Exercise 1.4

Question 1:

Determine whether or not each of the definition of given below gives a binary operation. In the event that * is not a binary operation, give justification for this.

(i) On $\mathbf{Z}^+$, define * by $a * b = a - b$

(ii) On $\mathbf{Z}^+$, define * by $a * b = ab$

(iii) On $\mathbf{R}$, define * by $a * b = ab^2$

(iv) On $\mathbf{Z}^+$, define * by $a * b = |a - b|$

(v) On $\mathbf{Z}^+$, define * by $a * b = a$

Answer

(i) On $\mathbf{Z}^+$, * is defined by $a * b = a - b$.

It is not a binary operation as the image of (1, 2) under * is $1 * 2 = 1 - 2 = -1 \notin \mathbf{Z}^+$.

(ii) On $\mathbf{Z}^+$, * is defined by $a * b = ab$.

It is seen that for each $a, b \in \mathbf{Z}^+$, there is a unique element ab in $\mathbf{Z}^+$.

This means that * carries each pair (a, b) to a unique element $a * b = ab$ in $\mathbf{Z}^+$.

Therefore, * is a binary operation.

(iii) On $\mathbf{R}$, * is defined by $a * b = ab^2$.

It is seen that for each $a, b \in \mathbf{R}$, there is a unique element ab^2 in $\mathbf{R}$.

This means that * carries each pair (a, b) to a unique element $a * b = ab^2$ in $\mathbf{R}$.

Therefore, * is a binary operation.

(iv) On $\mathbf{Z}^+$, * is defined by $a * b = |a - b|$.

It is seen that for each $a, b \in \mathbf{Z}^+$, there is a unique element $|a - b|$ in $\mathbf{Z}^+$.

This means that * carries each pair (a, b) to a unique element $a * b = |a - b|$ in $\mathbf{Z}^+$.

Therefore, * is a binary operation.

(v) On $\mathbf{Z}^+$, * is defined by $a * b = a$.

* carries each pair (a, b) to a unique element $a * b = a$ in $\mathbf{Z}^+$.

Therefore, * is a binary operation.

Question 2:

For each binary operation * defined below, determine whether * is commutative or associative.

(i) On **Z**, define *a* * *b* = *a* − *b*

(ii) On **Q**, define *a* * *b* = *ab* + 1

(iii) On **Q**, define *a* * *b* $= \frac{ab}{2}$

(iv) On $\mathbf{Z}^+$, define *a* * *b* = 2^{ab}

(v) On $\mathbf{Z}^+$, define *a* * *b* = a^b

(vi) On **R** − {−1}, define $a * b = \frac{a}{b+1}$

Answer

(i) On **Z**, * is defined by *a* * *b* = *a* − *b*.

It can be observed that 1 * 2 = 1 − 2 = 1 and 2 * 1 = 2 − 1 = 1.

∴1 * 2 ≠ 2 * 1; where 1, 2 ∈ **Z**

Hence, the operation * is not commutative.

Also we have:

(1 * 2) * 3 = (1 − 2) * 3 = −1 * 3 = −1 − 3 = −4

1 * (2 * 3) = 1 * (2 − 3) = 1 * −1 = 1 − (−1) = 2

∴(1 * 2) * 3 ≠ 1 * (2 * 3) ; where 1, 2, 3 ∈ **Z**

Hence, the operation * is not associative.

(ii) On **Q**, * is defined by *a* * *b* = *ab* + 1.

It is known that:

ab = *ba* ☐ *a, b* ∈ **Q**

⇒ *ab* + 1 = *ba* + 1 ☐ *a, b* ∈ **Q**

⇒ *a* * *b* = *a* * *b* ☐ *a, b* ∈ **Q**

Therefore, the operation * is commutative.

It can be observed that:

(1 * 2) * 3 = (1 × 2 + 1) * 3 = 3 * 3 = 3 × 3 + 1 = 10

1 * (2 * 3) = 1 * (2 × 3 + 1) = 1 * 7 = 1 × 7 + 1 = 8

∴(1 * 2) * 3 ≠ 1 * (2 * 3) ; where 1, 2, 3 ∈ **Q**

Therefore, the operation * is not associative.

