

## Exercise 4.1

## Question 1:

Evaluate the determinants in Exercises 1 and 2.

$$\begin{vmatrix} 2 & 4 \\ -5 & -1 \end{vmatrix}$$

Answer

$$\begin{vmatrix} 2 & 4 \\ -5 & -1 \end{vmatrix} = 2(-1) - 4(-5) = -2 + 20 = 18$$

## Question 2:

Evaluate the determinants in Exercises 1 and 2.

$$(i) \begin{vmatrix} \cos \theta & -\sin \theta \\ \sin \theta & \cos \theta \end{vmatrix} \quad (ii) \begin{vmatrix} x^2 - x + 1 & x - 1 \\ x + 1 & x + 1 \end{vmatrix}$$

Answer

$$(i) \begin{vmatrix} \cos \theta & -\sin \theta \\ \sin \theta & \cos \theta \end{vmatrix} = (\cos \theta)(\cos \theta) - (-\sin \theta)(\sin \theta) = \cos^2 \theta + \sin^2 \theta = 1$$

$$\begin{aligned} (ii) \begin{vmatrix} x^2 - x + 1 & x - 1 \\ x + 1 & x + 1 \end{vmatrix} &= (x^2 - x + 1)(x + 1) - (x - 1)(x + 1) \\ &= x^3 - x^2 + x + x^2 - x + 1 - (x^2 - 1) \\ &= x^3 + 1 - x^2 + 1 \\ &= x^3 - x^2 + 2 \end{aligned}$$

## Question 3:

$$A = \begin{bmatrix} 1 & 2 \\ 4 & 2 \end{bmatrix}$$

If  $A = \begin{bmatrix} 1 & 2 \\ 4 & 2 \end{bmatrix}$ , then show that  $|2A| = 4|A|$

Answer

$$A = \begin{bmatrix} 1 & 2 \\ 4 & 2 \end{bmatrix}$$

The given matrix is

$$\therefore 2A = 2 \begin{bmatrix} 1 & 2 \\ 4 & 2 \end{bmatrix} = \begin{bmatrix} 2 & 4 \\ 8 & 4 \end{bmatrix}$$

$$\therefore \text{L.H.S.} = |2A| = \begin{vmatrix} 2 & 4 \\ 8 & 4 \end{vmatrix} = 2 \times 4 - 4 \times 8 = 8 - 32 = -24$$

$$\text{Now, } |A| = \begin{vmatrix} 1 & 2 \\ 4 & 2 \end{vmatrix} = 1 \times 2 - 2 \times 4 = 2 - 8 = -6$$

$$\therefore \text{R.H.S.} = 4|A| = 4 \times (-6) = -24$$

$$\therefore \text{L.H.S.} = \text{R.H.S.}$$

**Question 4:**

$$A = \begin{bmatrix} 1 & 0 & 1 \\ 0 & 1 & 2 \\ 0 & 0 & 4 \end{bmatrix}, \text{ then show that } |3A| = 27|A|.$$

Answer

$$A = \begin{bmatrix} 1 & 0 & 1 \\ 0 & 1 & 2 \\ 0 & 0 & 4 \end{bmatrix}.$$

The given matrix is

It can be observed that in the first column, two entries are zero. Thus, we expand along the first column ( $C_1$ ) for easier calculation.

$$|A| = 1 \begin{vmatrix} 2 & 1 \\ 4 & 4 \end{vmatrix} - 0 \begin{vmatrix} 0 & 1 \\ 0 & 4 \end{vmatrix} + 0 \begin{vmatrix} 0 & 1 \\ 1 & 2 \end{vmatrix} = 1(4 - 0) - 0 + 0 = 4$$

$$\therefore 27|A| = 27(4) = 108 \quad \dots(i)$$

$$\text{Now, } 3A = 3 \begin{bmatrix} 1 & 0 & 1 \\ 0 & 1 & 2 \\ 0 & 0 & 4 \end{bmatrix} = \begin{bmatrix} 3 & 0 & 3 \\ 0 & 3 & 6 \\ 0 & 0 & 12 \end{bmatrix}$$

$$\begin{aligned} \therefore |3A| &= 3 \begin{vmatrix} 3 & 6 \\ 0 & 12 \end{vmatrix} - 0 \begin{vmatrix} 0 & 3 \\ 0 & 12 \end{vmatrix} + 0 \begin{vmatrix} 0 & 3 \\ 3 & 6 \end{vmatrix} \\ &= 3(36 - 0) = 3(36) = 108 \quad \dots(ii) \end{aligned}$$

From equations (i) and (ii), we have:

$$|3A| = 27|A|$$

Hence, the given result is proved.

### Question 5:

Evaluate the determinants

$$(i) \begin{vmatrix} 3 & -1 & -2 \\ 0 & 0 & -1 \\ 3 & -5 & 0 \end{vmatrix} \quad (iii) \begin{vmatrix} 3 & -4 & 5 \\ 1 & 1 & -2 \\ 2 & 3 & 1 \end{vmatrix}$$

$$(ii) \begin{vmatrix} 0 & 1 & 2 \\ -1 & 0 & -3 \\ -2 & 3 & 0 \end{vmatrix} \quad (iv) \begin{bmatrix} 2 & -1 & -2 \\ 0 & 2 & -1 \\ 3 & -5 & 0 \end{bmatrix}$$

Answer

$$(i) \text{ Let } A = \begin{vmatrix} 3 & -1 & -2 \\ 0 & 0 & -1 \\ 3 & -5 & 0 \end{vmatrix}.$$

It can be observed that in the second row, two entries are zero. Thus, we expand along the second row for easier calculation.

$$|A| = -0 \begin{vmatrix} -1 & -2 \\ -5 & 0 \end{vmatrix} + 0 \begin{vmatrix} 3 & -2 \\ 3 & 0 \end{vmatrix} - (-1) \begin{vmatrix} 3 & -1 \\ 3 & -5 \end{vmatrix} = (-15 + 3) = -12$$

$$(ii) \text{ Let } A = \begin{bmatrix} 3 & -4 & 5 \\ 1 & 1 & -2 \\ 2 & 3 & 1 \end{bmatrix}.$$

By expanding along the first row, we have:

$$\begin{aligned} |A| &= 3 \begin{vmatrix} 1 & -2 \\ 2 & 1 \end{vmatrix} + 4 \begin{vmatrix} 1 & -2 \\ 2 & 1 \end{vmatrix} + 5 \begin{vmatrix} 1 & 1 \\ 2 & 3 \end{vmatrix} \\ &= 3(1+6) + 4(1+4) + 5(3-2) \\ &= 3(7) + 4(5) + 5(1) \\ &= 21 + 20 + 5 = 46 \end{aligned}$$

$$A = \begin{bmatrix} 0 & 1 & 2 \\ -1 & 0 & -3 \\ -2 & 3 & 0 \end{bmatrix}.$$

(iii) Let

By expanding along the first row, we have:

$$\begin{aligned} |A| &= 0 \begin{vmatrix} -3 & -1 \\ 0 & -2 \end{vmatrix} - 1 \begin{vmatrix} -1 & -3 \\ -2 & 0 \end{vmatrix} + 2 \begin{vmatrix} -1 & 0 \\ -2 & 3 \end{vmatrix} \\ &= 0 - 1(0 - 6) + 2(-3 - 0) \\ &= -1(-6) + 2(-3) \\ &= 6 - 6 = 0 \end{aligned}$$

$$A = \begin{bmatrix} 2 & -1 & -2 \\ 0 & 2 & -1 \\ 3 & -5 & 0 \end{bmatrix}.$$

(iv) Let

By expanding along the first column, we have:

$$\begin{aligned} |A| &= 2 \begin{vmatrix} -1 & -2 \\ 0 & -1 \end{vmatrix} - 0 \begin{vmatrix} -1 & -2 \\ -5 & 0 \end{vmatrix} + 3 \begin{vmatrix} -1 & -2 \\ 2 & -1 \end{vmatrix} \\ &= 2(0 - 5) - 0 + 3(1 + 4) \\ &= -10 + 15 = 5 \end{aligned}$$

**Question 6:**

$$A = \begin{bmatrix} 1 & 1 & -2 \\ 2 & 1 & -3 \\ 5 & 4 & -9 \end{bmatrix}, \text{ find } |A|.$$

If

Answer

$$A = \begin{bmatrix} 1 & 1 & -2 \\ 2 & 1 & -3 \\ 5 & 4 & -9 \end{bmatrix}.$$

Let

By expanding along the first row, we have:

$$\begin{aligned}
 |A| &= 1 \begin{vmatrix} 1 & -3 \\ 4 & -9 \end{vmatrix} - 1 \begin{vmatrix} 2 & -3 \\ 5 & -9 \end{vmatrix} - 2 \begin{vmatrix} 2 & 1 \\ 5 & 4 \end{vmatrix} \\
 &= 1(-9+12) - 1(-18+15) - 2(8-5) \\
 &= 1(3) - 1(-3) - 2(3) \\
 &= 3+3-6 \\
 &= 6-6 \\
 &= 0
 \end{aligned}$$

**Question 7:**Find values of  $x$ , if

$$(i) \begin{vmatrix} 2 & 4 \\ 2 & 1 \end{vmatrix} = \begin{vmatrix} 2x & 4 \\ 6 & x \end{vmatrix} \quad (ii) \begin{vmatrix} 2 & 3 \\ 4 & 5 \end{vmatrix} = \begin{vmatrix} x & 3 \\ 2x & 5 \end{vmatrix}$$

Answer

$$\begin{aligned}
 (i) \quad & \begin{vmatrix} 2 & 4 \\ 5 & 1 \end{vmatrix} = \begin{vmatrix} 2x & 4 \\ 6 & x \end{vmatrix} \\
 \Rightarrow & 2 \times 1 - 5 \times 4 = 2x \times x - 6 \times 4 \\
 \Rightarrow & 2 - 20 = 2x^2 - 24 \\
 \Rightarrow & 2x^2 = 6 \\
 \Rightarrow & x^2 = 3 \\
 \Rightarrow & x = \pm\sqrt{3}
 \end{aligned}$$

$$\begin{aligned}
 (ii) \quad & \begin{vmatrix} 2 & 3 \\ 4 & 5 \end{vmatrix} = \begin{vmatrix} x & 3 \\ 2x & 5 \end{vmatrix} \\
 \Rightarrow & 2 \times 5 - 3 \times 4 = x \times 5 - 3 \times 2x \\
 \Rightarrow & 10 - 12 = 5x - 6x \\
 \Rightarrow & -2 = -x \\
 \Rightarrow & x = 2
 \end{aligned}$$

**Question 8:**

$$\text{If } \begin{vmatrix} x & 2 \\ 18 & x \end{vmatrix} = \begin{vmatrix} 6 & 2 \\ 18 & 6 \end{vmatrix}, \text{ then } x \text{ is equal to}$$

(A) 6 (B)  $\pm 6$  (C)  $-6$  (D) 0

Answer

**Answer: B**

$$\begin{vmatrix} x & 2 \\ 18 & x \end{vmatrix} = \begin{vmatrix} 6 & 2 \\ 18 & 6 \end{vmatrix}$$

$$\Rightarrow x^2 - 36 = 36 - 36$$

$$\Rightarrow x^2 - 36 = 0$$

$$\Rightarrow x^2 = 36$$

$$\Rightarrow x = \pm 6$$

Hence, the correct answer is B.

