Miscellaneous Solutions

Question 1:

Prove that the determinant $\begin{vmatrix} x & \sin\theta & \cos\theta \\ -\sin\theta & -x & 1 \\ \cos\theta & 1 & x \end{vmatrix}$ is independent of θ.

Answer

$$\Delta = \begin{vmatrix} x & \sin\theta & \cos\theta \\ -\sin\theta & -x & 1 \\ \cos\theta & 1 & x \end{vmatrix}$$

$$= x\left(x^2 - 1\right) - \sin\theta\left(-x\sin\theta - \cos\theta\right) + \cos\theta\left(-\sin\theta + x\cos\theta\right)$$

$$= x^3 - x + x\sin^2\theta + \sin\theta\cos\theta - \sin\theta\cos\theta + x\cos^2\theta$$

$$= x^3 - x + x\left(\sin^2\theta + \cos^2\theta\right)$$

$$= x^3 - x + x$$

$$= x^3 \text{ (Independent of } \theta\text{)}$$

Hence, Δ is independent of θ.

Question 2:

Without expanding the determinant, prove that

$$\begin{vmatrix} a & a^2 & bc \\ b & b^2 & ca \\ c & c^2 & ab \end{vmatrix} = \begin{vmatrix} 1 & a^2 & a^3 \\ 1 & b^2 & b^3 \\ 1 & c^2 & c^3 \end{vmatrix}$$

Answer

$$\text{L.H.S.} = \begin{vmatrix} a & a^2 & bc \\ b & b^2 & ca \\ c & c^2 & ab \end{vmatrix}$$

$$= \frac{1}{abc}\begin{vmatrix} a^2 & a^3 & abc \\ b^2 & b^3 & abc \\ c^2 & c^3 & abc \end{vmatrix} \qquad \left[R_1 \to aR_1, R_2 \to bR_2, \text{and } R_3 \to cR_3\right]$$

$$= \frac{1}{abc}\cdot abc\begin{vmatrix} a^2 & a^3 & 1 \\ b^2 & b^3 & 1 \\ c^2 & c^3 & 1 \end{vmatrix} \qquad \left[\text{Taking out factor } abc \text{ from } C_3\right]$$

$$= \begin{vmatrix} a^2 & a^3 & 1 \\ b^2 & b^3 & 1 \\ c^2 & c^3 & 1 \end{vmatrix}$$

$$= \begin{vmatrix} 1 & a^2 & a^3 \\ 1 & b^2 & b^3 \\ 1 & c^2 & c^3 \end{vmatrix} \qquad \left[\text{Applying } C_1 \leftrightarrow C_3 \text{ and } C_2 \leftrightarrow C_3\right]$$

$$= \text{R.H.S.}$$

Hence, the given result is proved.

Question 3:

Evaluate $\begin{vmatrix} \cos\alpha\cos\beta & \cos\alpha\sin\beta & -\sin\alpha \\ -\sin\beta & \cos\beta & 0 \\ \sin\alpha\cos\beta & \sin\alpha\sin\beta & \cos\alpha \end{vmatrix}$

Answer

$$\Delta = \begin{vmatrix} \cos\alpha\cos\beta & \cos\alpha\sin\beta & -\sin\alpha \\ -\sin\beta & \cos\beta & 0 \\ \sin\alpha\cos\beta & \sin\alpha\sin\beta & \cos\alpha \end{vmatrix}$$

Expanding along C_3, we have:

$$\Delta = -\sin\alpha\left(-\sin\alpha\sin^2\beta - \cos^2\beta\sin\alpha\right) + \cos\alpha\left(\cos\alpha\cos^2\beta + \cos\alpha\sin^2\beta\right)$$
$$= \sin^2\alpha\left(\sin^2\beta + \cos^2\beta\right) + \cos^2\alpha\left(\cos^2\beta + \sin^2\beta\right)$$
$$= \sin^2\alpha(1) + \cos^2\alpha(1)$$
$$= 1$$

Question 4:

If a, b and c are real numbers, and $\Delta = \begin{vmatrix} b+c & c+a & a+b \\ c+a & a+b & b+c \\ a+b & b+c & c+a \end{vmatrix} = 0$,

Show that either $a + b + c = 0$ or $a = b = c$.