(iii) On **Q**, * is defined by $a * b = \frac{ab}{2}$.

It is known that:

ab = *ba* ☐ *a, b* ∈ **Q**

⇒ $\frac{ab}{2} = \frac{ba}{2}$ ☐ *a, b* ∈ **Q**

⇒ *a* * *b* = *b* * *a* ☐ *a, b* ∈ **Q**

Therefore, the operation * is commutative.

For all *a, b*, *c* ∈ **Q**, we have:

$$(a * b) * c = \left(\frac{ab}{2}\right) * c = \frac{\left(\frac{ab}{2}\right)c}{2} = \frac{abc}{4}$$

$$a * (b * c) = a * \left(\frac{bc}{2}\right) = \frac{a\left(\frac{bc}{2}\right)}{2} = \frac{abc}{4}$$

$\therefore (a * b) * c = a * (b * c)$

Therefore, the operation * is associative.

(iv) On $\mathbf{Z}^+$, * is defined by $a * b = 2^{ab}$.

It is known that:

ab = *ba* ☐ *a, b* ∈ $\mathbf{Z}^+$

⇒ $2^{ab} = 2^{ba}$ ☐ *a, b* ∈ $\mathbf{Z}^+$

⇒ *a* * *b* = *b* * *a* ☐ *a, b* ∈ $\mathbf{Z}^+$

Therefore, the operation * is commutative.

It can be observed that:

$$(1 * 2) * 3 = 2^{(1\times2)} * 3 = 4 * 3 = 2^{4\times3} = 2^{12}$$

$$1 * (2 * 3) = 1 * 2^{2\times3} = 1 * 2^6 = 1 * 64 = 2^{64}$$

∴(1 * 2) * 3 ≠ 1 * (2 * 3) ; where 1, 2, 3 ∈ $\mathbf{Z}^+$

Therefore, the operation * is not associative.

(v) On $\mathbf{Z}^+$, * is defined by $a * b = a^b$.

It can be observed that:

$1 * 2 = 1^2 = 1$ and $2 * 1 = 2^1 = 2$

∴ 1 * 2 ≠ 2 * 1 ; where 1, 2 ∈ $\mathbf{Z}^+$

Therefore, the operation * is not commutative.

It can also be observed that:

$$(2*3)*4 = 2^3 * 4 = 8*4 = 8^4 = \left(2^3\right)^4 = 2^{12}$$

$$2*(3*4) = 2*3^4 = 2*81 = 2^{81}$$

∴(2 * 3) * 4 ≠ 2 * (3 * 4) ; where 2, 3, 4 ∈ $\mathbf{Z}^+$

Therefore, the operation * is not associative.

(vi) On **R**, * − {−1} is defined by $a*b = \dfrac{a}{b+1}$.

It can be observed that $1*2 = \dfrac{1}{2+1} = \dfrac{1}{3}$ and $2*1 = \dfrac{2}{1+1} = \dfrac{2}{2} = 1$.

∴1 * 2 ≠ 2 * 1 ; where 1, 2 ∈ **R** − {−1}

Therefore, the operation * is not commutative.

It can also be observed that:

$$(1*2)*3 = \frac{1}{3}*3 = \frac{\frac{1}{3}}{3+1} = \frac{1}{12}$$

$$1*(2*3) = 1*\frac{2}{3+1} = 1*\frac{2}{4} = 1*\frac{1}{2} = \frac{1}{\frac{1}{2}+1} = \frac{1}{\frac{3}{2}} = \frac{2}{3}$$

∴ (1 * 2) * 3 ≠ 1 * (2 * 3) ; where 1, 2, 3 ∈ **R** − {−1}

Therefore, the operation * is not associative.

Question 3:

Consider the binary operation ∨ on the set {1, 2, 3, 4, 5} defined by *a* ∨*b* = min {*a*, *b*}. Write the operation table of the operation∨.

Answer

The binary operation ∨ on the set {1, 2, 3, 4, 5} is defined as *a* ∨ *b* = min {*a*, *b*} ☐ *a*, *b* ∈ {1, 2, 3, 4, 5}.