## Exercise 4.2

**Question 1:**

Using the property of determinants and without expanding, prove that:

$$\begin{vmatrix} x & a & x+a \\ y & b & y+b \\ z & c & z+c \end{vmatrix} = 0$$

Answer

$$\begin{vmatrix} x & a & x+a \\ y & b & y+b \\ z & c & z+c \end{vmatrix} = \begin{vmatrix} x & a & x \\ y & b & y \\ z & c & z \end{vmatrix} + \begin{vmatrix} x & a & a \\ y & b & b \\ z & c & c \end{vmatrix} = 0 + 0 = 0$$

[Here, the two columns of the determinants are identical]

**Question 2:**

Using the property of determinants and without expanding, prove that:

$$\begin{vmatrix} a-b & b-c & c-a \\ b-c & c-a & a-b \\ c-a & a-b & b-c \end{vmatrix} = 0$$

Answer

$$\Delta = \begin{vmatrix} a-b & b-c & c-a \\ b-c & c-a & a-b \\ c-a & a-b & b-c \end{vmatrix}$$

Applying  $R_1 \rightarrow R_1 + R_2$ , we have:

$$\begin{aligned} \Delta &= \begin{vmatrix} a-c & b-a & c-b \\ b-c & c-a & a-b \\ -(a-c) & -(b-a) & -(c-b) \end{vmatrix} \\ &= - \begin{vmatrix} a-c & b-a & c-b \\ b-c & c-a & a-b \\ a-c & b-a & c-b \end{vmatrix} \end{aligned}$$

Here, the two rows  $R_1$  and  $R_3$  are identical.

$\therefore \Delta = 0$ .

**Question 3:**

Using the property of determinants and without expanding, prove that:

$$\begin{vmatrix} 2 & 7 & 65 \\ 3 & 8 & 75 \\ 5 & 9 & 86 \end{vmatrix} = 0$$

Answer

$$\begin{aligned} \begin{vmatrix} 2 & 7 & 65 \\ 3 & 8 & 75 \\ 5 & 9 & 86 \end{vmatrix} &= \begin{vmatrix} 2 & 7 & 63+2 \\ 3 & 8 & 72+3 \\ 5 & 9 & 81+5 \end{vmatrix} \\ &= \begin{vmatrix} 2 & 7 & 63 \\ 3 & 8 & 72 \\ 5 & 9 & 81 \end{vmatrix} + \begin{vmatrix} 2 & 7 & 2 \\ 3 & 8 & 3 \\ 5 & 9 & 5 \end{vmatrix} \\ &= \begin{vmatrix} 2 & 7 & 9(7) \\ 3 & 8 & 9(8) \\ 5 & 9 & 9(9) \end{vmatrix} + 0 \quad \text{[Two columns are identical]} \\ &= 9 \begin{vmatrix} 2 & 7 & 7 \\ 3 & 8 & 8 \\ 5 & 9 & 9 \end{vmatrix} \\ &= 0 \quad \text{[Two columns are identical]} \end{aligned}$$

**Question 4:**

Using the property of determinants and without expanding, prove that:

$$\begin{vmatrix} 1 & bc & a(b+c) \\ 1 & ca & b(c+a) \\ 1 & ab & c(a+b) \end{vmatrix} = 0$$

Answer

$$\Delta = \begin{vmatrix} 1 & bc & a(b+c) \\ 1 & ca & b(c+a) \\ 1 & ab & c(a+b) \end{vmatrix}$$

By applying  $C_3 \rightarrow C_3 + C_2$ , we have:

$$\Delta = \begin{vmatrix} 1 & bc & ab+bc+ca \\ 1 & ca & ab+bc+ca \\ 1 & ab & ab+bc+ca \end{vmatrix}$$

Here, two columns  $C_1$  and  $C_3$  are proportional.

$$\therefore \Delta = 0.$$

### Question 5:

Using the property of determinants and without expanding, prove that:

$$\begin{vmatrix} b+c & q+r & y+z \\ c+a & r+p & z+x \\ a+b & p+q & x+y \end{vmatrix} = 2 \begin{vmatrix} a & p & x \\ b & q & y \\ c & r & z \end{vmatrix}$$

Answer

$$\begin{aligned} \Delta &= \begin{vmatrix} b+c & q+r & y+z \\ c+a & r+p & z+x \\ a+b & p+q & x+y \end{vmatrix} \\ &= \begin{vmatrix} b+c & q+r & y+z \\ c+a & r+p & z+x \\ a & p & x \end{vmatrix} + \begin{vmatrix} b+c & q+r & y+z \\ c+a & r+p & z+x \\ b & q & y \end{vmatrix} \\ &= \Delta_1 + \Delta_2 \text{ (say)} \end{aligned} \quad \dots(1)$$

$$\text{Now, } \Delta_1 = \begin{vmatrix} b+c & q+r & y+z \\ c+a & r+p & z+x \\ a & p & x \end{vmatrix}$$

Applying  $R_2 \rightarrow R_2 - R_3$ , we have:

$$\Delta_1 = \begin{vmatrix} b+c & q+r & y+z \\ c & r & z \\ a & p & x \end{vmatrix}$$

Applying  $R_1 \rightarrow R_1 - R_2$ , we have:

$$\Delta_1 = \begin{vmatrix} b & q & y \\ c & r & z \\ a & p & x \end{vmatrix}$$

Applying  $R_1 \leftrightarrow R_3$  and  $R_2 \leftrightarrow R_3$ , we have:

$$\Delta_1 = (-1)^2 \begin{vmatrix} a & p & x \\ b & q & y \\ c & r & z \end{vmatrix} = \begin{vmatrix} a & p & x \\ b & q & y \\ c & r & z \end{vmatrix} \quad \dots(2)$$

$$\Delta_2 = \begin{vmatrix} b+c & q+r & y+z \\ c+a & r+p & z+x \\ b & q & y \end{vmatrix}$$

Applying  $R_1 \rightarrow R_1 - R_3$ , we have:

$$\Delta_2 = \begin{vmatrix} c & r & z \\ c+a & r+p & z+x \\ b & q & y \end{vmatrix}$$

Applying  $R_2 \rightarrow R_2 - R_1$ , we have:

$$\Delta_2 = \begin{vmatrix} c & r & z \\ a & p & x \\ b & q & y \end{vmatrix}$$

Applying  $R_1 \leftrightarrow R_2$  and  $R_2 \leftrightarrow R_3$ , we have:

$$\Delta_2 = (-1)^2 \begin{vmatrix} a & p & x \\ b & q & y \\ c & r & z \end{vmatrix} = \begin{vmatrix} a & p & x \\ b & q & y \\ c & r & z \end{vmatrix} \quad \dots(3)$$

From (1), (2), and (3), we have:

$$\Delta = 2 \begin{vmatrix} a & p & x \\ b & q & y \\ c & r & z \end{vmatrix}$$

Hence, the given result is proved.

#### Question 6:

By using properties of determinants, show that:

$$\begin{vmatrix} 0 & a & -b \\ -a & 0 & -c \\ b & c & 0 \end{vmatrix} = 0$$

Answer

We have,

$$\Delta = \begin{vmatrix} 0 & a & -b \\ -a & 0 & -c \\ b & c & 0 \end{vmatrix}$$

Applying  $R_1 \rightarrow cR_1$ , we have:

$$\Delta = \frac{1}{c} \begin{vmatrix} 0 & ac & -bc \\ -a & 0 & -c \\ b & c & 0 \end{vmatrix}$$

Applying  $R_1 \rightarrow R_1 - bR_2$ , we have:

$$\begin{aligned} \Delta &= \frac{1}{c} \begin{vmatrix} ab & ac & 0 \\ -a & 0 & -c \\ b & c & 0 \end{vmatrix} \\ &= \frac{a}{c} \begin{vmatrix} b & c & 0 \\ -a & 0 & -c \\ b & c & 0 \end{vmatrix} \end{aligned}$$

Here, the two rows  $R_1$  and  $R_3$  are identical.

$$\therefore \Delta = 0.$$

#### Question 7:

By using properties of determinants, show that:

$$\begin{vmatrix} -a^2 & ab & ac \\ ba & -b^2 & bc \\ ca & cb & -c^2 \end{vmatrix} = 4a^2b^2c^2$$

Answer

$$\Delta = \begin{vmatrix} -a^2 & ab & ac \\ ba & -b^2 & bc \\ ca & cb & -c^2 \end{vmatrix}$$

$$= abc \begin{vmatrix} -a & b & c \\ a & -b & c \\ a & b & -c \end{vmatrix} \quad \left[ \text{Taking out factors } a, b, c \text{ from } R_1, R_2, \text{ and } R_3 \right]$$

$$= a^2 b^2 c^2 \begin{vmatrix} -1 & 1 & 1 \\ 1 & -1 & 1 \\ 1 & 1 & -1 \end{vmatrix} \quad \left[ \text{Taking out factors } a, b, c \text{ from } C_1, C_2, \text{ and } C_3 \right]$$

Applying  $R_2 \rightarrow R_2 + R_1$  and  $R_3 \rightarrow R_3 + R_1$ , we have:

$$\Delta = a^2 b^2 c^2 \begin{vmatrix} -1 & 1 & 1 \\ 0 & 0 & 2 \\ 0 & 2 & 0 \end{vmatrix}$$

$$= a^2 b^2 c^2 (-1) \begin{vmatrix} 0 & 2 \\ 2 & 0 \end{vmatrix}$$

$$= -a^2 b^2 c^2 (0 - 4) = 4a^2 b^2 c^2$$

### Question 8:

By using properties of determinants, show that:

$$(i) \begin{vmatrix} 1 & a & a^2 \\ 1 & b & b^2 \\ 1 & c & c^2 \end{vmatrix} = (a-b)(b-c)(c-a)$$

$$(ii) \begin{vmatrix} 1 & 1 & 1 \\ a & b & c \\ a^3 & b^3 & c^3 \end{vmatrix} = (a-b)(b-c)(c-a)(a+b+c)$$

Answer

$$(i) \text{ Let } \Delta = \begin{vmatrix} 1 & a & a^2 \\ 1 & b & b^2 \\ 1 & c & c^2 \end{vmatrix}$$

Applying  $R_1 \rightarrow R_1 - R_3$  and  $R_2 \rightarrow R_2 - R_3$ , we have:

$$\Delta = \begin{vmatrix} 0 & a-c & a^2-c^2 \\ 0 & b-c & b^2-c^2 \\ 1 & c & c^2 \end{vmatrix}$$

$$= (c-a)(b-c) \begin{vmatrix} 0 & -1 & -a-c \\ 0 & 1 & b+c \\ 1 & c & c^2 \end{vmatrix}$$

