Exercise 1.2

Question 1:

Show that the function *f*: **R**$_*$ → **R**$_*$ defined by $f(x)=\frac{1}{x}$ is one-one and onto, where **R**$_*$ is the set of all non-zero real numbers. Is the result true, if the domain **R**$_*$ is replaced by **N** with co-domain being same as **R**$_*$?

Answer

It is given that *f*: **R**$_*$ → **R**$_*$ is defined by $f(x)=\frac{1}{x}$.

One-one:

$f(x)=f(y)$

$\Rightarrow \frac{1}{x}=\frac{1}{y}$

$\Rightarrow x=y$

∴*f* is one-one.

Onto:

It is clear that for *y*∈ **R**$_*$, there exists $x=\frac{1}{y}\in \mathrm{R}_* \ (\text{Exists as } y\neq 0)$ such that

$f(x)=\frac{1}{\left(\frac{1}{y}\right)}=y.$

∴*f* is onto.

Thus, the given function *(f)* is one-one and onto.

Now, consider function *g*: **N** → **R**$_*$defined by

$g(x)=\frac{1}{x}.$

We have,

$g(x_1)=g(x_2)\Rightarrow \frac{1}{x_1}=\frac{1}{x_2}\Rightarrow x_1=x_2$

∴*g* is one-one.

Further, it is clear that g is not onto as for $1.2 \in \mathbf{R}_*$ there does not exit any x in **N** such that $g(x) = \frac{1}{1.2}$.

Hence, function g is one-one but not onto.

Question 2:

Check the injectivity and surjectivity of the following functions:

(i) f: $\mathbf{N} \rightarrow \mathbf{N}$ given by $f(x) = x^2$

(ii) f: $\mathbf{Z} \rightarrow \mathbf{Z}$ given by $f(x) = x^2$

(iii) f: $\mathbf{R} \rightarrow \mathbf{R}$ given by $f(x) = x^2$

(iv) f: $\mathbf{N} \rightarrow \mathbf{N}$ given by $f(x) = x^3$

(v) f: $\mathbf{Z} \rightarrow \mathbf{Z}$ given by $f(x) = x^3$

Answer

(i) f: $\mathbf{N} \rightarrow \mathbf{N}$ is given by,

$f(x) = x^2$

It is seen that for $x, y \in \mathbf{N}$, $f(x) = f(y) \Rightarrow x^2 = y^2 \Rightarrow x = y$.

$\therefore f$ is injective.

Now, $2 \in \mathbf{N}$. But, there does not exist any x in **N** such that $f(x) = x^2 = 2$.

$\therefore$ f is not surjective.

Hence, function f is injective but not surjective.

(ii) f: $\mathbf{Z} \rightarrow \mathbf{Z}$ is given by,

$f(x) = x^2$

It is seen that $f(-1) = f(1) = 1$, but $-1 \neq 1$.

$\therefore$ f is not injective.

Now,$-2 \in \mathbf{Z}$. But, there does not exist any element $x \in \mathbf{Z}$ such that $f(x) = x^2 = -2$.

$\therefore$ f is not surjective.

Hence, function f is neither injective nor surjective.

(iii) f: $\mathbf{R} \rightarrow \mathbf{R}$ is given by,

$f(x) = x^2$

It is seen that $f(-1) = f(1) = 1$, but $-1 \neq 1$.

$\therefore$ f is not injective.

Now,$-2 \in \mathbf{R}$. But, there does not exist any element $x \in \mathbf{R}$ such that $f(x) = x^2 = -2$.

∴ f is not surjective.

Hence, function f is neither injective nor surjective.

(iv) f: **N** → **N** given by,

$f(x) = x^3$

It is seen that for $x, y \in$ **N**, $f(x) = f(y) \Rightarrow x^3 = y^3 \Rightarrow x = y$.

∴ f is injective.

Now, 2 ∈ **N**. But, there does not exist any element x in domain **N** such that $f(x) = x^3 = 2$.

∴ f is not surjective

Hence, function f is injective but not surjective.

(v) f: **Z** → **Z** is given by,

$f(x) = x^3$

It is seen that for $x, y \in$ **Z**, $f(x) = f(y) \Rightarrow x^3 = y^3 \Rightarrow x = y$.

∴ f is injective.

Now, 2 ∈ **Z**. But, there does not exist any element x in domain **Z** such that $f(x) = x^3 = 2$.

∴ f is not surjective.

Hence, function f is injective but not surjective.

Question 3:

Prove that the Greatest Integer Function f: **R** → **R** given by $f(x) = [x]$, is neither one-once nor onto, where $[x]$ denotes the greatest integer less than or equal to x.

Answer

f: **R** → **R** is given by,

$f(x) = [x]$

It is seen that $f(1.2) = [1.2] = 1$, $f(1.9) = [1.9] = 1$.

∴ $f(1.2) = f(1.9)$, but $1.2 \neq 1.9$.

∴ f is not one-one.

Now, consider 0.7 ∈ **R**.

It is known that $f(x) = [x]$ is always an integer. Thus, there does not exist any element $x \in$ **R** such that $f(x) = 0.7$.