Answer

$$\Delta = \begin{vmatrix} b+c & c+a & a+b \\ c+a & a+b & b+c \\ a+b & b+c & c+a \end{vmatrix}$$

Applying $R_1 \to R_1 + R_2 + R_3$, we have:

$$\Delta = \begin{vmatrix} 2(a+b+c) & 2(a+b+c) & 2(a+b+c) \\ c+a & a+b & b+c \\ a+b & b+c & c+a \end{vmatrix}$$

$$= 2(a+b+c)\begin{vmatrix} 1 & 1 & 1 \\ c+a & a+b & b+c \\ a+b & b+c & c+a \end{vmatrix}$$

Applying $C_2 \to C_2 - C_1$ and $C_3 \to C_3 - C_1$, we have:

$$\Delta = 2(a+b+c)\begin{vmatrix} 1 & 0 & 0 \\ c+a & b-c & b-a \\ a+b & c-a & c-b \end{vmatrix}$$

Expanding along R_1, we have:

$$\Delta = 2(a+b+c)(1)\left[(b-c)(c-b)-(b-a)(c-a)\right]$$
$$= 2(a+b+c)\left[-b^2-c^2+2bc-bc+ba+ac-a^2\right]$$
$$= 2(a+b+c)\left[ab+bc+ca-a^2-b^2-c^2\right]$$

It is given that $\Delta = 0$.

$$(a+b+c)\left[ab+bc+ca-a^2-b^2-c^2\right]=0$$

$\Rightarrow$ Either $a+b+c=0,$ or $ab+bc+ca-a^2-b^2-c^2=0.$

Now,

$$ab+bc+ca-a^2-b^2-c^2=0$$
$$\Rightarrow -2ab-2bc-2ca+2a^2+2b^2+2c^2=0$$
$$\Rightarrow (a-b)^2+(b-c)^2+(c-a)^2=0$$
$$\Rightarrow (a-b)^2=(b-c)^2=(c-a)^2=0 \qquad \left[(a-b)^2,(b-c)^2,(c-a)^2 \text{ are non-negative}\right]$$
$$\Rightarrow (a-b)=(b-c)=(c-a)=0$$
$$\Rightarrow a=b=c$$

Hence, if $\Delta = 0$, then either $a + b + c = 0$ or $a = b = c$.

Question 5:

Solve the equations $\begin{vmatrix} x+a & x & x \\ x & x+a & x \\ x & x & x+a \end{vmatrix}=0, a\neq 0$

Answer

$$\begin{vmatrix} x+a & x & x \\ x & x+a & x \\ x & x & x+a \end{vmatrix} = 0$$

Applying $R_1 \to R_1 + R_2 + R_3$, we get:

$$\begin{vmatrix} 3x+a & 3x+a & 3x+a \\ x & x+a & x \\ x & x & x+a \end{vmatrix} = 0$$

$$\Rightarrow (3x+a)\begin{vmatrix} 1 & 1 & 1 \\ x & x+a & x \\ x & x & x+a \end{vmatrix} = 0$$

Applying $C_2 \to C_2 - C_1$ and $C_3 \to C_3 - C_1$, we have:

$$(3x+a)\begin{vmatrix} 1 & 0 & 0 \\ x & a & 0 \\ x & 0 & a \end{vmatrix} = 0$$

Expanding along R_1, we have:

$$(3x+a)\left[1 \times a^2\right] = 0$$

$$\Rightarrow a^2(3x+a) = 0$$

But $a \neq 0$.

Therefore, we have:

$$3x + a = 0$$

$$\Rightarrow x = -\frac{a}{3}$$

Question 6:

Prove that $\begin{vmatrix} a^2 & bc & ac+c^2 \\ a^2+ab & b^2 & ac \\ ab & b^2+bc & c^2 \end{vmatrix} = 4a^2b^2c^2$

Answer

$$\Delta = \begin{vmatrix} a^2 & bc & ac+c^2 \\ a^2+ab & b^2 & ac \\ ab & b^2+bc & c^2 \end{vmatrix}$$

Taking out common factors a, b, and c from C_1, C_2, and C_3, we have:

$$\Delta = abc\begin{vmatrix} a & c & a+c \\ a+b & b & a \\ b & b+c & c \end{vmatrix}$$