Applying  $R_1 \rightarrow R_1 + R_2$ , we have:

$$\Delta = (b-c)(c-a) \begin{vmatrix} 0 & 0 & -a+b \\ 0 & 1 & b+c \\ 1 & c & c^2 \end{vmatrix}$$

$$= (a-b)(b-c)(c-a) \begin{vmatrix} 0 & 0 & -1 \\ 0 & 1 & b+c \\ 1 & c & c^2 \end{vmatrix}$$

Expanding along  $C_1$ , we have:

$$\Delta = (a-b)(b-c)(c-a) \begin{vmatrix} 0 & -1 \\ 1 & b+c \end{vmatrix} = (a-b)(b-c)(c-a)$$

Hence, the given result is proved.

$$\Delta = \begin{vmatrix} 1 & 1 & 1 \\ a & b & c \\ a^3 & b^3 & c^3 \end{vmatrix}$$

(ii) Let

Applying  $C_1 \rightarrow C_1 - C_3$  and  $C_2 \rightarrow C_2 - C_3$ , we have:

$$\begin{aligned}
 \Delta &= \begin{vmatrix} 0 & 0 & 1 \\ a-c & b-c & c \\ a^3-c^3 & b^3-c^3 & c^3 \end{vmatrix} \\
 &= \begin{vmatrix} 0 & 0 & 1 \\ a-c & b-c & c \\ (a-c)(a^2+ac+c^2) & (b-c)(b^2+bc+c^2) & c^3 \end{vmatrix} \\
 &= (c-a)(b-c) \begin{vmatrix} 0 & 0 & 1 \\ -1 & 1 & c \\ -(a^2+ac+c^2) & (b^2+bc+c^2) & c^3 \end{vmatrix}
 \end{aligned}$$

Applying  $C_1 \rightarrow C_1 + C_2$ , we have:

$$\begin{aligned}
 \Delta &= (c-a)(b-c) \begin{vmatrix} 0 & 0 & 1 \\ 0 & 1 & c \\ (b^2-a^2)+(bc-ac) & (b^2+bc+c^2) & c^3 \end{vmatrix} \\
 &= (b-c)(c-a)(a-b) \begin{vmatrix} 0 & 0 & 1 \\ 0 & 1 & c \\ -(a+b+c) & (b^2+bc+c^2) & c^3 \end{vmatrix} \\
 &= (a-b)(b-c)(c-a)(a+b+c) \begin{vmatrix} 0 & 0 & 1 \\ 0 & 1 & c \\ -1 & (b^2+bc+c^2) & c^3 \end{vmatrix}
 \end{aligned}$$

Expanding along  $C_1$ , we have:

$$\begin{aligned}
 \Delta &= (a-b)(b-c)(c-a)(a+b+c)(-1) \begin{vmatrix} 0 & 1 \\ 1 & c \end{vmatrix} \\
 &= (a-b)(b-c)(c-a)(a+b+c)
 \end{aligned}$$

Hence, the given result is proved.

### Question 9:

By using properties of determinants, show that:

$$\begin{vmatrix} x & x^2 & yz \\ y & y^2 & zx \\ z & z^2 & xy \end{vmatrix} = (x-y)(y-z)(z-x)(xy + yz + zx)$$

Answer

$$\text{Let } \Delta = \begin{vmatrix} x & x^2 & yz \\ y & y^2 & zx \\ z & z^2 & xy \end{vmatrix}.$$

Applying  $R_2 \rightarrow R_2 - R_1$  and  $R_3 \rightarrow R_3 - R_1$ , we have:

$$\begin{aligned} \Delta &= \begin{vmatrix} x & x^2 & yz \\ y-x & y^2-x^2 & zx-yz \\ z-x & z^2-x^2 & xy-yz \end{vmatrix} \\ &= \begin{vmatrix} x & x^2 & yz \\ -(x-y) & -(x-y)(x+y) & z(x-y) \\ (z-x) & (z-x)(z+x) & -y(z-x) \end{vmatrix} \\ &= (x-y)(z-x) \begin{vmatrix} x & x^2 & yz \\ -1 & -x-y & z \\ 1 & z+x & -y \end{vmatrix} \end{aligned}$$

Applying  $R_3 \rightarrow R_3 + R_2$ , we have:

$$\begin{aligned} \Delta &= (x-y)(z-x) \begin{vmatrix} x & x^2 & yz \\ -1 & -x-y & z \\ 0 & z-y & z-y \end{vmatrix} \\ &= (x-y)(z-x)(z-y) \begin{vmatrix} x & x^2 & yz \\ -1 & -x-y & z \\ 0 & 1 & 1 \end{vmatrix} \end{aligned}$$

Expanding along  $R_3$ , we have:

$$\begin{aligned}
 \Delta &= [(x-y)(z-x)(z-y)] \left[ (-1) \begin{vmatrix} x & yz \\ -1 & z \end{vmatrix} + 1 \begin{vmatrix} x & x^2 \\ -1 & -x-y \end{vmatrix} \right] \\
 &= (x-y)(z-x)(z-y) [(-xz - yz) + (-x^2 - xy + x^2)] \\
 &= -(x-y)(z-x)(z-y)(xy + yz + zx) \\
 &= (x-y)(y-z)(z-x)(xy + yz + zx)
 \end{aligned}$$

Hence, the given result is proved.

**Question 10:**

By using properties of determinants, show that:

$$(i) \begin{vmatrix} x+4 & 2x & 2x \\ 2x & x+4 & 2x \\ 2x & 2x & x+4 \end{vmatrix} = (5x+4)(4-x)^2$$

$$(ii) \begin{vmatrix} y+k & y & y \\ y & y+k & y \\ y & y & y+k \end{vmatrix} = k^2(3y+k)$$

Answer

$$(i) \Delta = \begin{vmatrix} x+4 & 2x & 2x \\ 2x & x+4 & 2x \\ 2x & 2x & x+4 \end{vmatrix}$$

Applying  $R_1 \rightarrow R_1 + R_2 + R_3$ , we have:

$$\begin{aligned}
 \Delta &= \begin{vmatrix} 5x+4 & 5x+4 & 5x+4 \\ 2x & x+4 & 2x \\ 2x & 2x & x+4 \end{vmatrix} \\
 &= (5x+4) \begin{vmatrix} 1 & 1 & 1 \\ 2x & x+4 & 2x \\ 2x & 2x & x+4 \end{vmatrix}
 \end{aligned}$$

Applying  $C_2 \rightarrow C_2 - C_1$ ,  $C_3 \rightarrow C_3 - C_1$ , we have:

$$\Delta = (5x+4) \begin{vmatrix} 1 & 0 & 0 \\ 2x & -x+4 & 0 \\ 2x & 0 & -x+4 \end{vmatrix}$$

$$= (5x+4)(4-x)(4-x) \begin{vmatrix} 1 & 0 & 0 \\ 2x & 1 & 0 \\ 2x & 0 & 1 \end{vmatrix}$$

Expanding along  $C_3$ , we have:

$$\Delta = (5x+4)(4-x)^2 \begin{vmatrix} 1 & 0 \\ 2x & 1 \end{vmatrix}$$

$$= (5x+4)(4-x)^2$$

Hence, the given result is proved.

$$\Delta = \begin{vmatrix} y+k & y & y \\ y & y+k & y \\ y & y & y+k \end{vmatrix}$$

(ii)

Applying  $R_1 \rightarrow R_1 + R_2 + R_3$ , we have:

$$\Delta = \begin{vmatrix} 3y+k & 3y+k & 3y+k \\ y & y+k & y \\ y & y & y+k \end{vmatrix}$$

$$= (3y+k) \begin{vmatrix} 1 & 1 & 1 \\ y & y+k & y \\ y & y & y+k \end{vmatrix}$$

Applying  $C_2 \rightarrow C_2 - C_1$  and  $C_3 \rightarrow C_3 - C_1$ , we have:

$$\Delta = (3y+k) \begin{vmatrix} 1 & 0 & 0 \\ y & k & 0 \\ y & 0 & k \end{vmatrix}$$

$$= k^2 (3y+k) \begin{vmatrix} 1 & 0 & 0 \\ y & 1 & 0 \\ y & 0 & 1 \end{vmatrix}$$

Expanding along  $C_3$ , we have:

$$\Delta = k^2(3y+k) \begin{vmatrix} 1 & 0 \\ y & 1 \end{vmatrix} = k^2(3y+k)$$

Hence, the given result is proved.

**Question 11:**

By using properties of determinants, show that:

$$(i) \begin{vmatrix} a-b-c & 2a & 2a \\ 2b & b-c-a & 2b \\ 2c & 2c & c-a-b \end{vmatrix} = (a+b+c)^3$$

$$(ii) \begin{vmatrix} x+y+2z & x & y \\ z & y+z+2x & y \\ z & x & z+x+2y \end{vmatrix} = 2(x+y+z)^3$$

Answer

$$(i) \Delta = \begin{vmatrix} a-b-c & 2a & 2a \\ 2b & b-c-a & 2b \\ 2c & 2c & c-a-b \end{vmatrix}$$

Applying  $R_1 \rightarrow R_1 + R_2 + R_3$ , we have:

$$\begin{aligned} \Delta &= \begin{vmatrix} a+b+c & a+b+c & a+b+c \\ 2b & b-c-a & 2b \\ 2c & 2c & c-a-b \end{vmatrix} \\ &= (a+b+c) \begin{vmatrix} 1 & 1 & 1 \\ 2b & b-c-a & 2b \\ 2c & 2c & c-a-b \end{vmatrix} \end{aligned}$$

Applying  $C_2 \rightarrow C_2 - C_1$ ,  $C_3 \rightarrow C_3 - C_1$ , we have:

$$\begin{aligned} \Delta &= (a+b+c) \begin{vmatrix} 1 & 0 & 0 \\ 2b & -(a+b+c) & 0 \\ 2c & 0 & -(a+b+c) \end{vmatrix} \\ &= (a+b+c)^3 \begin{vmatrix} 1 & 0 & 0 \\ 2b & -1 & 0 \\ 2c & 0 & -1 \end{vmatrix} \end{aligned}$$

Expanding along  $C_3$ , we have:

$$\Delta = (a+b+c)^3 (-1)(-1) = (a+b+c)^3$$

Hence, the given result is proved.

$$(ii) \quad \Delta = \begin{vmatrix} x+y+2z & x & y \\ z & y+z+2x & y \\ z & x & z+x+2y \end{vmatrix}$$

Applying  $C_1 \rightarrow C_1 + C_2 + C_3$ , we have:

$$\begin{aligned} \Delta &= \begin{vmatrix} 2(x+y+z) & x & y \\ 2(x+y+z) & y+z+2x & y \\ 2(x+y+z) & x & z+x+2y \end{vmatrix} \\ &= 2(x+y+z) \begin{vmatrix} 1 & x & y \\ 1 & y+z+2x & y \\ 1 & x & z+x+2y \end{vmatrix} \end{aligned}$$

Applying  $R_2 \rightarrow R_2 - R_1$  and  $R_3 \rightarrow R_3 - R_1$ , we have:

$$\begin{aligned} \Delta &= 2(x+y+z) \begin{vmatrix} 1 & x & y \\ 0 & x+y+z & 0 \\ 0 & 0 & x+y+z \end{vmatrix} \\ &= 2(x+y+z)^3 \begin{vmatrix} 1 & x & y \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{vmatrix} \end{aligned}$$

Expanding along  $R_3$ , we have:

$$\Delta = 2(x+y+z)^3 (1)(1-0) = 2(x+y+z)^3$$

Hence, the given result is proved.

