}^{n}C_{n-x} q^{n-x} p^{x}$, where $n = 0, 1, 2 \ldots n$

$= {}^{6}C_{x} \left(\frac{1}{2}\right)^{6-x} \cdot \left(\frac{1}{2}\right)^{x}$

$= {}^{6}C_{x} \left(\frac{1}{2}\right)^{6}$

(i) P (5 successes) = P (X = 5)

$= {}^{6}C_{5} \left(\frac{1}{2}\right)^{6}$

$= 6 \cdot \frac{1}{64}$

$= \frac{3}{32}$

(ii) P(at least 5 successes) = P(X ≥ 5)

$= P(X=5)+P(X=6)$

$= {}^6C_5\left(\frac{1}{2}\right)^6 + {}^6C_6\left(\frac{1}{2}\right)^6$

$= 6\cdot\frac{1}{64}+1\cdot\frac{1}{64}$

$= \frac{7}{64}$

(iii) P (at most 5 successes) = P(X ≤ 5)

$= 1-P(X>5)$

$= 1-P(X=6)$

$= 1-{}^6C_6\left(\frac{1}{2}\right)^6$

$= 1-\frac{1}{64}$

$= \frac{63}{64}$

Question 2:

A pair of dice is thrown 4 times. If getting a doublet is considered a success, find the probability of two successes.

Answer

The repeated tosses of a pair of dice are Bernoulli trials. Let X denote the number of times of getting doublets in an experiment of throwing two dice simultaneously four times.

Probability of getting doublets in a single throw of the pair of dice is

$p = \frac{6}{36} = \frac{1}{6}$

$\therefore q = 1-p = 1-\frac{1}{6} = \frac{5}{6}$

Clearly, X has the binomial distribution with n = 4, $p = \frac{1}{6}$, and $q = \frac{5}{6}$

$\therefore P(X=x) = {}^nC_x q^{n-x} p^x$, where $x = 0, 1, 2, 3 \dots n$

$= {}^4C_x \left(\frac{5}{6}\right)^{4-x} \cdot \left(\frac{1}{6}\right)^x$

$= {}^4C_x \cdot \frac{5^{4-x}}{6^4}$

∴ P (2 successes) = P (X = 2)

$= {}^4C_2 \cdot \frac{5^{4-2}}{6^4}$

$= 6 \cdot \frac{25}{1296}$

$= \frac{25}{216}$

Question 3:

There are 5% defective items in a large bulk of items. What is the probability that a sample of 10 items will include not more than one defective item?

Answer

Let X denote the number of defective items in a sample of 10 items drawn successively. Since the drawing is done with replacement, the trials are Bernoulli trials.

$\Rightarrow p = \frac{5}{100} = \frac{1}{20}$

$\therefore q = 1 - \frac{1}{20} = \frac{19}{20}$

X has a binomial distribution with n = 10 and $p = \frac{1}{20}$

P(X = x) = ${}^nC_x q^{n-x} p^x$, where $x = 0, 1, 2 \ldots n$

$= {}^{10}C_x \left(\frac{19}{20}\right)^{10-x} \cdot \left(\frac{1}{20}\right)^x$

P (not more than 1 defective item) = P (X ≤ 1)

$= P(X=0)+P(X=1)$

$= {}^{10}C_0\left(\frac{19}{20}\right)^{10}\cdot\left(\frac{1}{20}\right)^0 + {}^{10}C_1\left(\frac{19}{20}\right)^9\cdot\left(\frac{1}{20}\right)^1$

$=\left(\frac{19}{20}\right)^{10}+10\left(\frac{19}{20}\right)^9\cdot\left(\frac{1}{20}\right)$

$=\left(\frac{19}{20}\right)^9\cdot\left[\frac{19}{20}+\frac{10}{20}\right]$

$=\left(\frac{19}{20}\right)^9\cdot\left(\frac{29}{20}\right)$

$=\left(\frac{29}{20}\right)\cdot\left(\frac{19}{20}\right)^9$

Question 4:

Five cards are drawn successively with replacement from a well-shuffled deck of 52 cards. What is the probability that

(i) all the five cards are spades?

(ii) only 3 cards are spades?

(iii) none is a spade?

Answer

Let X represent the number of spade cards among the five cards drawn. Since the drawing of card is with replacement, the trials are Bernoulli trials.

In a well shuffled deck of 52 cards, there are 13 spade cards.