∴ f is not onto.

Hence, the greatest integer function is neither one-one nor onto.

Question 4:

Show that the Modulus Function *f*: **R** → **R** given by $f(x)=|x|$, is neither one-one nor onto, where $|x|$ is *x*, if *x* is positive or 0 and $|x|$ is − *x*, if *x* is negative.

Answer

f: **R** → **R** is given by,

$$f(x)=|x|=\begin{cases}x, & \text{if } x\geq 0\\ -x, & \text{if } x<0\end{cases}$$

It is seen that $f(-1)=|-1|=1, f(1)=|1|=1$.

∴$f(-1) = f(1)$, but $-1 \neq 1$.

∴ *f* is not one-one.

Now, consider $-1 \in$ **R**.

It is known that $f(x) = |x|$ is always non-negative. Thus, there does not exist any element *x* in domain **R** such that $f(x) = |x| = -1$.

∴ *f* is not onto.

Hence, the modulus function is neither one-one nor onto.

Question 5:

Show that the Signum Function *f*: **R** → **R**, given by

$$f(x)=\begin{cases}1, & \text{if } x>0\\ 0, & \text{if } x=0\\ -1, & \text{if } x<0\end{cases}$$

is neither one-one nor onto.

Answer

f: **R** → **R** is given by,

$$f(x)=\begin{cases}1, & \text{if } x>0\\ 0, & \text{if } x=0\\ -1, & \text{if } x<0\end{cases}$$

It is seen that $f(1) = f(2) = 1$, but $1 \neq 2$.

∴*f* is not one-one.

Now, as $f(x)$ takes only 3 values (1, 0, or −1) for the element −2 in co-domain **R**, there does not exist any x in domain **R** such that $f(x) = -2$.

∴ f is not onto.

Hence, the signum function is neither one-one nor onto.

Question 6:

Let $A = \{1, 2, 3\}$, $B = \{4, 5, 6, 7\}$ and let $f = \{(1, 4), (2, 5), (3, 6)\}$ be a function from A to B. Show that f is one-one.

Answer

It is given that $A = \{1, 2, 3\}$, $B = \{4, 5, 6, 7\}$.

$f: A \to B$ is defined as $f = \{(1, 4), (2, 5), (3, 6)\}$.

∴ $f(1) = 4$, $f(2) = 5$, $f(3) = 6$

It is seen that the images of distinct elements of A under f are distinct.

Hence, function f is one-one.

Question 7:

In each of the following cases, state whether the function is one-one, onto or bijective. Justify your answer.

(i) f: **R** → **R** defined by $f(x) = 3 - 4x$

(ii) f: **R** → **R** defined by $f(x) = 1 + x^2$

Answer

(i) f: **R** → **R** is defined as $f(x) = 3 - 4x$.

Let $x_1, x_2 \in \mathbf{R}$ such that $f(x_1) = f(x_2)$.

$$\Rightarrow 3-4x_1 = 3-4x_2$$
$$\Rightarrow -4x_1 = -4x_2$$
$$\Rightarrow x_1 = x_2$$

∴ f is one-one.

For any real number (y) in **R**, there exists $\frac{3-y}{4}$ in **R** such that

$$f\left(\frac{3-y}{4}\right) = 3-4\left(\frac{3-y}{4}\right) = y.$$

∴ f is onto.

Hence, f is bijective.

(ii) f: **R** → **R** is defined as

$f(x) = 1 + x^2$.

Let $x_1, x_2 \in \mathbf{R}$ such that $f(x_1) = f(x_2)$.

$\Rightarrow 1 + x_1^2 = 1 + x_2^2$
$\Rightarrow x_1^2 = x_2^2$
$\Rightarrow x_1 = \pm x_2$

$\therefore f(x_1) = f(x_2)$ does not imply that $x_1 = x_2$.

For instance,

$f(1) = f(-1) = 2$

∴ f is not one-one.

Consider an element −2 in co-domain **R**.

It is seen that $f(x) = 1 + x^2$ is positive for all $x \in$ **R**.

Thus, there does not exist any x in domain **R** such that $f(x) = -2$.

∴ f is not onto.

Hence, f is neither one-one nor onto.

Question 8:

Let A and B be sets. Show that f: $A \times B \to B \times A$ such that $(a, b) = (b, a)$ is bijective function.

Answer

f: $A \times B \to B \times A$ is defined as $f(a, b) = (b, a)$.

Let $(a_1, b_1), (a_2, b_2) \in \mathrm{A} \times \mathrm{B}$ such that $f(a_1, b_1) = f(a_2, b_2)$.

$\Rightarrow (b_1, a_1) = (b_2, a_2)$
$\Rightarrow b_1 = b_2$ and $a_1 = a_2$
$\Rightarrow (a_1, b_1) = (a_2, b_2)$

∴ f is one-one.

Now, let $(b, a) \in B \times A$ be any element.

Then, there exists $(a, b) \in A \times B$ such that $f(a, b) = (b, a)$. [By definition of f]

∴ *f* is onto.

Hence, *f* is bijective.