Thus, the operation table for the given operation ∨ can be given as:

∨	1	2	3	4	5

1	1	1	1	1	1
2	1	2	2	2	2
3	1	2	3	3	3
4	1	2	3	4	4
5	1	2	3	4	5

Question 4:

Consider a binary operation * on the set {1, 2, 3, 4, 5} given by the following multiplication table.

(i) Compute (2 * 3) * 4 and 2 * (3 * 4)

(ii) Is * commutative?

(iii) Compute (2 * 3) * (4 * 5).

(Hint: use the following table)

*	1	2	3	4	5
1	1	1	1	1	1
2	1	2	1	2	1
3	1	1	3	1	1
4	1	2	1	4	1
5	1	1	1	1	5

Answer

(i) (2 * 3) * 4 = 1 * 4 = 1

2 * (3 * 4) = 2 * 1 = 1

(ii) For every *a, b* ∈{1, 2, 3, 4, 5}, we have *a* * *b* = *b* * *a*. Therefore, the operation * is commutative.

(iii) (2 * 3) = 1 and (4 * 5) = 1

∴(2 * 3) * (4 * 5) = 1 * 1 = 1

Question 5:

Let*′ be the binary operation on the set {1, 2, 3, 4, 5} defined by *a* *′ *b* = H.C.F. of *a* and *b*. Is the operation *′ same as the operation * defined in Exercise 4 above? Justify your answer.

Answer

The binary operation *′ on the set {1, 2, 3 4, 5} is defined as *a* *′ *b* = H.C.F of *a* and *b*.

The operation table for the operation *′ can be given as:

*′	1	2	3	4	5
1	1	1	1	1	1
2	1	2	1	2	1
3	1	1	3	1	1
4	1	2	1	4	1
5	1	1	1	1	5

We observe that the operation tables for the operations * and *′ are the same.

Thus, the operation *′ is same as the operation*.

Question 6:

Let * be the binary operation on **N** given by *a* * *b* = L.C.M. of *a* and *b*. Find

(i) 5 * 7, 20 * 16 (ii) Is * commutative?

(iii) Is * associative? (iv) Find the identity of * in **N**

(v) Which elements of **N** are invertible for the operation *?

Answer

The binary operation * on **N** is defined as *a* * *b* = L.C.M. of *a* and *b*.

(i) 5 * 7 = L.C.M. of 5 and 7 = 35

20 * 16 = L.C.M of 20 and 16 = 80

(ii) It is known that:

L.C.M of *a* and *b* = L.C.M of *b* and *a* □ *a, b* ∈ **N**.

∴ $a * b = b * a$

Thus, the operation * is commutative.

(iii) For $a, b, c \in \mathbf{N}$, we have:

$(a * b) * c$ = (L.C.M of a and b) * c = LCM of a, b, and c

$a * (b * c) = a$ * (LCM of b and c) = L.C.M of a, b, and c

∴ $(a * b) * c = a * (b * c)$

Thus, the operation * is associative.

(iv) It is known that:

L.C.M. of a and 1 = a = L.C.M. 1 and a ☐ $a \in \mathbf{N}$

⇒ $a * 1 = a = 1 * a$ ☐ $a \in \mathbf{N}$

Thus, 1 is the identity of * in **N**.

(v) An element a in **N** is invertible with respect to the operation * if there exists an element b in **N**, such that $a * b = e = b * a$.

Here, $e = 1$

This means that:

L.C.M of a and b = 1 = L.C.M of b and a

This case is possible only when a and b are equal to 1.

Thus, 1 is the only invertible element of **N** with respect to the operation *.

Question 7:

Is * defined on the set {1, 2, 3, 4, 5} by $a * b$ = L.C.M. of a and b a binary operation? Justify your answer.

Answer

The operation * on the set A = {1, 2, 3, 4, 5} is defined as

$a * b$ = L.C.M. of a and b.

Then, the operation table for the given operation * can be given as:

*	1	2	3	4	5
1	1	2	3	4	5
2	2	2	6	4	10
3	3	6	3	12	15

4	4	4	12	4	20
5	5	10	15	20	5

It can be observed from the obtained table that:

3 * 2 = 2 * 3 = 6 ∉ A, 5 * 2 = 2 * 5 = 10 ∉ A, 3 * 4 = 4 * 3 = 12 ∉ A

3 * 5 = 5 * 3 = 15 ∉ A, 4 * 5 = 5 * 4 = 20 ∉ A

Hence, the given operation * is not a binary operation.

