NCERT Miscellaneous Solution

Question 1: Prove that: $2\cos\frac{\pi}{13}\cos\frac{9\pi}{13}+\cos\frac{3\pi}{13}+\cos\frac{5\pi}{13}=0$

Answer

L.H.S.

$$=2\cos\frac{\pi}{13}\cos\frac{9\pi}{13}+\cos\frac{3\pi}{13}+\cos\frac{5\pi}{13}$$

$$=2\cos\frac{\pi}{13}\cos\frac{9\pi}{13}+2\cos\left(\frac{\frac{3\pi}{13}+\frac{5\pi}{13}}{2}\right)\cos\left(\frac{\frac{3\pi}{13}-\frac{5\pi}{13}}{2}\right)\quad\left[\cos x+\cos y=2\cos\left(\frac{x+y}{2}\right)\cos\left(\frac{x-y}{2}\right)\right]$$

$$=2\cos\frac{\pi}{13}\cos\frac{9\pi}{13}+2\cos\frac{4\pi}{13}\cos\left(\frac{-\pi}{13}\right)$$

$$=2\cos\frac{\pi}{13}\cos\frac{9\pi}{13}+2\cos\frac{4\pi}{13}\cos\frac{\pi}{13}$$

$$=2\cos\frac{\pi}{13}\left[\cos\frac{9\pi}{13}+\cos\frac{4\pi}{13}\right]$$

$$=2\cos\frac{\pi}{13}\left[2\cos\left(\frac{\frac{9\pi}{13}+\frac{4\pi}{13}}{2}\right)\cos\left(\frac{\frac{9\pi}{13}-\frac{4\pi}{13}}{2}\right)\right]$$

$$=2\cos\frac{\pi}{13}\left[2\cos\frac{\pi}{2}\cos\frac{5\pi}{26}\right]$$

$$=2\cos\frac{\pi}{13}\times 2\times 0\times\cos\frac{5\pi}{26}$$

= 0 = R.H.S

Question 2:

Prove that: $(\sin 3x + \sin x)\sin x + (\cos 3x - \cos x)\cos x = 0$

Answer

L.H.S.

$= (\sin 3x + \sin x)\sin x + (\cos 3x - \cos x)\cos x$

$= \sin 3x \sin x + \sin^2 x + \cos 3x \cos x - \cos^2 x$

$= \cos 3x \cos x + \sin 3x \sin x - \left(\cos^2 x - \sin^2 x\right)$

$= \cos\left(3x - x\right) - \cos 2x \qquad \left[\cos\left(A - B\right) = \cos A \cos B + \sin A \sin B\right]$

$= \cos 2x - \cos 2x$

$= 0$

= RH.S.

Question 3:

Prove that: $\left(\cos x + \cos y\right)^2 + \left(\sin x - \sin y\right)^2 = 4\cos^2 \frac{x + y}{2}$

Answer

L.H.S. = $\left(\cos x + \cos y\right)^2 + \left(\sin x - \sin y\right)^2$

$= \cos^2 x + \cos^2 y + 2\cos x \cos y + \sin^2 x + \sin^2 y - 2\sin x \sin y$

$= \left(\cos^2 x + \sin^2 x\right) + \left(\cos^2 y + \sin^2 y\right) + 2\left(\cos x \cos y - \sin x \sin y\right)$

$= 1 + 1 + 2\cos\left(x + y\right) \qquad \left[\cos\left(A + B\right) = \left(\cos A \cos B - \sin A \sin B\right)\right]$

$= 2 + 2\cos\left(x + y\right)$

$= 2\left[1 + \cos\left(x + y\right)\right]$

$= 2\left[1 + 2\cos^2\left(\frac{x + y}{2}\right) - 1\right] \qquad \left[\cos 2A = 2\cos^2 A - 1\right]$

$= 4\cos^2\left(\frac{x + y}{2}\right) = \text{R.H.S.}$

Question 4:

Prove that: $\left(\cos x - \cos y\right)^2 + \left(\sin x - \sin y\right)^2 = 4\sin^2 \frac{x - y}{2}$

Answer

L.H.S. = $\left(\cos x - \cos y\right)^2 + \left(\sin x - \sin y\right)^2$

$= \cos^2 x + \cos^2 y - 2\cos x \cos y + \sin^2 x + \sin^2 y - 2\sin x \sin y$

$= (\cos^2 x + \sin^2 x) + (\cos^2 y + \sin^2 y) - 2[\cos x \cos y + \sin x \sin y]$

$= 1 + 1 - 2[\cos(x - y)]$ $\quad [\cos(A - B) = \cos A \cos B + \sin A \sin B]$

$= 2[1 - \cos(x - y)]$

$= 2\left[1 - \left\{1 - 2\sin^2\left(\frac{x - y}{2}\right)\right\}\right]$ $\quad [\cos 2A = 1 - 2\sin^2 A]$

$= 4\sin^2\left(\frac{x - y}{2}\right) = \text{R.H.S.}$

Question 5:

Prove that: $\sin x + \sin 3x + \sin 5x + \sin 7x = 4\cos x \cos 2x \sin 4x$

Answer

It is known that $\sin A + \sin B = 2\sin\left(\frac{A + B}{2}\right) \cdot \cos\left(\frac{A - B}{2}\right)$.