Applying $R_2 \to R_2 - R_1$ and $R_3 \to R_3 - R_1$, we have:

$$\Delta = abc\begin{vmatrix} a & c & a+c \\ b & b-c & -c \\ b-a & b & -a \end{vmatrix}$$

Applying $R_2 \to R_2 + R_1$, we have:

$$\Delta = abc\begin{vmatrix} a & c & a+c \\ a+b & b & a \\ b-a & b & -a \end{vmatrix}$$

Applying $R_3 \to R_3 + R_2$, we have:

$$\Delta = abc\begin{vmatrix} a & c & a+c \\ a+b & b & a \\ 2b & 2b & 0 \end{vmatrix}$$

$$= 2ab^2c\begin{vmatrix} a & c & a+c \\ a+b & b & a \\ 1 & 1 & 0 \end{vmatrix}$$

Applying $C_2 \to C_2 - C_1$, we have:

$$\Delta = 2ab^2c\begin{vmatrix} a & c-a & a+c \\ a+b & -a & a \\ 1 & 0 & 0 \end{vmatrix}$$

Expanding along R_3, we have:

$$\Delta = 2ab^2c\left[a(c-a)+a(a+c)\right]$$

$$= 2ab^2c\left[ac-a^2+a^2+ac\right]$$

$$= 2ab^2c(2ac)$$

$$= 4a^2b^2c^2$$

Hence, the given result is proved.

Question 8:

Let $A=\begin{bmatrix}1 & -2 & 1\\ -2 & 3 & 1\\ 1 & 1 & 5\end{bmatrix}$ verify that

(i) $\left[adjA\right]^{-1}=adj\left(A^{-1}\right)$

(ii) $\left(A^{-1}\right)^{-1}=A$

Answer

$$A=\begin{bmatrix}1 & -2 & 1\\ -2 & 3 & 1\\ 1 & 1 & 5\end{bmatrix}$$

$$\therefore |A|=1(15-1)+2(-10-1)+1(-2-3)=14-22-5=-13$$

$$\begin{aligned}\text{Now, } & A_{11}=14, A_{12}=11, A_{13}=-5\\ & A_{21}=11, A_{22}=4, A_{23}=-3\\ & A_{31}=-5, A_{32}=-3, A_{13}=-1\end{aligned}$$

$$\therefore adjA=\begin{bmatrix}14 & 11 & -5\\ 11 & 4 & -3\\ -5 & -3 & -1\end{bmatrix}$$

$$\therefore\ A^{-1}=\frac{1}{|A|}(adjA)$$

$$=-\frac{1}{13}\begin{bmatrix}14 & 11 & -5\\ 11 & 4 & -3\\ -5 & -3 & -1\end{bmatrix}=\frac{1}{13}\begin{bmatrix}-14 & -11 & 5\\ -11 & -4 & 3\\ 5 & 3 & 1\end{bmatrix}$$

(i)

$$\begin{aligned}|adjA| &= 14(-4-9)-11(-11-15)-5(-33+20)\\ &= 14(-13)-11(-26)-5(-13)\\ &= -182+286+65=169\end{aligned}$$

We have,

$$adj\left(adjA\right)=\begin{bmatrix}-13 & 26 & -13\\ 26 & -39 & -13\\ -13 & -13 & -65\end{bmatrix}$$

$$\therefore \left[adjA\right]^{-1}=\frac{1}{\left|adjA\right|}\left(adj\left(adjA\right)\right)$$

$$=\frac{1}{169}\begin{bmatrix}-13 & 26 & -13\\ 26 & -39 & -13\\ -13 & -13 & -65\end{bmatrix}$$

$$=\frac{1}{13}\begin{bmatrix}-1 & 2 & -1\\ 2 & -3 & -1\\ -1 & -1 & -5\end{bmatrix}$$

$$\text{Now, } A^{-1}=\frac{1}{13}\begin{bmatrix}-14 & -11 & 5\\ -11 & -4 & 3\\ 5 & 3 & 1\end{bmatrix}=\begin{bmatrix}-\frac{14}{13} & -\frac{11}{13} & \frac{5}{13}\\ -\frac{11}{13} & -\frac{4}{13} & \frac{3}{13}\\ \frac{5}{13} & \frac{3}{13} & \frac{1}{13}\end{bmatrix}$$