Question 8:

Let * be the binary operation on **N** defined by *a* * *b* = H.C.F. of *a* and *b*. Is * commutative? Is * associative? Does there exist identity for this binary operation on **N**?

Answer

The binary operation * on **N** is defined as:

a * *b* = H.C.F. of *a* and *b*

It is known that:

H.C.F. of *a* and *b* = H.C.F. of *b* and *a* ☐ *a, b* ∈ **N**.

∴*a* * *b* = *b* * *a*

Thus, the operation * is commutative.

For *a*, *b*, *c* ∈ **N**, we have:

(*a* * *b*)* *c* = (H.C.F. of *a* and *b*) * *c* = H.C.F. of *a*, *b*, and *c*

a *(*b* * *c*)= *a* *(H.C.F. of *b* and *c*) = H.C.F. of *a*, *b*, and *c*

∴(*a* * *b*) * *c* = *a* * (*b* * *c*)

Thus, the operation * is associative.

Now, an element *e* ∈ **N** will be the identity for the operation * if *a* * *e* = *a* = e* *a* ∀ *a* ∈ **N**.

But this relation is not true for any *a* ∈ **N**.

Thus, the operation * does not have any identity in **N**.

Question 9:

Let * be a binary operation on the set **Q** of rational numbers as follows:

(i) $a * b = a - b$ (ii) $a * b = a^2 + b^2$

(iii) $a * b = a + ab$ (iv) $a * b = (a - b)^2$

(v) $a * b = \frac{ab}{4}$ (vi) $a * b = ab^2$

Find which of the binary operations are commutative and which are associative.

Answer

(i) On **Q**, the operation * is defined as $a * b = a - b$.

It can be observed that:

$\frac{1}{2}*\frac{1}{3}=\frac{1}{2}-\frac{1}{3}=\frac{3-2}{6}=\frac{1}{6}$ and $\frac{1}{3}*\frac{1}{2}=\frac{1}{3}-\frac{1}{2}=\frac{2-3}{6}=\frac{-1}{6}$

$\therefore \frac{1}{2}*\frac{1}{3} \neq \frac{1}{3}*\frac{1}{2}$; where $\frac{1}{2}, \frac{1}{3} \in \mathbf{Q}$

Thus, the operation * is not commutative.

It can also be observed that:

$$\left(\frac{1}{2}*\frac{1}{3}\right)*\frac{1}{4}=\left(\frac{1}{2}-\frac{1}{3}\right)*\frac{1}{4}=\frac{1}{6}*\frac{1}{4}=\frac{1}{6}-\frac{1}{4}=\frac{2-3}{12}=\frac{-1}{12}$$

$$\frac{1}{2}*\left(\frac{1}{3}*\frac{1}{4}\right)=\frac{1}{2}*\left(\frac{1}{3}-\frac{1}{4}\right)=\frac{1}{2}*\frac{1}{12}=\frac{1}{2}-\frac{1}{12}=\frac{6-1}{12}=\frac{5}{12}$$

$$\therefore \left(\frac{1}{2}*\frac{1}{3}\right)*\frac{1}{4} \neq \frac{1}{2}*\left(\frac{1}{3}*\frac{1}{4}\right) \text{ ; where } \frac{1}{2}, \frac{1}{3}, \frac{1}{4} \in \mathbf{Q}$$

Thus, the operation * is not associative.

(ii) On **Q**, the operation * is defined as $a * b = a^2 + b^2$.

For $a, b \in$ **Q**, we have:

$a*b=a^2+b^2=b^2+a^2=b*a$

$\therefore a * b = b * a$

Thus, the operation * is commutative.

It can be observed that:

$$(1*2)*3=\left(1^2+2^2\right)*3=(1+4)*4=5*4=5^2+4^2=41$$

$$1*(2*3)=1*\left(2^2+3^2\right)=1*(4+9)=1*13=1^2+13^2=169$$

$$\therefore (1*2)*3 \neq 1*(2*3) \text{ ; where } 1, 2, 3 \in \mathbf{Q}$$

Thus, ,the operation * is not associative.