Question 9:

Let *f*: **N** → **N** be defined by $f(n)=\begin{cases}\dfrac{n+1}{2}, & \text{if } n \text{ is odd}\\ \dfrac{n}{2}, & \text{if } n \text{ is even}\end{cases}$ for all $n\in\mathbf{N}$.

State whether the function f is bijective. Justify your answer.

Answer

f: **N** → **N** is defined as $f(n)=\begin{cases}\dfrac{n+1}{2}, & \text{if } n \text{ is odd}\\ \dfrac{n}{2}, & \text{if } n \text{ is even}\end{cases}$ for all $n\in\mathbf{N}$.

It can be observed that:

$$f(1)=\frac{1+1}{2}=1 \text{ and } f(2)=\frac{2}{2}=1 \qquad [\text{By definition of } f]$$

$$\therefore f(1)=f(2), \text{ where } 1\neq 2.$$

∴ *f* is not one-one.

Consider a natural number *(n)* in co-domain **N**.

Case **I:** *n* is odd

∴*n* = 2*r* + 1 for some *r* ∈ **N.** Then, there exists 4*r* + 1∈**N** such that

$$f(4r+1)=\frac{4r+1+1}{2}=2r+1.$$

Case **II:** *n* is even

∴*n* = 2*r* for some *r* ∈ **N.** Then,there exists 4*r* ∈**N** such that $f(4r)=\dfrac{4r}{2}=2r$.

∴ *f* is onto.

Hence, *f* is not a bijective function.

Question 10:

Let A = **R** − {3} and B = **R** − {1}. Consider the function *f*: A → B defined by

$f(x) = \left(\frac{x-2}{x-3}\right)$. Is *f* one-one and onto? Justify your answer.

Answer

A = **R** − {3}, B = **R** − {1}

f: A → B is defined as $f(x) = \left(\frac{x-2}{x-3}\right)$.

Let $x,\ y \in \text{A}$ such that $f(x) = f(y)$.

$$\Rightarrow \frac{x-2}{x-3} = \frac{y-2}{y-3}$$
$$\Rightarrow (x-2)(y-3) = (y-2)(x-3)$$
$$\Rightarrow xy - 3x - 2y + 6 = xy - 3y - 2x + 6$$
$$\Rightarrow -3x - 2y = -3y - 2x$$
$$\Rightarrow 3x - 2x = 3y - 2y$$
$$\Rightarrow x = y$$

∴ *f* is one-one.

Let *y* ∈B = **R** − {1}. Then, *y* ≠ 1.

The function *f* is onto if there exists *x* ∈A such that *f*(*x*) = *y.*

Now,

$$f(x) = y$$
$$\Rightarrow \frac{x-2}{x-3} = y$$
$$\Rightarrow x - 2 = xy - 3y$$
$$\Rightarrow x(1-y) = -3y + 2$$
$$\Rightarrow x = \frac{2-3y}{1-y} \in \text{A} \qquad [y \neq 1]$$

Thus, for any *y* ∈ B, there exists $\frac{2-3y}{1-y} \in \text{A}$ such that

$$f\left(\frac{2-3y}{1-y}\right)=\frac{\left(\frac{2-3y}{1-y}\right)-2}{\left(\frac{2-3y}{1-y}\right)-3}=\frac{2-3y-2+2y}{2-3y-3+3y}=\frac{-y}{-1}=y.$$

$\therefore f$ is onto.

Hence, function *f* is one-one and onto.

Question 11:

Let *f*: **R** → **R** be defined as $f(x) = x^4$. Choose the correct answer.

(A) *f* is one-one onto (B) *f* is many-one onto

(C) *f* is one-one but not onto (D) *f* is neither one-one nor onto

Answer

f: **R** → **R** is defined as $f(x)=x^4.$

Let *x*, *y* ∈ **R** such that *f*(*x*) = *f*(*y*).

$\Rightarrow x^4 = y^4$

$\Rightarrow x = \pm y$

∴ $f(x_1)=f(x_2)$ does not imply that $x_1 = x_2$.

For instance,

$f(1)=f(-1)=1$

∴ *f* is not one-one.

Consider an element 2 in co-domain **R**. It is clear that there does not exist any *x* in domain **R** such that *f*(*x*) = 2.

∴ *f* is not onto.

Hence, function *f* is neither one-one nor onto.

The correct answer is D.

Question 12:

Let *f*: **R** → **R** be defined as *f*(*x*) = 3*x*. Choose the correct answer.

(A) *f* is one-one onto (B) *f* is many-one onto

(C) *f* is one-one but not onto (D) *f* is neither one-one nor onto

Answer

f: **R** → **R** is defined as $f(x) = 3x$.

Let $x, y \in$ **R** such that $f(x) = f(y)$.

$\Rightarrow 3x = 3y$

$\Rightarrow x = y$

$\therefore f$ is one-one.

Also, for any real number (y) in co-domain **R**, there exists $\frac{y}{3}$ in **R** such that

$$f\left(\frac{y}{3}\right) = 3\left(\frac{y}{3}\right) = y.$$

$\therefore f$ is onto.

Hence, function f is one-one and onto.

The correct answer is A.