$$\therefore adj\left(A^{-1}\right)=\begin{bmatrix}-\frac{4}{169}-\frac{9}{169} & -\left(-\frac{11}{169}-\frac{15}{169}\right) & -\frac{33}{169}+\frac{20}{169}\\ -\left(-\frac{11}{169}-\frac{15}{169}\right) & -\frac{14}{169}-\frac{25}{169} & -\left(-\frac{42}{169}+\frac{55}{169}\right)\\ -\frac{33}{169}+\frac{20}{169} & -\left(-\frac{42}{169}+\frac{55}{169}\right) & \frac{56}{169}-\frac{121}{169}\end{bmatrix}$$

$$=\frac{1}{169}\begin{bmatrix}-13 & 26 & -13\\ 26 & -39 & -13\\ -13 & -13 & -65\end{bmatrix}=\frac{1}{13}\begin{bmatrix}-1 & 2 & -1\\ 2 & -3 & -1\\ -1 & -1 & -5\end{bmatrix}$$

Hence, $\left[adjA\right]^{-1}=adj\left(A^{-1}\right)$.

(ii)

We have shown that:

$$A^{-1}=\frac{1}{13}\begin{bmatrix}-14 & -11 & 5\\ -11 & -4 & 3\\ 5 & 3 & 1\end{bmatrix}$$

And, $adjA^{-1}=\frac{1}{13}\begin{bmatrix}-1 & 2 & -1\\ 2 & -3 & -1\\ -1 & -1 & -5\end{bmatrix}$

Now,

$$\left|A^{-1}\right|=\left(\frac{1}{13}\right)^3\left[-14\times(-13)+11\times(-26)+5\times(-13)\right]=\left(\frac{1}{13}\right)^3\times(-169)=-\frac{1}{13}$$

$$\therefore \left(A^{-1}\right)^{-1}=\frac{adjA^{-1}}{\left|A^{-1}\right|}=\frac{1}{\left(-\frac{1}{13}\right)}\times\frac{1}{13}\begin{bmatrix}-1 & 2 & -1\\ 2 & -3 & -1\\ -1 & -1 & -5\end{bmatrix}=\begin{bmatrix}1 & -2 & 1\\ -2 & 3 & 1\\ 1 & 1 & 5\end{bmatrix}=A$$

$$\therefore \left(A^{-1}\right)^{-1}=A$$

Question 9:

Evaluate $\begin{vmatrix}x & y & x+y\\ y & x+y & x\\ x+y & x & y\end{vmatrix}$

Answer

$$\Delta=\begin{vmatrix} x & y & x+y \\ y & x+y & x \\ x+y & x & y \end{vmatrix}$$

Applying $R_1 \to R_1+R_2+R_3$, we have:

$$\Delta=\begin{vmatrix} 2(x+y) & 2(x+y) & 2(x+y) \\ y & x+y & x \\ x+y & x & y \end{vmatrix}$$

$$=2(x+y)\begin{vmatrix} 1 & 1 & 1 \\ y & x+y & x \\ x+y & x & y \end{vmatrix}$$

Applying $C_2 \to C_2-C_1$ and $C_3 \to C_3-C_1$, we have:

$$\Delta=2(x+y)\begin{vmatrix} 1 & 0 & 0 \\ y & x & x-y \\ x+y & -y & -x \end{vmatrix}$$

Expanding along R_1, we have:

$$\Delta=2(x+y)\left[-x^2+y(x-y)\right]$$

$$=-2(x+y)\left(x^2+y^2-yx\right)$$

$$=-2\left(x^3+y^3\right)$$

Question 10:

Evaluate $\begin{vmatrix} 1 & x & y \\ 1 & x+y & y \\ 1 & x & x+y \end{vmatrix}$

Answer

$$\Delta=\begin{vmatrix}1 & x & y\\ 1 & x+y & y\\ 1 & x & x+y\end{vmatrix}$$

Applying $R_2 \to R_2 - R_1$ and $R_3 \to R_3 - R_1$, we have:

$$\Delta=\begin{vmatrix}1 & x & y\\ 0 & y & 0\\ 0 & 0 & x\end{vmatrix}$$

Expanding along C_1, we have:

$$\Delta = 1(xy-0) = xy$$

Question 11:

Using properties of determinants, prove that:

$$\begin{vmatrix}\alpha & \alpha^2 & \beta+\gamma\\ \beta & \beta^2 & \gamma+\alpha\\ \gamma & \gamma^2 & \alpha+\beta\end{vmatrix}=(\beta-\gamma)(\gamma-\alpha)(\alpha-\beta)(\alpha+\beta+\gamma)$$

Answer

$$\Delta=\begin{vmatrix}\alpha & \alpha^2 & \beta+\gamma\\ \beta & \beta^2 & \gamma+\alpha\\ \gamma & \gamma^2 & \alpha+\beta\end{vmatrix}$$

Applying $R_2 \to R_2 - R_1$ and $R_3 \to R_3 - R_1$, we have:

$$\Delta=\begin{vmatrix}\alpha & \alpha^2 & \beta+\gamma\\ \beta-\alpha & \beta^2-\alpha^2 & \alpha-\beta\\ \gamma-\alpha & \gamma^2-\alpha^2 & \alpha-\gamma\end{vmatrix}$$

$$=(\beta-\alpha)(\gamma-\alpha)\begin{vmatrix}\alpha & \alpha^2 & \beta+\gamma\\ 1 & \beta+\alpha & -1\\ 1 & \gamma+\alpha & -1\end{vmatrix}$$

Applying $R_3 \to R_3 - R_2$, we have:

$$\Delta = (\beta-\alpha)(\gamma-\alpha)\begin{vmatrix} \alpha & \alpha^2 & \beta+\gamma \\ 1 & \beta+\alpha & -1 \\ 0 & \gamma-\beta & 0 \end{vmatrix}$$

Expanding along R_3, we have:

$$\Delta = (\beta-\alpha)(\gamma-\alpha)\left[-(\gamma-\beta)(-\alpha-\beta-\gamma)\right]$$
$$= (\beta-\alpha)(\gamma-\alpha)(\gamma-\beta)(\alpha+\beta+\gamma)$$
$$= (\alpha-\beta)(\beta-\gamma)(\gamma-\alpha)(\alpha+\beta+\gamma)$$

Hence, the given result is proved.

Question 12:

Using properties of determinants, prove that:

$$\begin{vmatrix} x & x^2 & 1+px^3 \\ y & y^2 & 1+py^3 \\ z & z^2 & 1+pz^3 \end{vmatrix} = (1+pxyz)(x-y)(y-z)(z-x)$$

Answer

$$\Delta = \begin{vmatrix} x & x^2 & 1+px^3 \\ y & y^2 & 1+py^3 \\ z & z^2 & 1+pz^3 \end{vmatrix}$$

Applying $R_2 \to R_2 - R_1$ and $R_3 \to R_3 - R_1$, we have:

$$\Delta = \begin{vmatrix} x & x^2 & 1+px^3 \\ y-x & y^2-x^2 & p(y^3-x^3) \\ z-x & z^2-x^2 & p(z^3-x^3) \end{vmatrix}$$

$$= (y-x)(z-x)\begin{vmatrix} x & x^2 & 1+px^3 \\ 1 & y+x & p(y^2+x^2+xy) \\ 1 & z+x & p(z^2+x^2+xz) \end{vmatrix}$$

Applying $R_3 \to R_3 - R_2$, we have:

$$\Delta=(y-x)(z-x)\begin{vmatrix} x & x^2 & 1+px^3 \\ 1 & y+x & p(y^2+x^2+xy) \\ 0 & z-y & p(z-y)(x+y+z) \end{vmatrix}$$

$$=(y-x)(z-x)(z-y)\begin{vmatrix} x & x^2 & 1+px^3 \\ 1 & y+x & p(y^2+x^2+xy) \\ 0 & 1 & p(x+y+z) \end{vmatrix}$$

Expanding along R_3, we have:

$$\Delta=(x-y)(y-z)(z-x)\left[(-1)(p)(xy^2+x^3+x^2y)+1+px^3+p(x+y+z)(xy)\right]$$

$$=(x-y)(y-z)(z-x)\left[-pxy^2-px^3-px^2y+1+px^3+px^2y+pxy^2+pxyz\right]$$

$$=(x-y)(y-z)(z-x)(1+pxyz)$$

Hence, the given result is proved.