∴L.H.S. = $\sin x + \sin 3x + \sin 5x + \sin 7x$

$= (\sin x + \sin 5x) + (\sin 3x + \sin 7x)$

$= 2\sin\left(\frac{x + 5x}{2}\right) \cdot \cos\left(\frac{x - 5x}{2}\right) + 2\sin\left(\frac{3x + 7x}{2}\right)\cos\left(\frac{3x - 7x}{2}\right)$

$= 2\sin 3x \cos(-2x) + 2\sin 5x \ \cos(-2x)$

$= 2\sin 3x \cos 2x + 2\sin 5x \cos 2x$

$= 2\cos 2x[\sin 3x + \sin 5x]$

$= 2\cos 2x\left[2\sin\left(\frac{3x + 5x}{2}\right) \cdot \cos\left(\frac{3x - 5x}{2}\right)\right]$

$= 2\cos 2x[2\sin 4x \cdot \cos(-x)]$

$= 4\cos 2x \sin 4x \cos x = \text{R.H.S.}$

Question 6:

Prove that: $\frac{(\sin 7x + \sin 5x) + (\sin 9x + \sin 3x)}{(\cos 7x + \cos 5x) + (\cos 9x + \cos 3x)} = \tan 6x$

Answer

It is known that

$$\sin A+\sin B=2\sin\left(\frac{A+B}{2}\right)\cdot\cos\left(\frac{A-B}{2}\right),\ \cos A+\cos B=2\cos\left(\frac{A+B}{2}\right)\cdot\cos\left(\frac{A-B}{2}\right).$$

L.H.S. = $\dfrac{(\sin 7x+\sin 5x)+(\sin 9x+\sin 3x)}{(\cos 7x+\cos 5x)+(\cos 9x+\cos 3x)}$

$$=\frac{\left[2\sin\left(\frac{7x+5x}{2}\right)\cdot\cos\left(\frac{7x-5x}{2}\right)\right]+\left[2\sin\left(\frac{9x+3x}{2}\right)\cdot\cos\left(\frac{9x-3x}{2}\right)\right]}{\left[2\cos\left(\frac{7x+5x}{2}\right)\cdot\cos\left(\frac{7x-5x}{2}\right)\right]+\left[2\cos\left(\frac{9x+3x}{2}\right)\cdot\cos\left(\frac{9x-3x}{2}\right)\right]}$$

$$=\frac{[2\sin 6x\cdot\cos x]+[2\sin 6x\cdot\cos 3x]}{[2\cos 6x\cdot\cos x]+[2\cos 6x\cdot\cos 3x]}$$

$$=\frac{2\sin 6x[\cos x+\cos 3x]}{2\cos 6x[\cos x+\cos 3x]}$$

= tan 6*x*

= R.H.S.

Question 7:

Prove that: $\sin 3x+\sin 2x-\sin x=4\sin x\cos\frac{x}{2}\cos\frac{3x}{2}$

Answer

L.H.S. = $\sin 3x+\sin 2x-\sin x$

$= \sin 3x + (\sin 2x - \sin x)$

$= \sin 3x + \left[2\cos\left(\frac{2x+x}{2}\right)\sin\left(\frac{2x-x}{2}\right)\right]$ $\quad \left[\sin A - \sin B = 2\cos\left(\frac{A+B}{2}\right)\sin\left(\frac{A-B}{2}\right)\right]$

$= \sin 3x + \left[2\cos\left(\frac{3x}{2}\right)\sin\left(\frac{x}{2}\right)\right]$

$= \sin 3x + 2\cos\frac{3x}{2}\sin\frac{x}{2}$

$= 2\sin\frac{3x}{2}\cdot\cos\frac{3x}{2} + 2\cos\frac{3x}{2}\sin\frac{x}{2}$ $\quad [\sin 2A = 2\sin A\cdot\cos B]$

$= 2\cos\left(\frac{3x}{2}\right)\left[\sin\left(\frac{3x}{2}\right) + \sin\left(\frac{x}{2}\right)\right]$

$= 2\cos\left(\frac{3x}{2}\right)\left[2\sin\left\{\frac{\left(\frac{3x}{2}\right)+\left(\frac{x}{2}\right)}{2}\right\}\cos\left\{\frac{\left(\frac{3x}{2}\right)-\left(\frac{x}{2}\right)}{2}\right\}\right]$ $\quad \left[\sin A + \sin B = 2\sin\left(\frac{A+B}{2}\right)\cos\left(\frac{A-B}{2}\right)\right]$