$\Rightarrow p=\frac{13}{52}=\frac{1}{4}$

$\therefore q=1-\frac{1}{4}=\frac{3}{4}$

X has a binomial distribution with n = 5 and $p=\frac{1}{4}$

$P(X=x)={}^nC_x q^{n-x}p^x$, where $x=0, 1, \ldots n$

$= {}^5C_x\left(\frac{3}{4}\right)^{5-x}\left(\frac{1}{4}\right)^x$

(i) P (all five cards are spades) = P(X = 5)

$= {}^5C_5\left(\frac{3}{4}\right)^0\cdot\left(\frac{1}{4}\right)^5$

$=1\cdot\frac{1}{1024}$

$=\frac{1}{1024}$

(ii) P (only 3 cards are spades) = P(X = 3)

$= {}^5C_3\cdot\left(\frac{3}{4}\right)^2\cdot\left(\frac{1}{4}\right)^3$

$=10\cdot\frac{9}{16}\cdot\frac{1}{64}$

$=\frac{45}{512}$

(iii) P (none is a spade) = P(X = 0)

$= {}^5C_0\cdot\left(\frac{3}{4}\right)^5\cdot\left(\frac{1}{4}\right)^0$

$=1\cdot\frac{243}{1024}$

$=\frac{243}{1024}$

Question 5:

The probability that a bulb produced by a factory will fuse after 150 days of use is 0.05. What is the probability that out of 5 such bulbs

(i) none

(ii) not more than one

(iii) more than one

(iv) at least one

will fuse after 150 days of use.

Answer

Let X represent the number of bulbs that will fuse after 150 days of use in an experiment of 5 trials. The trials are Bernoulli trials.

It is given that, $p = 0.05$

$\therefore q = 1 - p = 1 - 0.05 = 0.95$

X has a binomial distribution with *n* = 5 and *p* = 0.05

$\therefore \mathrm{P}(\mathrm{X} = x) = {}^{n}\mathrm{C}_{x} q^{n-x} p^{x}$, where $x = 1, 2, \ldots n$

$= {}^{5}\mathrm{C}_{x} (0.95)^{5-x} \cdot (0.05)^{x}$

(i) P (none) = P(X = 0)

$= {}^{5}\mathrm{C}_{0} (0.95)^{5} \cdot (0.05)^{0}$

$= 1 \times (0.95)^{5}$

$= (0.95)^{5}$

(ii) P (not more than one) = P(X ≤ 1)

$= \mathrm{P}(\mathrm{X} = 0) + \mathrm{P}(\mathrm{X} = 1)$

$= {}^{5}\mathrm{C}_{0} (0.95)^{5} \times (0.05)^{0} + {}^{5}\mathrm{C}_{1} (0.95)^{4} \times (0.05)^{1}$

$= 1 \times (0.95)^{5} + 5 \times (0.95)^{4} \times (0.05)$

$= (0.95)^{5} + (0.25)(0.95)^{4}$

$= (0.95)^{4} [0.95 + 0.25]$

$= (0.95)^{4} \times 1.2$

(iii) P (more than 1) = P(X > 1)

$= 1 - \mathrm{P}(\mathrm{X} \le 1)$

$= 1 - \mathrm{P}(\text{not more than } 1)$

$= 1 - (0.95)^{4} \times 1.2$

(iv) P (at least one) = P(X ≥ 1)

$= 1 - \mathrm{P}(\mathrm{X} < 1)$

$= 1 - \mathrm{P}(\mathrm{X} = 0)$

$= 1 - {}^{5}\mathrm{C}_{0} (0.95)^{5} \times (0.05)^{0}$

$= 1 - 1 \times (0.95)^{5}$

$= 1 - (0.95)^{5}$

Question 6:

A bag consists of 10 balls each marked with one of the digits 0 to 9. If four balls are drawn successively with replacement from the bag, what is the probability that none is marked with the digit 0?

Answer

Let X denote the number of balls marked with the digit 0 among the 4 balls drawn.

Since the balls are drawn with replacement, the trials are Bernoulli trials.

X has a binomial distribution with n = 4 and $p=\frac{1}{10}$

$$\therefore q=1-p=1-\frac{1}{10}=\frac{9}{10}$$

$$\therefore P(X=x)={}^{n}C_{x}q^{n-x}\cdot p^{x}, x=1,2,...n$$

$$={}^{4}C_{x}\left(\frac{9}{10}\right)^{4-x}\cdot\left(\frac{1}{10}\right)^{x}$$

P (none marked with 0) = P (X = 0)

$$={}^{4}C_{0}\left(\frac{9}{10}\right)^{4}\cdot\left(\frac{1}{10}\right)^{0}$$

$$=1\cdot\left(\frac{9}{10}\right)^{4}$$

$$=\left(\frac{9}{10}\right)^{4}$$

Question 7:

In an examination, 20 questions of true-false type are asked. Suppose a student tosses a fair coin to determine his answer to each question. If the coin falls heads, he answers 'true'; if it falls tails, he answers 'false'. Find the probability that he answers at least 12 questions correctly.