(iii) On **Q**, the operation * is defined as $a * b = a + ab$.

It can be observed that:

$$1*2=1+1\times 2=1+2=3$$
$$2*1=2+2\times 1=2+2=4$$

$$\therefore 1*2\neq 2*1 \text{ ; where } 1, 2 \in \mathbf{Q}$$

Thus, the operation * is not commutative.

It can also be observed that:

$$(1*2)*3=(1+1\times 2)*3=3*3=3+3\times 3=3+9=12$$
$$1*(2*3)=1*(2+2\times 3)=1*8=1+1\times 8=9$$

$$\therefore (1*2)*3\neq 1*(2*3) \text{ ; where } 1, 2, 3 \in \mathbf{Q}$$

Thus, the operation * is not associative.

(iv) On **Q**, the operation * is defined by $a * b = (a - b)^2$.

For a, $b \in$ **Q**, we have:

$a * b = (a - b)^2$

$b * a = (b - a)^2 = [-(a - b)]^2 = (a - b)^2$

$\therefore a * b = b * a$

Thus, the operation * is commutative.

It can be observed that:

$$(1*2)*3=(1-2)^2*3=(-1)^2*3=1*3=(1-3)^2=(-2)^2=4$$
$$1*(2*3)=1*(2-3)^2=1*(-1)^2=1*1=(1-1)^2=0$$

$$\therefore (1*2)*3\neq 1*(2*3) \text{ ; where } 1, 2, 3 \in \mathbf{Q}$$

Thus, the operation * is not associative.

(v) On **Q**, the operation * is defined as $a*b=\frac{ab}{4}$.

For a, $b \in$ **Q**, we have:

$$a*b=\frac{ab}{4}=\frac{ba}{4}=b*a$$

$\therefore a * b = b * a$

Thus, the operation * is commutative.

For *a, b, c* ∈ **Q**, we have:

$$(a*b)*c = \frac{ab}{4}*c = \frac{\frac{ab}{4}\cdot c}{4} = \frac{abc}{16}$$

$$a*(b*c) = a*\frac{bc}{4} = \frac{a\cdot\frac{bc}{4}}{4} = \frac{abc}{16}$$

∴$(a * b) * c = a * (b * c)$

Thus, the operation * is associative.

(vi) On **Q**, the operation * is defined as $a * b = ab^2$

It can be observed that:

$$\frac{1}{2}*\frac{1}{3} = \frac{1}{2}\cdot\left(\frac{1}{3}\right)^2 = \frac{1}{2}\cdot\frac{1}{9} = \frac{1}{18}$$

$$\frac{1}{3}*\frac{1}{2} = \frac{1}{3}.\left(\frac{1}{2}\right)^2 = \frac{1}{3}.\frac{1}{4} = \frac{1}{12}$$

$$\therefore \frac{1}{2}*\frac{1}{3} \neq \frac{1}{3}*\frac{1}{2} \text{ ; where } \frac{1}{2},\frac{1}{3}\in \mathbf{Q}$$

Thus, the operation * is not commutative.

It can also be observed that:

$$\left(\frac{1}{2}*\frac{1}{3}\right)*\frac{1}{4} = \left[\frac{1}{2}\cdot\left(\frac{1}{3}\right)^2\right]*\frac{1}{4} = \frac{1}{18}*\frac{1}{4} = \frac{1}{18}\cdot\left(\frac{1}{4}\right)^2 = \frac{1}{18\times16}$$

$$\frac{1}{2}*\left(\frac{1}{3}*\frac{1}{4}\right) = \frac{1}{2}*\left[\frac{1}{3}\cdot\left(\frac{1}{4}\right)^2\right] = \frac{1}{2}*\frac{1}{48} = \frac{1}{2}.\left(\frac{1}{48}\right)^2 = \frac{1}{2\times(48)^2}$$

$$\therefore \left(\frac{1}{2}*\frac{1}{3}\right)*\frac{1}{4} \neq \frac{1}{2}*\left(\frac{1}{3}*\frac{1}{4}\right) \text{ ; where } \frac{1}{2},\ \frac{1}{3},\ \frac{1}{4}\in \mathbf{Q}$$

Thus, the operation * is not associative.