$= 2\cos\left(\frac{3x}{2}\right).2\sin x\cos\left(\frac{x}{2}\right)$

$= 4\sin x\cos\left(\frac{x}{2}\right)\cos\left(\frac{3x}{2}\right) = \text{R.H.S.}$

Question 8:

$\tan x = -\frac{4}{3}$, x in quadrant II

Answer

Here, x is in quadrant II.

i.e., $\frac{\pi}{2} < x < \pi$

$\Rightarrow \frac{\pi}{4} < \frac{x}{2} < \frac{\pi}{2}$

Therefore, $\sin\frac{x}{2},\ \cos\frac{x}{2}$ and $\tan\frac{x}{2}$ are all positive.

It is given that $\tan x = -\frac{4}{3}$.

$$\sec^2 x = 1 + \tan^2 x = 1 + \left(\frac{-4}{3}\right)^2 = 1 + \frac{16}{9} = \frac{25}{9}$$

$$\therefore \cos^2 x = \frac{9}{25}$$

$$\Rightarrow \cos x = \pm\frac{3}{5}$$

As x is in quadrant II, cos$x$ is negative.

$$\cos x = \frac{-3}{5}$$

$$\text{Now, } \cos x = 2\cos^2\frac{x}{2} - 1$$

$$\Rightarrow \frac{-3}{5} = 2\cos^2\frac{x}{2} - 1$$

$$\Rightarrow 2\cos^2\frac{x}{2} = 1 - \frac{3}{5}$$

$$\Rightarrow 2\cos^2\frac{x}{2} = \frac{2}{5}$$

$$\Rightarrow \cos^2\frac{x}{2} = \frac{1}{5}$$

$$\Rightarrow \cos\frac{x}{2} = \frac{1}{\sqrt{5}} \qquad \left[\because \cos\frac{x}{2} \text{ is positive}\right]$$

$$\therefore \cos\frac{x}{2} = \frac{\sqrt{5}}{5}$$

$$\sin^2\frac{x}{2} + \cos^2\frac{x}{2} = 1$$

$$\Rightarrow \sin^2\frac{x}{2} + \left(\frac{1}{\sqrt{5}}\right)^2 = 1$$

$$\Rightarrow \sin^2\frac{x}{2} = 1 - \frac{1}{5} = \frac{4}{5}$$

$\Rightarrow \sin\frac{x}{2} = \frac{2}{\sqrt{5}}$ $\left[\because \sin\frac{x}{2} \text{ is positive}\right]$

$\therefore \sin\frac{x}{2} = \frac{2\sqrt{5}}{5}$

$$\tan\frac{x}{2} = \frac{\sin\frac{x}{2}}{\cos\frac{x}{2}} = \frac{\left(\frac{2}{\sqrt{5}}\right)}{\left(\frac{1}{\sqrt{5}}\right)} = 2$$

Thus, the respective values of $\sin\frac{x}{2}, \cos\frac{x}{2}$ and $\tan\frac{x}{2}$ are $\frac{2\sqrt{5}}{5}, \frac{\sqrt{5}}{5},$ and 2.

Question 9:

Find $\sin\frac{x}{2}, \cos\frac{x}{2}$ and $\tan\frac{x}{2}$ for $\cos x = -\frac{1}{3}$, *x* in quadrant III

Answer

Here, *x* is in quadrant III.

$\text{i.e., } \pi < x < \frac{3\pi}{2}$

$\Rightarrow \frac{\pi}{2} < \frac{x}{2} < \frac{3\pi}{4}$

Therefore, $\cos\frac{x}{2}$ and $\tan\frac{x}{2}$ are negative, whereas $\sin\frac{x}{2}$ is positive.

It is given that $\cos x = -\frac{1}{3}$.

$\cos x = 1 - 2\sin^2\frac{x}{2}$

$\Rightarrow \sin^2\frac{x}{2} = \frac{1-\cos x}{2}$

$$\Rightarrow \sin^2\frac{x}{2} = \frac{1-\left(-\frac{1}{3}\right)}{2} = \frac{\left(1+\frac{1}{3}\right)}{2} = \frac{\frac{4}{3}}{2} = \frac{2}{3}$$

$$\Rightarrow \sin\frac{x}{2} = \frac{\sqrt{2}}{\sqrt{3}} \qquad \left[\because \ \sin\frac{x}{2} \text{ is positive}\right]$$

$$\therefore \sin\frac{x}{2} = \frac{\sqrt{2}}{\sqrt{3}} \times \frac{\sqrt{3}}{\sqrt{3}} = \frac{\sqrt{6}}{3}$$