Question 13:

Using properties of determinants, prove that:

$$\begin{vmatrix} 3a & -a+b & -a+c \\ -b+a & 3b & -b+c \\ -c+a & -c+b & 3c \end{vmatrix}=3(a+b+c)(ab+bc+ca)$$

Answer

$$\Delta=\begin{vmatrix} 3a & -a+b & -a+c \\ -b+a & 3b & -b+c \\ -c+a & -c+b & 3c \end{vmatrix}$$

Applying $C_1 \to C_1+C_2+C_3$, we have:

$$\Delta=\begin{vmatrix} a+b+c & -a+b & -a+c \\ a+b+c & 3b & -b+c \\ a+b+c & -c+b & 3c \end{vmatrix}$$

$$=(a+b+c)\begin{vmatrix}1 & -a+b & -a+c\\ 1 & 3b & -b+c\\ 1 & -c+b & 3c\end{vmatrix}$$

Applying $R_2 \to R_2 - R_1$ and $R_3 \to R_3 - R_1$, we have:

$$\Delta=(a+b+c)\begin{vmatrix}1 & -a+b & -a+c\\ 0 & 2b+a & a-b\\ 0 & a-c & 2c+a\end{vmatrix}$$

Expanding along C_1, we have:

$$\Delta=(a+b+c)\left[(2b+a)(2c+a)-(a-b)(a-c)\right]$$
$$=(a+b+c)\left[4bc+2ab+2ac+a^2-a^2+ac+ba-bc\right]$$
$$=(a+b+c)(3ab+3bc+3ac)$$
$$=3(a+b+c)(ab+bc+ca)$$

Hence, the given result is proved.

Question 14:

Using properties of determinants, prove that:

$$\begin{vmatrix}1 & 1+p & 1+p+q\\ 2 & 3+2p & 4+3p+2q\\ 3 & 6+3p & 10+6p+3q\end{vmatrix}=1$$

Answer

$$\Delta = \begin{vmatrix} 1 & 1+p & 1+p+q \\ 2 & 3+2p & 4+3p+2q \\ 3 & 6+3p & 10+6p+3q \end{vmatrix}$$

Applying $R_2 \to R_2 - 2R_1$ and $R_3 \to R_3 - 3R_1$, we have:

$$\Delta = \begin{vmatrix} 1 & 1+p & 1+p+q \\ 0 & 1 & 2+p \\ 0 & 3 & 7+3p \end{vmatrix}$$

Applying $R_3 \to R_3 - 3R_2$, we have:

$$\Delta = \begin{vmatrix} 1 & 1+p & 1+p+q \\ 0 & 1 & 2+p \\ 0 & 0 & 1 \end{vmatrix}$$

Expanding along C_1, we have:

$$\Delta = 1\begin{vmatrix} 1 & 2+p \\ 0 & 1 \end{vmatrix} = 1(1-0) = 1$$

Hence, the given result is proved.

Question 15:

Using properties of determinants, prove that:

$$\begin{vmatrix} \sin\alpha & \cos\alpha & \cos(\alpha+\delta) \\ \sin\beta & \cos\beta & \cos(\beta+\delta) \\ \sin\gamma & \cos\gamma & \cos(\gamma+\delta) \end{vmatrix} = 0$$

Answer

$$\Delta = \begin{vmatrix} \sin\alpha & \cos\alpha & \cos(\alpha+\delta) \\ \sin\beta & \cos\beta & \cos(\beta+\delta) \\ \sin\gamma & \cos\gamma & \cos(\gamma+\delta) \end{vmatrix}$$

$$= \frac{1}{\sin\delta\cos\delta}\begin{vmatrix} \sin\alpha\sin\delta & \cos\alpha\cos\delta & \cos\alpha\cos\delta - \sin\alpha\sin\delta \\ \sin\beta\sin\delta & \cos\beta\cos\delta & \cos\beta\cos\delta - \sin\beta\sin\delta \\ \sin\gamma\sin\delta & \cos\gamma\cos\delta & \cos\gamma\cos\delta - \sin\gamma\sin\delta \end{vmatrix}$$

Applying $C_1 \to C_1 + C_3$, we have:

$$\Delta = \frac{1}{\sin\delta\cos\delta}\begin{vmatrix} \cos\alpha\cos\delta & \cos\alpha\cos\delta & \cos\alpha\cos\delta - \sin\alpha\sin\delta \\ \cos\beta\cos\delta & \cos\beta\cos\delta & \cos\beta\cos\delta - \sin\beta\sin\delta \\ \cos\gamma\cos\delta & \cos\gamma\cos\delta & \cos\gamma\cos\delta - \sin\gamma\sin\delta \end{vmatrix}$$

Here, two columns C_1 and C_2 are identical.