### Question 12:

By using properties of determinants, show that:

$$\begin{vmatrix} 1 & x & x^2 \\ x^2 & 1 & x \\ x & x^2 & 1 \end{vmatrix} = (1-x^3)^2$$

Answer

$$\Delta = \begin{vmatrix} 1 & x & x^2 \\ x^2 & 1 & x \\ x & x^2 & 1 \end{vmatrix}$$

Applying  $R_1 \rightarrow R_1 + R_2 + R_3$ , we have:

$$\begin{aligned} \Delta &= \begin{vmatrix} 1+x+x^2 & 1+x+x^2 & 1+x+x^2 \\ x^2 & 1 & x \\ x & x^2 & 1 \end{vmatrix} \\ &= (1+x+x^2) \begin{vmatrix} 1 & 1 & 1 \\ x^2 & 1 & x \\ x & x^2 & 1 \end{vmatrix} \end{aligned}$$

Applying  $C_2 \rightarrow C_2 - C_1$  and  $C_3 \rightarrow C_3 - C_1$ , we have:

$$\begin{aligned} \Delta &= (1+x+x^2) \begin{vmatrix} 1 & 0 & 0 \\ x^2 & 1-x^2 & x-x^2 \\ x & x^2-x & 1-x \end{vmatrix} \\ &= (1+x+x^2)(1-x)(1-x) \begin{vmatrix} 1 & 0 & 0 \\ x^2 & 1+x & x \\ x & -x & 1 \end{vmatrix} \\ &= (1-x^3)(1-x) \begin{vmatrix} 1 & 0 & 0 \\ x^2 & 1+x & x \\ x & -x & 1 \end{vmatrix} \end{aligned}$$

Expanding along  $R_1$ , we have:

$$\begin{aligned} \Delta &= (1-x^3)(1-x)(1) \begin{vmatrix} 1+x & x \\ -x & 1 \end{vmatrix} \\ &= (1-x^3)(1-x)(1+x+x^2) \\ &= (1-x^3)(1-x^3) \\ &= (1-x^3)^2 \end{aligned}$$

Hence, the given result is proved.

**Question 13:**

By using properties of determinants, show that:

$$\begin{vmatrix} 1+a^2-b^2 & 2ab & -2b \\ 2ab & 1-a^2+b^2 & 2a \\ 2b & -2a & 1-a^2-b^2 \end{vmatrix} = (1+a^2+b^2)^3$$

Answer

$$\Delta = \begin{vmatrix} 1+a^2-b^2 & 2ab & -2b \\ 2ab & 1-a^2+b^2 & 2a \\ 2b & -2a & 1-a^2-b^2 \end{vmatrix}$$

Applying  $R_1 \rightarrow R_1 + bR_3$  and  $R_2 \rightarrow R_2 - aR_3$ , we have:

$$\begin{aligned} \Delta &= \begin{vmatrix} 1+a^2+b^2 & 0 & -b(1+a^2+b^2) \\ 0 & 1+a^2+b^2 & a(1+a^2+b^2) \\ 2b & -2a & 1-a^2-b^2 \end{vmatrix} \\ &= (1+a^2+b^2)^2 \begin{vmatrix} 1 & 0 & -b \\ 0 & 1 & a \\ 2b & -2a & 1-a^2-b^2 \end{vmatrix} \end{aligned}$$

Expanding along  $R_1$ , we have:

$$\begin{aligned} \Delta &= (1+a^2+b^2)^2 \left[ (1) \begin{vmatrix} 1 & a \\ -2a & 1-a^2-b^2 \end{vmatrix} - b \begin{vmatrix} 0 & 1 \\ 2b & -2a \end{vmatrix} \right] \\ &= (1+a^2+b^2)^2 [1-a^2-b^2+2a^2-b(-2b)] \\ &= (1+a^2+b^2)^2 (1+a^2+b^2) \\ &= (1+a^2+b^2)^3 \end{aligned}$$

**Question 14:**

By using properties of determinants, show that:

$$\begin{vmatrix} a^2+1 & ab & ac \\ ab & b^2+1 & bc \\ ca & cb & c^2+1 \end{vmatrix} = 1+a^2+b^2+c^2$$

Answer

$$\Delta = \begin{vmatrix} a^2+1 & ab & ac \\ ab & b^2+1 & bc \\ ca & cb & c^2+1 \end{vmatrix}$$

Taking out common factors  $a$ ,  $b$ , and  $c$  from  $R_1$ ,  $R_2$ , and  $R_3$  respectively, we have:

$$\Delta = abc \begin{vmatrix} a+\frac{1}{a} & b & c \\ a & b+\frac{1}{b} & c \\ a & b & c+\frac{1}{c} \end{vmatrix}$$

Applying  $R_2 \rightarrow R_2 - R_1$  and  $R_3 \rightarrow R_3 - R_1$ , we have:

$$\Delta = abc \begin{vmatrix} a+\frac{1}{a} & b & c \\ -\frac{1}{a} & \frac{1}{b} & 0 \\ -\frac{1}{a} & 0 & \frac{1}{c} \end{vmatrix}$$

Applying  $C_1 \rightarrow aC_1$ ,  $C_2 \rightarrow bC_2$ , and  $C_3 \rightarrow cC_3$ , we have:

$$\begin{aligned} \Delta &= abc \times \frac{1}{abc} \begin{vmatrix} a^2+1 & b^2 & c^2 \\ -1 & 1 & 0 \\ -1 & 0 & 1 \end{vmatrix} \\ &= \begin{vmatrix} a^2+1 & b^2 & c^2 \\ -1 & 1 & 0 \\ -1 & 0 & 1 \end{vmatrix} \end{aligned}$$

Expanding along  $R_3$ , we have:

$$\Delta = -1 \begin{vmatrix} b^2 & c^2 \\ 1 & 0 \end{vmatrix} + 1 \begin{vmatrix} a^2 + 1 & b^2 \\ -1 & 1 \end{vmatrix}$$

$$= -1(-c^2) + (a^2 + 1 + b^2) = 1 + a^2 + b^2 + c^2$$

Hence, the given result is proved.

**Question 15:**

Choose the correct answer.

Let  $A$  be a square matrix of order  $3 \times 3$ , then  $|kA|$  is equal to

**A.**  $k|A|$  **B.**  $k^2|A|$  **C.**  $k^3|A|$  **D.**  $3k|A|$

Answer

**Answer: C**

$A$  is a square matrix of order  $3 \times 3$ .

$$\text{Let } A = \begin{bmatrix} a_1 & b_1 & c_1 \\ a_2 & b_2 & c_2 \\ a_3 & b_3 & c_3 \end{bmatrix}.$$

$$\text{Then, } kA = \begin{bmatrix} ka_1 & kb_1 & kc_1 \\ ka_2 & kb_2 & kc_2 \\ ka_3 & kb_3 & kc_3 \end{bmatrix}.$$

$$\therefore |kA| = \begin{vmatrix} ka_1 & kb_1 & kc_1 \\ ka_2 & kb_2 & kc_2 \\ ka_3 & kb_3 & kc_3 \end{vmatrix}$$

$$= k^3 \begin{vmatrix} a_1 & b_1 & c_1 \\ a_2 & b_2 & c_2 \\ a_3 & b_3 & c_3 \end{vmatrix}$$

(Taking out common factors  $k$  from each row)

$$= k^3 |A|$$

$$\therefore |kA| = k^3 |A|$$

Hence, the correct answer is C.

**Question 16:**

Which of the following is correct?

- A.** Determinant is a square matrix.
- B.** Determinant is a number associated to a matrix.
- C.** Determinant is a number associated to a square matrix.
- D.** None of these

Answer

**Answer: C**

We know that to every square matrix,  $A = [a_{ij}]$  of order  $n$ . We can associate a number called the determinant of square matrix  $A$ , where  $a_{ij} = (i, j)^{\text{th}}$  element of  $A$ . Thus, the determinant is a number associated to a square matrix. Hence, the correct answer is C.

**Exercise 4.3****Question 1:**

Find area of the triangle with vertices at the point given in each of the following:

(i) (1, 0), (6, 0), (4, 3) (ii) (2, 7), (1, 1), (10, 8)

(iii) (-2, -3), (3, 2), (-1, -8)

Answer

(i) The area of the triangle with vertices (1, 0), (6, 0), (4, 3) is given by the relation,

$$\begin{aligned}\Delta &= \frac{1}{2} \begin{vmatrix} 1 & 0 & 1 \\ 6 & 0 & 1 \\ 4 & 3 & 1 \end{vmatrix} \\ &= \frac{1}{2} [1(0-3) - 0(6-4) + 1(18-0)] \\ &= \frac{1}{2} [-3 + 18] = \frac{15}{2} \text{ square units}\end{aligned}$$

(ii) The area of the triangle with vertices (2, 7), (1, 1), (10, 8) is given by the relation,

$$\begin{aligned}\Delta &= \frac{1}{2} \begin{vmatrix} 2 & 7 & 1 \\ 1 & 1 & 1 \\ 10 & 8 & 1 \end{vmatrix} \\ &= \frac{1}{2} [2(1-8) - 7(1-10) + 1(8-10)] \\ &= \frac{1}{2} [2(-7) - 7(-9) + 1(-2)] \\ &= \frac{1}{2} [-14 + 63 - 2] = \frac{1}{2} [-16 + 63] \\ &= \frac{47}{2} \text{ square units}\end{aligned}$$

(iii) The area of the triangle with vertices (-2, -3), (3, 2), (-1, -8) is given by the relation,

$$\begin{aligned}
 \Delta &= \frac{1}{2} \begin{vmatrix} -2 & -3 & 1 \\ 3 & 2 & 1 \\ -1 & -8 & 1 \end{vmatrix} \\
 &= \frac{1}{2} [-2(2+8) + 3(3+1) + 1(-24+2)] \\
 &= \frac{1}{2} [-2(10) + 3(4) + 1(-22)] \\
 &= \frac{1}{2} [-20 + 12 - 22] \\
 &= -\frac{30}{2} = -15
 \end{aligned}$$

Hence, the area of the triangle is  $|-15| = 15$  square units.