Now, $\cos x = 2\cos^2\frac{x}{2} - 1$

$$\Rightarrow \cos^2\frac{x}{2} = \frac{1+\cos x}{2} = \frac{1+\left(-\frac{1}{3}\right)}{2} = \frac{\left(\frac{3-1}{3}\right)}{2} = \frac{\left(\frac{2}{3}\right)}{2} = \frac{1}{3}$$

$$\Rightarrow \cos\frac{x}{2} = -\frac{1}{\sqrt{3}} \qquad \left[\because \ \cos\frac{x}{2} \text{ is negative}\right]$$

$$\therefore \cos\frac{x}{2} = -\frac{1}{\sqrt{3}} \times \frac{\sqrt{3}}{\sqrt{3}} = \frac{-\sqrt{3}}{3}$$

$$\tan\frac{x}{2} = \frac{\sin\frac{x}{2}}{\cos\frac{x}{2}} = \frac{\left(\frac{\sqrt{2}}{\sqrt{3}}\right)}{\left(\frac{-1}{\sqrt{3}}\right)} = -\sqrt{2}$$

Thus, the respective values of $\sin\frac{x}{2},\ \cos\frac{x}{2}$ and $\tan\frac{x}{2}$ are $\frac{\sqrt{6}}{3},\ \frac{-\sqrt{3}}{3}$, and $-\sqrt{2}$.

Question 10:

Find $\sin\frac{x}{2},\ \cos\frac{x}{2}$ and $\tan\frac{x}{2}$ for $\sin x = \frac{1}{4}$, *x* in quadrant II

Answer

Here, *x* is in quadrant II.

i.e., $\frac{\pi}{2} < x < \pi$

$$\Rightarrow \frac{\pi}{4} < \frac{x}{2} < \frac{\pi}{2}$$

Therefore, $\sin\frac{x}{2}, \cos\frac{x}{2}$, and $\tan\frac{x}{2}$ are all positive.

It is given that $\sin x = \frac{1}{4}$.

$$\cos^2 x = 1 - \sin^2 x = 1 - \left(\frac{1}{4}\right)^2 = 1 - \frac{1}{16} = \frac{15}{16}$$

$\Rightarrow \cos x = -\frac{\sqrt{15}}{4}$ [cos*x* is negative in quadrant II]

$$\sin^2 \frac{x}{2} = \frac{1 - \cos x}{2} = \frac{1 - \left(-\frac{\sqrt{15}}{4}\right)}{2} = \frac{4 + \sqrt{15}}{8}$$

$$\Rightarrow \sin \frac{x}{2} = \sqrt{\frac{4 + \sqrt{15}}{8}} \qquad \left[\because \sin \frac{x}{2} \text{ is positive}\right]$$

$$= \sqrt{\frac{4 + \sqrt{15}}{8} \times \frac{2}{2}}$$

$$= \sqrt{\frac{8 + 2\sqrt{15}}{16}}$$

$$= \frac{\sqrt{8 + 2\sqrt{15}}}{4}$$

$$\cos^2 \frac{x}{2} = \frac{1 + \cos x}{2} = \frac{1 + \left(-\frac{\sqrt{15}}{4}\right)}{2} = \frac{4 - \sqrt{15}}{8}$$

$$\Rightarrow \cos \frac{x}{2} = \sqrt{\frac{4 - \sqrt{15}}{8}} \qquad \left[\because \cos \frac{x}{2} \text{ is positive}\right]$$

$$= \sqrt{\frac{4 - \sqrt{15}}{8} \times \frac{2}{2}}$$

$$= \sqrt{\frac{8 - 2\sqrt{15}}{16}}$$

$$= \frac{\sqrt{8 - 2\sqrt{15}}}{4}$$

$$\tan\frac{x}{2} = \frac{\sin\frac{x}{2}}{\cos\frac{x}{2}} = \frac{\left(\frac{\sqrt{8+2\sqrt{15}}}{4}\right)}{\left(\frac{\sqrt{8-2\sqrt{15}}}{4}\right)} = \frac{\sqrt{8+2\sqrt{15}}}{\sqrt{8-2\sqrt{15}}}$$

$$= \sqrt{\frac{8+2\sqrt{15}}{8-2\sqrt{15}} \times \frac{8+2\sqrt{15}}{8+2\sqrt{15}}}$$

$$= \sqrt{\frac{\left(8+2\sqrt{15}\right)^2}{64-60}} = \frac{8+2\sqrt{15}}{2} = 4+\sqrt{15}$$

Thus, the respective values of $\sin\frac{x}{2},\ \cos\frac{x}{2}$ and $\tan\frac{x}{2}$ are $\frac{\sqrt{8+2\sqrt{15}}}{4},\ \frac{\sqrt{8-2\sqrt{15}}}{4},$. and $4+\sqrt{15}$