$\therefore \Delta = 0.$

Hence, the given result is proved.

Question 16:

Solve the system of the following equations

$$\frac{2}{x} + \frac{3}{y} + \frac{10}{z} = 4$$

$$\frac{4}{x} - \frac{6}{y} + \frac{5}{z} = 1$$

$$\frac{6}{x} + \frac{9}{y} - \frac{20}{z} = 2$$

Answer

Let $\frac{1}{x} = p, \frac{1}{y} = q, \frac{1}{z} = r.$

Then the given system of equations is as follows:

$$2p + 3q + 10r = 4$$

$$4p - 6q + 5r = 1$$

$$6p + 9q - 20r = 2$$

This system can be written in the form of $AX = B$, where

$$A=\begin{bmatrix}2 & 3 & 10\\ 4 & -6 & 5\\ 6 & 9 & -20\end{bmatrix}, X=\begin{bmatrix}p\\ q\\ r\end{bmatrix} \text{ and } B=\begin{bmatrix}4\\ 1\\ 2\end{bmatrix}.$$

Now,

$$\begin{aligned}|A| &= 2(120-45)-3(-80-30)+10(36+36)\\ &= 150+330+720\\ &= 1200\end{aligned}$$

A

Thus, A is non-singular. Therefore, its inverse exists.

Now,

A_{11} = 75, A_{12} = 110, A_{13} = 72

A_{21} = 150, A_{22} = −100, A_{23} = 0

A_{31} = 75, A_{32} = 30, A_{33} = − 24

$$\therefore A^{-1}=\frac{1}{|A|}adjA$$

$$=\frac{1}{1200}\begin{bmatrix}75 & 150 & 75\\ 110 & -100 & 30\\ 72 & 0 & -24\end{bmatrix}$$

Now,

$$X=A^{-1}B$$

$$\Rightarrow \begin{bmatrix}p\\ q\\ r\end{bmatrix}=\frac{1}{1200}\begin{bmatrix}75 & 150 & 75\\ 110 & -100 & 30\\ 72 & 0 & -24\end{bmatrix}\begin{bmatrix}4\\ 1\\ 2\end{bmatrix}$$

$$=\frac{1}{1200}\begin{bmatrix}300+150+150\\ 440-100+60\\ 288+0-48\end{bmatrix}$$

$$=\frac{1}{1200}\begin{bmatrix}600\\ 400\\ 240\end{bmatrix}=\begin{bmatrix}\frac{1}{2}\\ \frac{1}{3}\\ \frac{1}{5}\end{bmatrix}$$

$$\therefore p=\frac{1}{2}, q=\frac{1}{3}, \text{ and } r=\frac{1}{5}$$

Hence, $x=2, y=3, \text{and } z=5$.

Question 17:

Choose the correct answer.

If *a*, *b*, *c*, are in A.P., then the determinant

$$\begin{vmatrix} x+2 & x+3 & x+2a \\ x+3 & x+4 & x+2b \\ x+4 & x+5 & x+2c \end{vmatrix}$$

A. 0 **B.** 1 **C.** *x* **D.** 2*x*

Answer

Answer: A

$$\Delta = \begin{vmatrix} x+2 & x+3 & x+2a \\ x+3 & x+4 & x+2b \\ x+4 & x+5 & x+2c \end{vmatrix}$$

$$= \begin{vmatrix} x+2 & x+3 & x+2a \\ x+3 & x+4 & x+(a+c) \\ x+4 & x+5 & x+2c \end{vmatrix} \qquad (2b = a+c \text{ as } a, b, \text{and } c \text{ are in A.P.})$$

Applying $R_1 \to R_1 - R_2$ and $R_3 \to R_3 - R_2$, we have:

$$\Delta = \begin{vmatrix} -1 & -1 & a-c \\ x+3 & x+4 & x+(a+c) \\ 1 & 1 & c-a \end{vmatrix}$$

Applying $R_1 \to R_1 + R_3$, we have:

$$\Delta = \begin{vmatrix} 0 & 0 & 0 \\ x+3 & x+4 & x+a+c \\ 1 & 1 & c-a \end{vmatrix}$$

Here, all the elements of the first row (R_1) are zero.