**Question 2:**

Show that points

$A(a, b+c), B(b, c+a), C(c, a+b)$  are collinear

Answer

Area of  $\Delta ABC$  is given by the relation,

$$\begin{aligned}
 \Delta &= \frac{1}{2} \begin{vmatrix} a & b+c & 1 \\ b & c+a & 1 \\ c & a+b & 1 \end{vmatrix} \\
 &= \frac{1}{2} \begin{vmatrix} a & b+c & 1 \\ b-a & a-b & 0 \\ c-a & a-c & 0 \end{vmatrix} \quad (\text{Applying } R_2 \rightarrow R_2 - R_1 \text{ and } R_3 \rightarrow R_3 - R_1) \\
 &= \frac{1}{2} (a-b)(c-a) \begin{vmatrix} a & b+c & 1 \\ -1 & 1 & 0 \\ 1 & -1 & 0 \end{vmatrix} \\
 &= \frac{1}{2} (a-b)(c-a) \begin{vmatrix} a & b+c & 1 \\ -1 & 1 & 0 \\ 0 & 0 & 0 \end{vmatrix} \quad (\text{Applying } R_3 \rightarrow R_3 + R_2) \\
 &= 0 \quad (\text{All elements of } R_3 \text{ are } 0)
 \end{aligned}$$

Thus, the area of the triangle formed by points A, B, and C is zero.

Hence, the points A, B, and C are collinear.

**Question 3:**

Find values of  $k$  if area of triangle is 4 square units and vertices are

(i)  $(k, 0), (4, 0), (0, 2)$  (ii)  $(-2, 0), (0, 4), (0, k)$

Answer

We know that the area of a triangle whose vertices are  $(x_1, y_1), (x_2, y_2)$ , and  $(x_3, y_3)$  is the absolute value of the determinant ( $\Delta$ ), where

$$\Delta = \frac{1}{2} \begin{vmatrix} x_1 & y_1 & 1 \\ x_2 & y_2 & 1 \\ x_3 & y_3 & 1 \end{vmatrix}$$

It is given that the area of triangle is 4 square units.

$$\therefore \Delta = \pm 4.$$

(i) The area of the triangle with vertices  $(k, 0), (4, 0), (0, 2)$  is given by the relation,

$$\begin{aligned} \Delta &= \frac{1}{2} \begin{vmatrix} k & 0 & 1 \\ 4 & 0 & 1 \\ 0 & 2 & 1 \end{vmatrix} \\ &= \frac{1}{2} [k(0-2) - 0(4-0) + 1(8-0)] \\ &= \frac{1}{2} [-2k + 8] = -k + 4 \end{aligned}$$

$$\therefore -k + 4 = \pm 4$$

When  $-k + 4 = -4$ ,  $k = 8$ .

When  $-k + 4 = 4$ ,  $k = 0$ .

Hence,  $k = 0, 8$ .

(ii) The area of the triangle with vertices  $(-2, 0)$ ,  $(0, 4)$ ,  $(0, k)$  is given by the relation,

$$\begin{aligned}\Delta &= \frac{1}{2} \begin{vmatrix} -2 & 0 & 1 \\ 0 & 4 & 1 \\ 0 & k & 1 \end{vmatrix} \\ &= \frac{1}{2} [-2(4-k)] \\ &= k - 4\end{aligned}$$

$$\therefore k - 4 = \pm 4$$

When  $k - 4 = -4$ ,  $k = 0$ .

When  $k - 4 = 4$ ,  $k = 8$ .

Hence,  $k = 0, 8$ .

#### Question 4:

(i) Find equation of line joining  $(1, 2)$  and  $(3, 6)$  using determinants

(ii) Find equation of line joining  $(3, 1)$  and  $(9, 3)$  using determinants

Answer

(i) Let  $P(x, y)$  be any point on the line joining points  $A(1, 2)$  and  $B(3, 6)$ . Then, the points  $A$ ,  $B$ , and  $P$  are collinear. Therefore, the area of triangle  $ABP$  will be zero.

$$\begin{aligned}\therefore \frac{1}{2} \begin{vmatrix} 1 & 2 & 1 \\ 3 & 6 & 1 \\ x & y & 1 \end{vmatrix} &= 0 \\ \Rightarrow \frac{1}{2} [1(6-y) - 2(3-x) + 1(3y-6x)] &= 0 \\ \Rightarrow 6 - y - 6 + 2x + 3y - 6x &= 0 \\ \Rightarrow 2y - 4x &= 0 \\ \Rightarrow y &= 2x\end{aligned}$$

Hence, the equation of the line joining the given points is  $y = 2x$ .

(ii) Let  $P(x, y)$  be any point on the line joining points  $A(3, 1)$  and

B (9, 3). Then, the points A, B, and P are collinear. Therefore, the area of triangle ABP will be zero.

$$\therefore \frac{1}{2} \begin{vmatrix} 3 & 1 & 1 \\ 9 & 3 & 1 \\ x & y & 1 \end{vmatrix} = 0$$

$$\Rightarrow \frac{1}{2} [3(3-y) - 1(9-x) + 1(9y-3x)] = 0$$

$$\Rightarrow 9 - 3y - 9 + x + 9y - 3x = 0$$

$$\Rightarrow 6y - 2x = 0$$

$$\Rightarrow x - 3y = 0$$

Hence, the equation of the line joining the given points is  $x - 3y = 0$ .

#### Question 5:

If area of triangle is 35 square units with vertices (2, -6), (5, 4), and (k, 4). Then k is

**A.** 12 **B.** -2 **C.** -12, -2 **D.** 12, -2

Answer

**Answer: D**

The area of the triangle with vertices (2, -6), (5, 4), and (k, 4) is given by the relation,

$$\begin{aligned} \Delta &= \frac{1}{2} \begin{vmatrix} 2 & -6 & 1 \\ 5 & 4 & 1 \\ k & 4 & 1 \end{vmatrix} \\ &= \frac{1}{2} [2(4-4) + 6(5-k) + 1(20-4k)] \\ &= \frac{1}{2} [30 - 6k + 20 - 4k] \\ &= \frac{1}{2} [50 - 10k] \\ &= 25 - 5k \end{aligned}$$

It is given that the area of the triangle is  $\pm 35$ .

Therefore, we have:

$$\Rightarrow 25 - 5k = \pm 35$$

$$\Rightarrow 5(5 - k) = \pm 35$$

$$\Rightarrow 5 - k = \pm 7$$

When  $5 - k = -7$ ,  $k = 5 + 7 = 12$ .

When  $5 - k = 7$ ,  $k = 5 - 7 = -2$ .

Hence,  $k = 12, -2$ .

The correct answer is D.

## Exercise 4.4

## Question 1:

Write Minors and Cofactors of the elements of following determinants:

$$(i) \begin{vmatrix} 2 & -4 \\ 0 & 3 \end{vmatrix} \quad (ii) \begin{vmatrix} a & c \\ b & d \end{vmatrix}$$

Answer

$$(i) \text{ The given determinant is } \begin{vmatrix} 2 & -4 \\ 0 & 3 \end{vmatrix}.$$

Minor of element  $a_{ij}$  is  $M_{ij}$ .

$$\therefore M_{11} = \text{minor of element } a_{11} = 3$$

$$M_{12} = \text{minor of element } a_{12} = 0$$

$$M_{21} = \text{minor of element } a_{21} = -4$$

$$M_{22} = \text{minor of element } a_{22} = 2$$

$$\text{Cofactor of } a_{ij} \text{ is } A_{ij} = (-1)^{i+j} M_{ij}.$$

$$\therefore A_{11} = (-1)^{1+1} M_{11} = (-1)^2 (3) = 3$$

$$A_{12} = (-1)^{1+2} M_{12} = (-1)^3 (0) = 0$$

$$A_{21} = (-1)^{2+1} M_{21} = (-1)^3 (-4) = 4$$

$$A_{22} = (-1)^{2+2} M_{22} = (-1)^4 (2) = 2$$

$$(ii) \text{ The given determinant is } \begin{vmatrix} a & c \\ b & d \end{vmatrix}.$$

Minor of element  $a_{ij}$  is  $M_{ij}$ .

$\therefore M_{11}$  = minor of element  $a_{11} = d$

$M_{12}$  = minor of element  $a_{12} = b$

$M_{21}$  = minor of element  $a_{21} = c$

$M_{22}$  = minor of element  $a_{22} = a$

Cofactor of  $a_{ij}$  is  $A_{ij} = (-1)^{i+j} M_{ij}$ .

$$\therefore A_{11} = (-1)^{1+1} M_{11} = (-1)^2 (d) = d$$

$$A_{12} = (-1)^{1+2} M_{12} = (-1)^3 (b) = -b$$

$$A_{21} = (-1)^{2+1} M_{21} = (-1)^3 (c) = -c$$

$$A_{22} = (-1)^{2+2} M_{22} = (-1)^4 (a) = a$$

### Question 2:

$$(i) \begin{vmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{vmatrix} \quad (ii) \begin{vmatrix} 1 & 0 & 4 \\ 3 & 5 & -1 \\ 0 & 1 & 2 \end{vmatrix}$$

Answer

$$(i) \text{ The given determinant is } \begin{vmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{vmatrix}.$$

By the definition of minors and cofactors, we have:

$$M_{11} = \text{minor of } a_{11} = \begin{vmatrix} 1 & 0 \\ 0 & 1 \end{vmatrix} = 1$$

$$M_{12} = \text{minor of } a_{12} = \begin{vmatrix} 0 & 0 \\ 0 & 1 \end{vmatrix} = 0$$

$$M_{13} = \text{minor of } a_{13} = \begin{vmatrix} 0 & 1 \\ 0 & 0 \end{vmatrix} = 0$$

$$M_{21} = \text{minor of } a_{21} = \begin{vmatrix} 0 & 0 \\ 0 & 1 \end{vmatrix} = 0$$

$$M_{22} = \text{minor of } a_{22} = \begin{vmatrix} 1 & 0 \\ 0 & 1 \end{vmatrix} = 1$$

$$M_{23} = \text{minor of } a_{23} = \begin{vmatrix} 1 & 0 \\ 0 & 0 \end{vmatrix} = 0$$

$$M_{31} = \text{minor of } a_{31} = \begin{vmatrix} 0 & 0 \\ 1 & 0 \end{vmatrix} = 0$$

$$M_{32} = \text{minor of } a_{32} = \begin{vmatrix} 1 & 0 \\ 0 & 0 \end{vmatrix} = 0$$

$$M_{33} = \text{minor of } a_{33} = \begin{vmatrix} 1 & 0 \\ 0 & 1 \end{vmatrix} = 1$$

$$A_{11} = \text{cofactor of } a_{11} = (-1)^{1+1} M_{11} = 1$$

$$A_{12} = \text{cofactor of } a_{12} = (-1)^{1+2} M_{12} = 0$$

$$A_{13} = \text{cofactor of } a_{13} = (-1)^{1+3} M_{13} = 0$$

$$A_{21} = \text{cofactor of } a_{21} = (-1)^{2+1} M_{21} = 0$$

$$A_{22} = \text{cofactor of } a_{22} = (-1)^{2+2} M_{22} = 1$$

$$A_{23} = \text{cofactor of } a_{23} = (-1)^{2+3} M_{23} = 0$$

$$A_{31} = \text{cofactor of } a_{31} = (-1)^{3+1} M_{31} = 0$$

$$A_{32} = \text{cofactor of } a_{32} = (-1)^{3+2} M_{32} = 0$$

$$A_{33} = \text{cofactor of } a_{33} = (-1)^{3+3} M_{33} = 1$$

$$(ii) \text{ The given determinant is } \begin{vmatrix} 1 & 0 & 4 \\ 3 & 5 & -1 \\ 0 & 1 & 2 \end{vmatrix}.$$