Hence, the operations defined in (ii), (iv), (v) are commutative and the operation defined in (v) is associative.

Question 10:

Find which of the operations given above has identity.

Answer

An element $e \in \mathbf{Q}$ will be the identity element for the operation * if

$a * e = a = e * a,\ \forall a \in \mathbf{Q}$.

However, there is no such element $e \in \mathbf{Q}$ with respect to each of the six operations satisfying the above condition.

Thus, none of the six operations has identity.

Question 11:

Let A = **N** × **N** and * be the binary operation on A defined by

$(a, b) * (c, d) = (a + c, b + d)$

Show that * is commutative and associative. Find the identity element for * on A, if any.

Answer

A = **N** × **N**

* is a binary operation on A and is defined by:

$(a, b) * (c, d) = (a + c, b + d)$

Let $(a, b), (c, d) \in$ A

Then, $a, b, c, d \in \mathbf{N}$

We have:

$(a, b) * (c, d) = (a + c, b + d)$

$(c, d) * (a, b) = (c + a, d + b) = (a + c, b + d)$

[Addition is commutative in the set of natural numbers]

$\therefore (a, b) * (c, d) = (c, d) * (a, b)$

Therefore, the operation * is commutative.

Now, let $(a, b), (c, d), (e, f) \in$A

Then, $a, b, c, d, e, f \in \mathbf{N}$

We have:

$$\big((a,b)*(c,d)\big)*(e,f) = (a+c,\ b+d)*(e,f) = (a+c+e,\ b+d+f)$$

$$(a,b)*((c,d)*(e,f))=(a,b)*(c+e,\ d+f)=(a+c+e,\ b+d+f)$$

$$\therefore ((a,b)*(c,d))*(e,f)=(a,b)*((c,d)*(e,f))$$

Therefore, the operation * is associative.

An element $e=(e_1,e_2)\in A$ will be an identity element for the operation * if $a*e=a=e*a\ \forall\ a=(a_1,a_2)\in A$, i.e., $(a_1+e_1,a_2+e_2)=(a_1,a_2)=(e_1+a_1,e_2+a_2)$, which is not true for any element in A.

Therefore, the operation * does not have any identity element.

Question 12:

State whether the following statements are true or false. Justify.

(i) For an arbitrary binary operation * on a set **N**, $a * a = a\ \forall a * \mathbf{N}$.

(ii) If * is a commutative binary operation on **N**, then $a * (b * c) = (c * b) * a$

Answer

(i) Define an operation * on **N** as:

$a * b = a + b\ \forall a, b \in \mathbf{N}$

Then, in particular, for $b = a = 3$, we have:

$3 * 3 = 3 + 3 = 6 \neq 3$

Therefore, statement (i) is false.

(ii) R.H.S. = $(c * b) * a$

$= (b * c) * a$ [* is commutative]

$= a * (b * c)$ [Again, as * is commutative]

= L.H.S.

$\therefore a * (b * c) = (c * b) * a$

Therefore, statement (ii) is true.

Question 13:

Consider a binary operation * on **N** defined as $a * b = a^3 + b^3$. Choose the correct answer.

(A) Is * both associative and commutative?

(B) Is * commutative but not associative?

(C) Is * associative but not commutative?

(D) Is * neither commutative nor associative?

Answer

On **N**, the operation * is defined as $a * b = a^3 + b^3$.

For, a, b, ∈ **N**, we have:

$a * b = a^3 + b^3 = b^3 + a^3 = b * a$ [Addition is commutative in **N**]

Therefore, the operation * is commutative.

It can be observed that:

$$(1*2)*3=(1^3+2^3)*3=9*3=9^3+3^3=729+27=756$$

$$1*(2*3)=1*(2^3+3^3)=1*(8+27)=1\times 35=1^3+35^3=1+(35)^3=1+42875=42876$$

∴(1 * 2) * 3 ≠ 1 * (2 * 3) ; where 1, 2, 3 ∈ **N**

Therefore, the operation * is not associative.

Hence, the operation * is commutative, but not associative. Thus, the correct answer is B.