Answer

Let X represent the number of correctly answered questions out of 20 questions.

The repeated tosses of a coin are Bernoulli trails. Since "head" on a coin represents the true answer and "tail" represents the false answer, the correctly answered questions are Bernoulli trials.

$\therefore p = \frac{1}{2}$

$\therefore q = 1 - p = 1 - \frac{1}{2} = \frac{1}{2}$

X has a binomial distribution with n = 20 and $p = \frac{1}{2}$

$\therefore P(X = x) = {}^nC_x q^{n-x} p^x$, where $x = 0, 1, 2, \ldots n$

$$= {}^{20}C_x \left(\frac{1}{2}\right)^{20-x} \cdot \left(\frac{1}{2}\right)^x$$

$$= {}^{20}C_x \left(\frac{1}{2}\right)^{20}$$

P (at least 12 questions answered correctly) = P(X ≥ 12)

$= P(X = 12) + P(X = 13) + \ldots + P(X = 20)$

$= {}^{20}C_{12}\left(\frac{1}{2}\right)^{20} + {}^{20}C_{13}\left(\frac{1}{2}\right)^{20} + \ldots + {}^{20}C_{20}\left(\frac{1}{2}\right)^{20}$

$= \left(\frac{1}{2}\right)^{20} \cdot \left[{}^{20}C_{12} + {}^{20}C_{13} + \ldots + {}^{20}C_{20}\right]$

Question 8:

Suppose X has a binomial distribution $B\left(6, \frac{1}{2}\right)$. Show that X = 3 is the most likely outcome.

(Hint: P(X = 3) is the maximum among all P (x_i), x_i = 0, 1, 2, 3, 4, 5, 6)

Answer

X is the random variable whose binomial distribution is $B\left(6,\frac{1}{2}\right)$.

Therefore, n = 6 and $p=\frac{1}{2}$

$\therefore q=1-p=1-\frac{1}{2}=\frac{1}{2}$

$$\text{Then, } P(X=x)={}^{n}C_{x}q^{n-x}p^{x}$$

$$={}^{6}C_{x}\left(\frac{1}{2}\right)^{6-x}\cdot\left(\frac{1}{2}\right)^{x}$$

$$={}^{6}C_{x}\left(\frac{1}{2}\right)^{6}$$

It can be seen that P(X = x) will be maximum, if ${}^{6}C_{x}$ will be maximum.

$$\text{Then, } {}^{6}C_{0}={}^{6}C_{6}=\frac{6!}{0!\cdot 6!}=1$$

$${}^{6}C_{1}={}^{6}C_{5}=\frac{6!}{1!\cdot 5!}=6$$

$${}^{6}C_{2}={}^{6}C_{4}=\frac{6!}{2!\cdot 4!}=15$$

$${}^{6}C_{3}=\frac{6!}{3!\cdot 3!}=20$$

The value of ${}^{6}C_{3}$ is maximum. Therefore, for x = 3, P(X = x) is maximum.

Thus, X = 3 is the most likely outcome.

Question 9:

On a multiple choice examination with three possible answers for each of the five questions, what is the probability that a candidate would get four or more correct answers just by guessing?

Answer

The repeated guessing of correct answers from multiple choice questions are Bernoulli trials. Let X represent the number of correct answers by guessing in the set of 5 multiple choice questions.

Probability of getting a correct answer is, $p = \frac{1}{3}$

$\therefore q = 1 - p = 1 - \frac{1}{3} = \frac{2}{3}$

Clearly, X has a binomial distribution with n = 5 and $p = \frac{1}{3}$

$$\therefore P(X = x) = {}^{n}C_{x} q^{n-x} p^{x}$$
$$= {}^{5}C_{x} \left(\frac{2}{3}\right)^{5-x} \cdot \left(\frac{1}{3}\right)^{x}$$

P (guessing more than 4 correct answers) = P(X ≥ 4)

$$= P(X = 4) + P(X = 5)$$
$$= {}^{5}C_{4}\left(\frac{2}{3}\right)\cdot\left(\frac{1}{3}\right)^{4} + {}^{5}C_{5}\left(\frac{1}{3}\right)^{5}$$
$$= 5 \cdot \frac{2}{3} \cdot \frac{1}{81} + 1 \cdot \frac{1}{243}$$
$$= \frac{10}{243} + \frac{1}{243}$$
$$= \frac{11}{243}$$

Question 10:

A person buys a lottery ticket in 50 lotteries, in each of which his chance of winning a prize is $\frac{1}{100}$. What is the probability that he will in a prize (a) at least once (b) exactly once (c) at least twice?