By definition of minors and cofactors, we have:

$$M_{11} = \text{minor of } a_{11} = \begin{vmatrix} 5 & -1 \\ 1 & 2 \end{vmatrix} = 10 + 1 = 11$$

$$M_{12} = \text{minor of } a_{12} = \begin{vmatrix} 3 & -1 \\ 0 & 2 \end{vmatrix} = 6 - 0 = 6$$

$$M_{13} = \text{minor of } a_{13} = \begin{vmatrix} 3 & 5 \\ 0 & 1 \end{vmatrix} = 3 - 0 = 3$$

$$M_{21} = \text{minor of } a_{21} = \begin{vmatrix} 0 & 4 \\ 1 & 2 \end{vmatrix} = 0 - 4 = -4$$

$$M_{22} = \text{minor of } a_{22} = \begin{vmatrix} 1 & 4 \\ 0 & 2 \end{vmatrix} = 2 - 0 = 2$$

$$M_{23} = \text{minor of } a_{23} = \begin{vmatrix} 1 & 0 \\ 0 & 1 \end{vmatrix} = 1 - 0 = 1$$

$$M_{31} = \text{minor of } a_{31} = \begin{vmatrix} 0 & 4 \\ 5 & -1 \end{vmatrix} = 0 - 20 = -20$$

$$M_{32} = \text{minor of } a_{32} = \begin{vmatrix} 1 & 4 \\ 3 & -1 \end{vmatrix} = -1 - 12 = -13$$

$$M_{33} = \text{minor of } a_{33} = \begin{vmatrix} 1 & 0 \\ 3 & 5 \end{vmatrix} = 5 - 0 = 5$$

$$A_{11} = \text{cofactor of } a_{11} = (-1)^{1+1} M_{11} = 11$$

$$A_{12} = \text{cofactor of } a_{12} = (-1)^{1+2} M_{12} = -6$$

$$A_{13} = \text{cofactor of } a_{13} = (-1)^{1+3} M_{13} = 3$$

$$A_{21} = \text{cofactor of } a_{21} = (-1)^{2+1} M_{21} = 4$$

$$A_{22} = \text{cofactor of } a_{22} = (-1)^{2+2} M_{22} = 2$$

$$A_{23} = \text{cofactor of } a_{23} = (-1)^{2+3} M_{23} = -1$$

$$A_{31} = \text{cofactor of } a_{31} = (-1)^{3+1} M_{31} = -20$$

$$A_{32} = \text{cofactor of } a_{32} = (-1)^{3+2} M_{32} = 13$$

$$A_{33} = \text{cofactor of } a_{33} = (-1)^{3+3} M_{33} = 5$$

**Question 3:**

$$\Delta = \begin{vmatrix} 5 & 3 & 8 \\ 2 & 0 & 1 \\ 1 & 2 & 3 \end{vmatrix}$$

Using Cofactors of elements of second row, evaluate

Answer

$$\begin{vmatrix} 5 & 3 & 8 \\ 2 & 0 & 1 \\ 1 & 2 & 3 \end{vmatrix}$$

The given determinant is

We have:

$$M_{21} = \begin{vmatrix} 3 & 8 \\ 2 & 3 \end{vmatrix} = 9 - 16 = -7$$

$$\therefore A_{21} = \text{cofactor of } a_{21} = (-1)^{2+1} M_{21} = 7$$

$$M_{22} = \begin{vmatrix} 5 & 8 \\ 1 & 3 \end{vmatrix} = 15 - 8 = 7$$

$$\therefore A_{22} = \text{cofactor of } a_{22} = (-1)^{2+2} M_{22} = 7$$

$$M_{23} = \begin{vmatrix} 5 & 3 \\ 1 & 2 \end{vmatrix} = 10 - 3 = 7$$

$$\therefore A_{23} = \text{cofactor of } a_{23} = (-1)^{2+3} M_{23} = -7$$

We know that  $\Delta$  is equal to the sum of the product of the elements of the second row with their corresponding cofactors.

$$\therefore \Delta = a_{21}A_{21} + a_{22}A_{22} + a_{23}A_{23} = 2(7) + 0(7) + 1(-7) = 14 - 7 = 7$$

**Question 4:**

$$\Delta = \begin{vmatrix} 1 & x & yz \\ 1 & y & zx \\ 1 & z & xy \end{vmatrix}$$

Using Cofactors of elements of third column, evaluate

Answer

$$\begin{vmatrix} 1 & x & yz \\ 1 & y & zx \\ 1 & z & xy \end{vmatrix}$$

The given determinant is

We have:

$$M_{13} = \begin{vmatrix} 1 & y \\ 1 & z \end{vmatrix} = z - y$$

$$M_{23} = \begin{vmatrix} 1 & x \\ 1 & z \end{vmatrix} = z - x$$

$$M_{33} = \begin{vmatrix} 1 & x \\ 1 & y \end{vmatrix} = y - x$$

$$\therefore A_{13} = \text{cofactor of } a_{13} = (-1)^{1+3} M_{13} = (z - y)$$

$$A_{23} = \text{cofactor of } a_{23} = (-1)^{2+3} M_{23} = -(z - x) = (x - z)$$

$$A_{33} = \text{cofactor of } a_{33} = (-1)^{3+3} M_{33} = (y - x)$$

We know that  $\Delta$  is equal to the sum of the product of the elements of the second row with their corresponding cofactors.

$$\begin{aligned}
 \therefore \Delta &= a_{13}A_{13} + a_{23}A_{23} + a_{33}A_{33} \\
 &= yz(z-y) + zx(x-z) + xy(y-x) \\
 &= yz^2 - y^2z + x^2z - xz^2 + xy^2 - x^2y \\
 &= (x^2z - y^2z) + (yz^2 - xz^2) + (xy^2 - x^2y) \\
 &= z(x^2 - y^2) + z^2(y - x) + xy(y - x) \\
 &= z(x - y)(x + y) + z^2(y - x) + xy(y - x) \\
 &= (x - y)[zx + zy - z^2 - xy] \\
 &= (x - y)[z(x - z) + y(z - x)] \\
 &= (x - y)(z - x)[-z + y] \\
 &= (x - y)(y - z)(z - x)
 \end{aligned}$$

Hence,  $\Delta = (x - y)(y - z)(z - x)$ .

**Question 5:**

For the matrices  $A$  and  $B$ , verify that  $(AB)' = B'A'$  where

(i)  $A = \begin{bmatrix} 1 \\ -4 \\ 3 \end{bmatrix}, B = \begin{bmatrix} -1 & 2 & 1 \end{bmatrix}$

(ii)  $A = \begin{bmatrix} 0 \\ 1 \\ 2 \end{bmatrix}, B = \begin{bmatrix} 1 & 5 & 7 \end{bmatrix}$

Answer

**(i)**

$$AB = \begin{bmatrix} 1 \\ -4 \\ 3 \end{bmatrix} \begin{bmatrix} -1 & 2 & 1 \end{bmatrix} = \begin{bmatrix} -1 & 2 & 1 \\ 4 & -8 & -4 \\ -3 & 6 & 3 \end{bmatrix}$$

$$\therefore (AB)' = \begin{bmatrix} -1 & 4 & -3 \\ 2 & -8 & 6 \\ 1 & -4 & 3 \end{bmatrix}$$

$$\text{Now, } A' = \begin{bmatrix} 1 & -4 & 3 \end{bmatrix}, B' = \begin{bmatrix} -1 \\ 2 \\ 1 \end{bmatrix}$$

$$\therefore B'A' = \begin{bmatrix} -1 \\ 2 \\ 1 \end{bmatrix} \begin{bmatrix} 1 & -4 & 3 \end{bmatrix} = \begin{bmatrix} -1 & 4 & -3 \\ 2 & -8 & 6 \\ 1 & -4 & 3 \end{bmatrix}$$

Hence, we have verified that  $(AB)' = B'A'$ .

**(ii)**

$$AB = \begin{bmatrix} 0 \\ 1 \\ 2 \end{bmatrix} \begin{bmatrix} 1 & 5 & 7 \end{bmatrix} = \begin{bmatrix} 0 & 0 & 0 \\ 1 & 5 & 7 \\ 2 & 10 & 14 \end{bmatrix}$$

$$\therefore (AB)' = \begin{bmatrix} 0 & 1 & 2 \\ 0 & 5 & 10 \\ 0 & 7 & 14 \end{bmatrix}$$

$$\text{Now, } A' = \begin{bmatrix} 0 & 1 & 2 \end{bmatrix}, B' = \begin{bmatrix} 1 \\ 5 \\ 7 \end{bmatrix}$$

$$\therefore B'A' = \begin{bmatrix} 1 \\ 5 \\ 7 \end{bmatrix} \begin{bmatrix} 0 & 1 & 2 \end{bmatrix} = \begin{bmatrix} 0 & 1 & 2 \\ 0 & 5 & 10 \\ 0 & 7 & 14 \end{bmatrix}$$

Hence, we have verified that  $(AB)' = B'A'$ .

**Exercise 4.5****Question 1:**

Find adjoint of each of the matrices.

$$\begin{bmatrix} 1 & 2 \\ 3 & 4 \end{bmatrix}$$

Answer

$$\text{Let } A = \begin{bmatrix} 1 & 2 \\ 3 & 4 \end{bmatrix}.$$

We have,

$$A_{11} = 4, A_{12} = -3, A_{21} = -2, A_{22} = 1$$

$$\therefore \text{adj}A = \begin{bmatrix} A_{11} & A_{21} \\ A_{12} & A_{22} \end{bmatrix} = \begin{bmatrix} 4 & -2 \\ -3 & 1 \end{bmatrix}$$

**Question 2:**

Find adjoint of each of the matrices.

$$\begin{bmatrix} 1 & -1 & 2 \\ 2 & 3 & 5 \\ -2 & 0 & 1 \end{bmatrix}$$

Answer

$$\text{Let } A = \begin{bmatrix} 1 & -1 & 2 \\ 2 & 3 & 5 \\ -2 & 0 & 1 \end{bmatrix}.$$