Hence, we have $\Delta = 0$.

The correct answer is A.

Question 18:

Choose the correct answer.

If x, y, z are nonzero real numbers, then the inverse of matrix $A=\begin{bmatrix} x & 0 & 0 \\ 0 & y & 0 \\ 0 & 0 & z \end{bmatrix}$ is

A. $\begin{bmatrix} x^{-1} & 0 & 0 \\ 0 & y^{-1} & 0 \\ 0 & 0 & z^{-1} \end{bmatrix}$ **B.** $xyz\begin{bmatrix} x^{-1} & 0 & 0 \\ 0 & y^{-1} & 0 \\ 0 & 0 & z^{-1} \end{bmatrix}$

C. $\frac{1}{xyz}\begin{bmatrix} x & 0 & 0 \\ 0 & y & 0 \\ 0 & 0 & z \end{bmatrix}$ **D.** $\frac{1}{xyz}\begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}$

Answer

Answer: A

$$A=\begin{bmatrix} x & 0 & 0 \\ 0 & y & 0 \\ 0 & 0 & z \end{bmatrix}$$

$$\therefore |A| = x(yz-0) = xyz \neq 0$$

$$\begin{aligned} \text{Now, } & A_{11}=yz, A_{12}=0, A_{13}=0 \\ & A_{21}=0, A_{22}=xz, A_{23}=0 \\ & A_{31}=0, A_{32}=0, A_{33}=xy \end{aligned}$$

$$\therefore adjA=\begin{bmatrix} yz & 0 & 0 \\ 0 & xz & 0 \\ 0 & 0 & xy \end{bmatrix}$$

$$\therefore A^{-1}=\frac{1}{|A|}adjA$$

$$=\frac{1}{xyz}\begin{bmatrix} yz & 0 & 0 \\ 0 & xz & 0 \\ 0 & 0 & xy \end{bmatrix}$$

$$=\begin{bmatrix} \frac{yz}{xyz} & 0 & 0 \\ 0 & \frac{xz}{xyz} & 0 \\ 0 & 0 & \frac{xy}{xyz} \end{bmatrix}$$

$$=\begin{bmatrix} \frac{1}{x} & 0 & 0 \\ 0 & \frac{1}{y} & 0 \\ 0 & 0 & \frac{1}{z} \end{bmatrix}=\begin{bmatrix} x^{-1} & 0 & 0 \\ 0 & y^{-1} & 0 \\ 0 & 0 & z^{-1} \end{bmatrix}$$

The correct answer is A.

Question 19:

Choose the correct answer.

Let $A=\begin{bmatrix} 1 & \sin\theta & 1 \\ -\sin\theta & 1 & \sin\theta \\ -1 & -\sin\theta & 1 \end{bmatrix}$, where $0 \le \theta \le 2\pi$, then

A. Det (A) = 0

B. Det (A) ∈ $(2, \infty)$

C. Det (A) ∈ (2, 4)

D. Det (A)∈ [2, 4]

Answer

sAnswer: D

$$A=\begin{bmatrix} 1 & \sin\theta & 1 \\ -\sin\theta & 1 & \sin\theta \\ -1 & -\sin\theta & 1 \end{bmatrix}$$

$$\therefore |A| = 1\left(1+\sin^2\theta\right)-\sin\theta\left(-\sin\theta+\sin\theta\right)+1\left(\sin^2\theta+1\right)$$

$$=1+\sin^2\theta+\sin^2\theta+1$$

$$=2+2\sin^2\theta$$

$$=2\left(1+\sin^2\theta\right)$$

Now, $0\le\theta\le2\pi$

$\Rightarrow 0\le\sin\theta\le1$

$\Rightarrow 0\le\sin^2\theta\le1$

$\Rightarrow 1\le1+\sin^2\theta\le2$

$\Rightarrow 2\le2\left(1+\sin^2\theta\right)\le4$

$\therefore \text{Det}(A)\in[2,4]$

The correct answer is D.