Answer

Let X represent the number of winning prizes in 50 lotteries. The trials are Bernoulli trials.

Clearly, X has a binomial distribution with n = 50 and $p = \frac{1}{100}$

$$\therefore q = 1 - p = 1 - \frac{1}{100} = \frac{99}{100}$$

$$\therefore P(X = x) = {}^{n}C_{x} q^{n-x} p^{x} = {}^{50}C_{x} \left(\frac{99}{100}\right)^{50-x} \cdot \left(\frac{1}{100}\right)^{x}$$

(a) P (winning at least once) = P (X ≥ 1)

$$= 1 - P(X < 1)$$

$$= 1 - P(X = 0)$$

$$= 1 - {}^{50}C_{0} \left(\frac{99}{100}\right)^{50}$$

$$= 1 - 1 \cdot \left(\frac{99}{100}\right)^{50}$$

$$= 1 - \left(\frac{99}{100}\right)^{50}$$

(b) P (winning exactly once) = P(X = 1)

$$= {}^{50}C_{1} \left(\frac{99}{100}\right)^{49} \cdot \left(\frac{1}{100}\right)^{1}$$

$$= 50\left(\frac{1}{100}\right)\left(\frac{99}{100}\right)^{49}$$

$$= \frac{1}{2}\left(\frac{99}{100}\right)^{49}$$

(c) P (at least twice) = P(X ≥ 2)

$$= 1 - P(X < 2)$$
$$= 1 - P(X \le 1)$$
$$= 1 - [P(X = 0) + P(X = 1)]$$
$$= [1 - P(X = 0)] - P(X = 1)$$
$$= 1 - \left(\frac{99}{100}\right)^{50} - \frac{1}{2} \cdot \left(\frac{99}{100}\right)^{49}$$
$$= 1 - \left(\frac{99}{100}\right)^{49} \cdot \left[\frac{99}{100} + \frac{1}{2}\right]$$
$$= 1 - \left(\frac{99}{100}\right)^{49} \cdot \left(\frac{149}{100}\right)$$
$$= 1 - \left(\frac{149}{100}\right)\left(\frac{99}{100}\right)^{49}$$

Question 11:

Find the probability of getting 5 exactly twice in 7 throws of a die.

Answer

The repeated tossing of a die are Bernoulli trials. Let X represent the number of times of getting 5 in 7 throws of the die.

Probability of getting 5 in a single throw of the die, $p = \frac{1}{6}$

$$\therefore q = 1 - p = 1 - \frac{1}{6} = \frac{5}{6}$$

Clearly, X has the probability distribution with n = 7 and $p = \frac{1}{6}$

$$\therefore P(X = x) = {}^{n}C_{x} q^{n-x} p^{x} = {}^{7}C_{x} \left(\frac{5}{6}\right)^{7-x} \cdot \left(\frac{1}{6}\right)^{x}$$

P (getting 5 exactly twice) = P(X = 2)

$= {}^7C_2 \left(\frac{5}{6}\right)^5 \cdot \left(\frac{1}{6}\right)^2$

$= 21 \cdot \left(\frac{5}{6}\right)^5 \cdot \frac{1}{36}$

$= \left(\frac{7}{12}\right)\left(\frac{5}{6}\right)^5$

Question 12:

Find the probability of throwing at most 2 sixes in 6 throws of a single die.

Answer

The repeated tossing of the die are Bernoulli trials. Let X represent the number of times of getting sixes in 6 throws of the die.

Probability of getting six in a single throw of die, $p = \frac{1}{6}$

$\therefore q = 1 - p = 1 - \frac{1}{6} = \frac{5}{6}$

Clearly, X has a binomial distribution with $n = 6$

$\therefore P(X = x) = {}^nC_x q^{n-x} p^x = {}^6C_x \left(\frac{5}{6}\right)^{6-x} \cdot \left(\frac{1}{6}\right)^x$

P (at most 2 sixes) = P(X ≤ 2)

$= P(X=0)+P(X=1)+P(X=2)$

$= {}^6C_0\left(\frac{5}{6}\right)^6 + {}^6C_1\cdot\left(\frac{5}{6}\right)^5\cdot\left(\frac{1}{6}\right) + {}^6C_2\left(\frac{5^4}{6}\right)\cdot\left(\frac{1}{6}\right)^2$