We have,

$$A_{11} = \begin{vmatrix} 3 & 5 \\ 0 & 1 \end{vmatrix} = 3 - 0 = 3$$

$$A_{12} = - \begin{vmatrix} 2 & 5 \\ -2 & 1 \end{vmatrix} = -(2 + 10) = -12$$

$$A_{13} = \begin{vmatrix} 2 & 3 \\ -2 & 0 \end{vmatrix} = 0 + 6 = 6$$

$$A_{21} = - \begin{vmatrix} -1 & 2 \\ 0 & 1 \end{vmatrix} = -(-1-0) = 1$$

$$A_{22} = \begin{vmatrix} 1 & 2 \\ -2 & 1 \end{vmatrix} = 1+4 = 5$$

$$A_{23} = - \begin{vmatrix} 1 & -1 \\ -2 & 0 \end{vmatrix} = -(0-2) = 2$$

$$A_{31} = \begin{vmatrix} -1 & 2 \\ 3 & 5 \end{vmatrix} = -5-6 = -11$$

$$A_{32} = - \begin{vmatrix} 1 & 2 \\ 2 & 5 \end{vmatrix} = -(5-4) = -1$$

$$A_{33} = \begin{vmatrix} 1 & -1 \\ 2 & 3 \end{vmatrix} = 3+2 = 5$$

$$\text{Hence, } \text{adj} A = \begin{bmatrix} A_{11} & A_{21} & A_{31} \\ A_{12} & A_{22} & A_{32} \\ A_{13} & A_{23} & A_{33} \end{bmatrix} = \begin{bmatrix} 3 & 1 & -11 \\ -12 & 5 & -1 \\ 6 & 2 & 5 \end{bmatrix}.$$

**Question 3:**

Verify  $A (\text{adj } A) = (\text{adj } A) A = |A| I$ .

$$\begin{bmatrix} 2 & 3 \\ -4 & -6 \end{bmatrix}$$

Answer

$$A = \begin{bmatrix} 2 & 3 \\ -4 & -6 \end{bmatrix}$$

we have,

$$|A| = -12 - (-12) = -12 + 12 = 0$$

$$\therefore |A|I = 0 \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix} = \begin{bmatrix} 0 & 0 \\ 0 & 0 \end{bmatrix}$$

Now,

$$A_{11} = -6, A_{12} = 4, A_{21} = -3, A_{22} = 2$$

$$\therefore \text{adj}A = \begin{bmatrix} -6 & -3 \\ 4 & 2 \end{bmatrix}$$

Now,

$$\begin{aligned} A(\text{adj}A) &= \begin{bmatrix} 2 & 3 \\ -4 & -6 \end{bmatrix} \begin{bmatrix} -6 & -3 \\ 4 & 2 \end{bmatrix} \\ &= \begin{bmatrix} -12+12 & -6+6 \\ 24-24 & 12-12 \end{bmatrix} = \begin{bmatrix} 0 & 0 \\ 0 & 0 \end{bmatrix} \end{aligned}$$

$$\begin{aligned} \text{Also, } (\text{adj}A)A &= \begin{bmatrix} -6 & -3 \\ 4 & 2 \end{bmatrix} \begin{bmatrix} 2 & 3 \\ -4 & -6 \end{bmatrix} \\ &= \begin{bmatrix} -12+12 & -18+18 \\ 8-8 & 12-12 \end{bmatrix} = \begin{bmatrix} 0 & 0 \\ 0 & 0 \end{bmatrix} \end{aligned}$$

$$\text{Hence, } A(\text{adj}A) = (\text{adj}A)A = |A|I.$$

#### Question 4:

Verify  $A(\text{adj}A) = (\text{adj}A)A = |A|I$ .

$$\begin{bmatrix} 1 & -1 & 2 \\ 3 & 0 & -2 \\ 1 & 0 & 3 \end{bmatrix}$$

Answer

$$A = \begin{bmatrix} 1 & -1 & 2 \\ 3 & 0 & -2 \\ 1 & 0 & 3 \end{bmatrix}$$

$$|A| = 1(0-0) + 1(9+2) + 2(0-0) = 11$$

$$\therefore |A|I = 11 \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix} = \begin{bmatrix} 11 & 0 & 0 \\ 0 & 11 & 0 \\ 0 & 0 & 11 \end{bmatrix}$$

Now,

$$A_{11} = 0, A_{12} = -(9+2) = -11, A_{13} = 0$$

$$A_{21} = -(-3-0) = 3, A_{22} = 3-2 = 1, A_{23} = -(0+1) = -1$$

$$A_{31} = 2-0 = 2, A_{32} = -(-2-6) = 8, A_{33} = 0+3 = 3$$

$$\therefore \text{adj}A = \begin{bmatrix} 0 & 3 & 2 \\ -11 & 1 & 8 \\ 0 & -1 & 3 \end{bmatrix}$$

Now,

$$\begin{aligned} A(\text{adj}A) &= \begin{bmatrix} 1 & -1 & 2 \\ 3 & 0 & -2 \\ 1 & 0 & 3 \end{bmatrix} \begin{bmatrix} 0 & 3 & 2 \\ -11 & 1 & 8 \\ 0 & -1 & 3 \end{bmatrix} \\ &= \begin{bmatrix} 0+11+0 & 3-1-2 & 2-8+6 \\ 0+0+0 & 9+0+2 & 6+0-6 \\ 0+0+0 & 3+0-3 & 2+0+9 \end{bmatrix} \\ &= \begin{bmatrix} 11 & 0 & 0 \\ 0 & 11 & 0 \\ 0 & 0 & 11 \end{bmatrix} \end{aligned}$$

Also,

$$\begin{aligned} (\text{adj}A) \cdot A &= \begin{bmatrix} 0 & 3 & 2 \\ -11 & 1 & 8 \\ 0 & -1 & 3 \end{bmatrix} \begin{bmatrix} 1 & -1 & 2 \\ 3 & 0 & -2 \\ 1 & 0 & 3 \end{bmatrix} \\ &= \begin{bmatrix} 0+9+2 & 0+0+0 & 0-6+6 \\ -11+3+8 & 11+0+0 & -22-2+24 \\ 0-3+3 & 0+0+0 & 0+2+9 \end{bmatrix} \\ &= \begin{bmatrix} 11 & 0 & 0 \\ 0 & 11 & 0 \\ 0 & 0 & 11 \end{bmatrix} \end{aligned}$$

Hence,  $A(\text{adj}A) = (\text{adj}A)A = |A|I$ .

#### Question 6:

Find the inverse of each of the matrices (if it exists).

$$\begin{bmatrix} -1 & 5 \\ -3 & 2 \end{bmatrix}$$

Answer

$$\text{Let } A = \begin{bmatrix} -1 & 5 \\ -3 & 2 \end{bmatrix}.$$

we have,

$$|A| = -2 + 15 = 13$$

Now,

$$A_{11} = 2, A_{12} = 3, A_{21} = -5, A_{22} = -1$$

$$\therefore \text{adj}A = \begin{bmatrix} 2 & -5 \\ 3 & -1 \end{bmatrix}$$

$$\therefore A^{-1} = \frac{1}{|A|} \text{adj}A = \frac{1}{13} \begin{bmatrix} 2 & -5 \\ 3 & -1 \end{bmatrix}$$

**Question 7:**

Find the inverse of each of the matrices (if it exists).

$$\begin{bmatrix} 1 & 2 & 3 \\ 0 & 2 & 4 \\ 0 & 0 & 5 \end{bmatrix}$$

Answer

$$\text{Let } A = \begin{bmatrix} 1 & 2 & 3 \\ 0 & 2 & 4 \\ 0 & 0 & 5 \end{bmatrix}.$$

We have,

$$|A| = 1(10 - 0) - 2(0 - 0) + 3(0 - 0) = 10$$

Now,

$$A_{11} = 10 - 0 = 10, A_{12} = -(0 - 0) = 0, A_{13} = 0 - 0 = 0$$

$$A_{21} = -(10 - 0) = -10, A_{22} = 5 - 0 = 5, A_{23} = -(0 - 0) = 0$$

$$A_{31} = 8 - 6 = 2, A_{32} = -(4 - 0) = -4, A_{33} = 2 - 0 = 2$$

$$\therefore \text{adj}A = \begin{bmatrix} 10 & -10 & 2 \\ 0 & 5 & -4 \\ 0 & 0 & 2 \end{bmatrix}$$

$$\therefore A^{-1} = \frac{1}{|A|} \text{adj}A = \frac{1}{10} \begin{bmatrix} 10 & -10 & 2 \\ 0 & 5 & -4 \\ 0 & 0 & 2 \end{bmatrix}$$

**Question 8:**

Find the inverse of each of the matrices (if it exists).

$$\begin{bmatrix} 1 & 0 & 0 \\ 3 & 3 & 0 \\ 5 & 2 & -1 \end{bmatrix}$$

Answer

$$\text{Let } A = \begin{bmatrix} 1 & 0 & 0 \\ 3 & 3 & 0 \\ 5 & 2 & -1 \end{bmatrix}.$$

We have,

$$|A| = 1(-3-0) - 0 + 0 = -3$$

Now,

$$A_{11} = -3 - 0 = -3, A_{12} = -(-3 - 0) = 3, A_{13} = 6 - 15 = -9$$

$$A_{21} = -(0 - 0) = 0, A_{22} = -1 - 0 = -1, A_{23} = -(2 - 0) = -2$$

$$A_{31} = 0 - 0 = 0, A_{32} = -(0 - 0) = 0, A_{33} = 3 - 0 = 3$$

$$\therefore \text{adj}A = \begin{bmatrix} -3 & 0 & 0 \\ 3 & -1 & 0 \\ -9 & -2 & 3 \end{bmatrix}$$

$$\therefore A^{-1} = \frac{1}{|A|} \text{adj}A = -\frac{1}{3} \begin{bmatrix} -3 & 0 & 0 \\ 3 & -1 & 0 \\ -9 & -2 & 3 \end{bmatrix}$$

**Question 9:**

Find the inverse of each of the matrices (if it exists).

$$\begin{bmatrix} 2 & 1 & 3 \\ 4 & -1 & 0 \\ -7 & 2 & 1 \end{bmatrix}$$

Answer

$$\text{Let } A = \begin{bmatrix} 2 & 1 & 3 \\ 4 & -1 & 0 \\ -7 & 2 & 1 \end{bmatrix}.$$

We have,

$$\begin{aligned} |A| &= 2(-1-0) - 1(4-0) + 3(8-7) \\ &= 2(-1) - 1(4) + 3(1) \\ &= -2 - 4 + 3 \\ &= -3 \end{aligned}$$

Now,

$$\begin{aligned} A_{11} &= -1 - 0 = -1, A_{12} = -(4 - 0) = -4, A_{13} = 8 - 7 = 1 \\ A_{21} &= -(1 - 6) = 5, A_{22} = 2 + 21 = 23, A_{23} = -(4 + 7) = -11 \\ A_{31} &= 0 + 3 = 3, A_{32} = -(0 - 12) = 12, A_{33} = -2 - 4 = -6 \end{aligned}$$

$$\therefore \text{adj}A = \begin{bmatrix} -1 & 5 & 3 \\ -4 & 23 & 12 \\ 1 & -11 & -6 \end{bmatrix}$$

$$\therefore A^{-1} = \frac{1}{|A|} \text{adj}A = -\frac{1}{3} \begin{bmatrix} -1 & 5 & 3 \\ -4 & 23 & 12 \\ 1 & -11 & -6 \end{bmatrix}$$

**Question 10:**

Find the inverse of each of the matrices (if it exists).