$= 1\cdot\left(\frac{5}{6}\right)^6 + 6\cdot\frac{1}{6}\cdot\left(\frac{5}{6}\right)^5 + 15.\frac{1}{36}\cdot\left(\frac{5}{6}\right)^4$

$= \left(\frac{5}{6}\right)^6 + \left(\frac{5}{6}\right)^5 + \frac{5}{12}\cdot\left(\frac{5}{6}\right)^4$

$= \left(\frac{5}{6}\right)^4\left[\left(\frac{5}{6}\right)^2 + \left(\frac{5}{6}\right) + \left(\frac{5}{12}\right)\right]$

$= \left(\frac{5}{6}\right)^4\cdot\left[\frac{25}{36}+\frac{5}{6}+\frac{5}{12}\right]$

$= \left(\frac{5}{6}\right)^4\cdot\left[\frac{25+30+15}{36}\right]$

$= \frac{70}{36}\cdot\left(\frac{5}{6}\right)^4$

$= \frac{35}{18}\cdot\left(\frac{5}{6}\right)^4$

Question 13:

It is known that 10% of certain articles manufactured are defective. What is the probability that in a random sample of 12 such articles, 9 are defective?

Answer

The repeated selections of articles in a random sample space are Bernoulli trails. Let X denote the number of times of selecting defective articles in a random sample space of 12 articles.

Clearly, X has a binomial distribution with n = 12 and p = 10% = $\frac{10}{100} = \frac{1}{10}$

$\therefore q = 1 - p = 1 - \frac{1}{10} = \frac{9}{10}$

$\therefore P(X=x) = {}^nC_x q^{n-x} p^x = {}^{12}C_x\left(\frac{9}{10}\right)^{12-x}\cdot\left(\frac{1}{10}\right)^x$

P (selecting 9 defective articles) = $^{12}C_9\left(\frac{9}{10}\right)^3\left(\frac{1}{10}\right)^9$

$= 220\cdot\frac{9^3}{10^3}\cdot\frac{1}{10^9}$

$= \frac{22\times 9^3}{10^{11}}$

Question 14:

In a box containing 100 bulbs, 10 are defective. The probability that out of a sample of 5 bulbs, none is defective is

(A) 10^{-1}

(B) $\left(\frac{1}{2}\right)^5$

(C) $\left(\frac{9}{10}\right)^5$

(D) $\frac{9}{10}$

Answer

The repeated selections of defective bulbs from a box are Bernoulli trials. Let X denote the number of defective bulbs out of a sample of 5 bulbs.

Probability of getting a defective bulb, *p* $= \frac{10}{100} = \frac{1}{10}$

$\therefore q = 1 - p = 1 - \frac{1}{10} = \frac{9}{10}$

Clearly, X has a binomial distribution with *n* = 5 and $p = \frac{1}{10}$

$\therefore \mathrm{P}(\mathrm{X} = x) = {}^n\mathrm{C}_x q^{n-x} p^x = {}^5\mathrm{C}_x\left(\frac{9}{10}\right)^{5-x}\left(\frac{1}{10}\right)^x$

P (none of the bulbs is defective) = P(X = 0)

$= {}^5C_0 \cdot \left(\frac{9}{10}\right)^5$

$= 1 \cdot \left(\frac{9}{10}\right)^5$

$= \left(\frac{9}{10}\right)^5$

The correct answer is C.

Question 15:

The probability that a student is not a swimmer is $\frac{1}{5}$. Then the probability that out of five students, four are swimmers is

(A) ${}^5C_4\left(\frac{4}{5}\right)^4\frac{1}{5}$ **(B)** $\left(\frac{4}{5}\right)^4\frac{1}{5}$

(C) ${}^5C_1\frac{1}{5}\left(\frac{4}{5}\right)^4$ **(D)** None of these

Answer

The repeated selection of students who are swimmers are Bernoulli trials. Let X denote the number of students, out of 5 students, who are swimmers.

Probability of students who are not swimmers, $q = \frac{1}{5}$

$\therefore p = 1 - q = 1 - \frac{1}{5} = \frac{4}{5}$

Clearly, X has a binomial distribution with n = 5 and $p = \frac{4}{5}$

$P(X = x) = {}^nC_x q^{n-x} p^x = {}^5C_x \left(\frac{1}{5}\right)^{5-x} \cdot \left(\frac{4}{5}\right)^x$

P (four students are swimmers) = P(X = 4) $= {}^5C_4\left(\frac{1}{5}\right)\cdot\left(\frac{4}{5}\right)^4$

Therefore, the correct answer is A.