$$\begin{bmatrix} 1 & -1 & 2 \\ 0 & 2 & -3 \\ 3 & -2 & 4 \end{bmatrix}.$$

Answer

$$\text{Let } A = \begin{bmatrix} 1 & -1 & 2 \\ 0 & 2 & -3 \\ 3 & -2 & 4 \end{bmatrix}.$$

By expanding along  $C_1$ , we have:

$$|A| = 1(8-6) - 0 + 3(3-4) = 2 - 3 = -1$$

Now,

$$A_{11} = 8 - 6 = 2, A_{12} = -(0 + 9) = -9, A_{13} = 0 - 6 = -6$$

$$A_{21} = -(-4 + 4) = 0, A_{22} = 4 - 6 = -2, A_{23} = -(-2 + 3) = -1$$

$$A_{31} = 3 - 4 = -1, A_{32} = -(-3 - 0) = 3, A_{33} = 2 - 0 = 2$$

$$\therefore \text{adj}A = \begin{bmatrix} 2 & 0 & -1 \\ -9 & -2 & 3 \\ -6 & -1 & 2 \end{bmatrix}$$

$$\therefore A^{-1} = \frac{1}{|A|} \text{adj}A = - \begin{bmatrix} 2 & 0 & -1 \\ -9 & -2 & 3 \\ -6 & -1 & 2 \end{bmatrix} = \begin{bmatrix} -2 & 0 & 1 \\ 9 & 2 & -3 \\ 6 & 1 & -2 \end{bmatrix}$$

**Question 11:**

Find the inverse of each of the matrices (if it exists).

$$\begin{bmatrix} 1 & 0 & 0 \\ 0 & \cos \alpha & \sin \alpha \\ 0 & \sin \alpha & -\cos \alpha \end{bmatrix}$$

Answer

$$\text{Let } A = \begin{bmatrix} 1 & 0 & 0 \\ 0 & \cos \alpha & \sin \alpha \\ 0 & \sin \alpha & -\cos \alpha \end{bmatrix}.$$

We have,

$$|A| = 1(-\cos^2 \alpha - \sin^2 \alpha) = -(\cos^2 \alpha + \sin^2 \alpha) = -1$$

Now,

$$A_{11} = -\cos^2 \alpha - \sin^2 \alpha = -1, A_{12} = 0, A_{13} = 0$$

$$A_{21} = 0, A_{22} = -\cos \alpha, A_{23} = -\sin \alpha$$

$$A_{31} = 0, A_{32} = -\sin \alpha, A_{33} = \cos \alpha$$

$$\therefore \text{adj}A = \begin{bmatrix} -1 & 0 & 0 \\ 0 & -\cos \alpha & -\sin \alpha \\ 0 & -\sin \alpha & \cos \alpha \end{bmatrix}$$

$$\therefore A^{-1} = \frac{1}{|A|} \cdot \text{adj}A = - \begin{bmatrix} -1 & 0 & 0 \\ 0 & -\cos \alpha & -\sin \alpha \\ 0 & -\sin \alpha & \cos \alpha \end{bmatrix} = \begin{bmatrix} 1 & 0 & 0 \\ 0 & \cos \alpha & \sin \alpha \\ 0 & \sin \alpha & -\cos \alpha \end{bmatrix}$$

#### Question 12:

$$\text{Let } A = \begin{bmatrix} 3 & 7 \\ 2 & 5 \end{bmatrix} \text{ and } B = \begin{bmatrix} 6 & 8 \\ 7 & 9 \end{bmatrix}. \text{ Verify that } (AB)^{-1} = B^{-1}A^{-1}$$

Answer

$$\text{Let } A = \begin{bmatrix} 3 & 7 \\ 2 & 5 \end{bmatrix}.$$

We have,

$$|A| = 15 - 14 = 1$$

Now,

$$A_{11} = 5, A_{12} = -2, A_{21} = -7, A_{22} = 3$$

$$\therefore \text{adj}A = \begin{bmatrix} 5 & -7 \\ -2 & 3 \end{bmatrix}$$

$$\therefore A^{-1} = \frac{1}{|A|} \cdot \text{adj}A = \begin{bmatrix} 5 & -7 \\ -2 & 3 \end{bmatrix}$$

Now, let  $B = \begin{bmatrix} 6 & 8 \\ 7 & 9 \end{bmatrix}$ .

We have,

$$|B| = 54 - 56 = -2$$

$$\therefore \text{adj}B = \begin{bmatrix} 9 & -8 \\ -7 & 6 \end{bmatrix}$$

$$\therefore B^{-1} = \frac{1}{|B|} \text{adj}B = -\frac{1}{2} \begin{bmatrix} 9 & -8 \\ -7 & 6 \end{bmatrix} = \begin{bmatrix} -\frac{9}{2} & 4 \\ \frac{7}{2} & -3 \end{bmatrix}$$

Now,

$$\begin{aligned} B^{-1}A^{-1} &= \begin{bmatrix} -\frac{9}{2} & 4 \\ \frac{7}{2} & -3 \end{bmatrix} \begin{bmatrix} 5 & -7 \\ -2 & 3 \end{bmatrix} \\ &= \begin{bmatrix} -\frac{45}{2} - 8 & \frac{63}{2} + 12 \\ \frac{35}{2} + 6 & -\frac{49}{2} - 9 \end{bmatrix} = \begin{bmatrix} -\frac{61}{2} & \frac{87}{2} \\ \frac{47}{2} & -\frac{67}{2} \end{bmatrix} \quad \dots(1) \end{aligned}$$

Then,

$$\begin{aligned} AB &= \begin{bmatrix} 3 & 7 \\ 2 & 5 \end{bmatrix} \begin{bmatrix} 6 & 8 \\ 7 & 9 \end{bmatrix} \\ &= \begin{bmatrix} 18+49 & 24+63 \\ 12+35 & 16+45 \end{bmatrix} \\ &= \begin{bmatrix} 67 & 87 \\ 47 & 61 \end{bmatrix} \end{aligned}$$

Therefore, we have  $|AB| = 67 \times 61 - 87 \times 47 = 4087 - 4089 = -2$ .

Also,

$$\begin{aligned} \text{adj}(AB) &= \begin{bmatrix} 61 & -87 \\ -47 & 67 \end{bmatrix} \\ \therefore (AB)^{-1} &= \frac{1}{|AB|} \text{adj}(AB) = -\frac{1}{2} \begin{bmatrix} 61 & -87 \\ -47 & 67 \end{bmatrix} \\ &= \begin{bmatrix} -\frac{61}{2} & \frac{87}{2} \\ \frac{47}{2} & -\frac{67}{2} \end{bmatrix} \quad \dots(2) \end{aligned}$$

From (1) and (2), we have:

$$(AB)^{-1} = B^{-1}A^{-1}$$

Hence, the given result is proved.

### Question 13:

If  $A = \begin{bmatrix} 3 & 1 \\ -1 & 2 \end{bmatrix}$ , show that  $A^2 - 5A + 7I = O$ . Hence find  $A^{-1}$ .

Answer

$$A = \begin{bmatrix} 3 & 1 \\ -1 & 2 \end{bmatrix}$$

$$A^2 = A \cdot A = \begin{bmatrix} 3 & 1 \\ -1 & 2 \end{bmatrix} \begin{bmatrix} 3 & 1 \\ -1 & 2 \end{bmatrix} = \begin{bmatrix} 9-1 & 3+2 \\ -3-2 & -1+4 \end{bmatrix} = \begin{bmatrix} 8 & 5 \\ -5 & 3 \end{bmatrix}$$

$$\therefore A^2 - 5A + 7I$$

$$= \begin{bmatrix} 8 & 5 \\ -5 & 3 \end{bmatrix} - 5 \begin{bmatrix} 3 & 1 \\ -1 & 2 \end{bmatrix} + 7 \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}$$

$$= \begin{bmatrix} 8 & 5 \\ -5 & 3 \end{bmatrix} - \begin{bmatrix} 15 & 5 \\ -5 & 10 \end{bmatrix} + \begin{bmatrix} 7 & 0 \\ 0 & 7 \end{bmatrix}$$

$$= \begin{bmatrix} -7 & 0 \\ 0 & -7 \end{bmatrix} + \begin{bmatrix} 7 & 0 \\ 0 & 7 \end{bmatrix} = \begin{bmatrix} 0 & 0 \\ 0 & 0 \end{bmatrix}$$

Hence,  $A^2 - 5A + 7I = O$ .

$$\therefore A \cdot A - 5A = -7I$$

$$\Rightarrow A \cdot A(A^{-1}) - 5AA^{-1} = -7IA^{-1} \quad \left[ \text{Post-multiplying by } A^{-1} \text{ as } |A| \neq 0 \right]$$

$$\Rightarrow A(AA^{-1}) - 5I = -7A^{-1}$$

$$\Rightarrow AI - 5I = -7A^{-1}$$

$$\Rightarrow A^{-1} = -\frac{1}{7}(A - 5I)$$

$$\Rightarrow A^{-1} = \frac{1}{7}(5I - A)$$

$$= \frac{1}{7} \left( \begin{bmatrix} 5 & 0 \\ 0 & 5 \end{bmatrix} - \begin{bmatrix} 3 & 1 \\ -1 & 2 \end{bmatrix} \right) = \frac{1}{7} \begin{bmatrix} 2 & -1 \\ 1 & 3 \end{bmatrix}$$

$$\therefore A^{-1} = \frac{1}{7} \begin{bmatrix} 2 & -1 \\ 1 & 3 \end{bmatrix}$$

**Question 14:**

$$A = \begin{bmatrix} 3 & 2 \\ 1 & 1 \end{bmatrix}$$

For the matrix  $A = \begin{bmatrix} 3 & 2 \\ 1 & 1 \end{bmatrix}$ , find the numbers  $a$  and  $b$  such that  $A^2 + aA + bI = O$ .

Answer

$$A = \begin{bmatrix} 3 & 2 \\ 1 & 1 \end{bmatrix}$$

$$\therefore A^2 = \begin{bmatrix} 3 & 2 \\ 1 & 1 \end{bmatrix} \begin{bmatrix} 3 & 2 \\ 1 & 1 \end{bmatrix} = \begin{bmatrix} 9+2 & 6+2 \\ 3+1 & 2+1 \end{bmatrix} = \begin{bmatrix} 11 & 8 \\ 4 & 3 \end{bmatrix}$$

Now,

$$A^2 + aA + bI = O$$

$$\Rightarrow (AA)A^{-1} + aAA^{-1} + bIA^{-1} = O \quad \left[ \text{Post-multiplying by } A^{-1} \text{ as } |A| \neq 0 \right]$$

$$\Rightarrow A(AA^{-1}) + aI + b(AA^{-1}) = O$$

$$\Rightarrow AI + aI + bA^{-1} = O$$

$$\Rightarrow A + aI = -bA^{-1}$$

$$\Rightarrow A^{-1} = -\frac{1}{b}(A + aI)$$

Now,

$$A^{-1} = \frac{1}{|A|} \text{adj}A = \frac{1}{1} \begin{bmatrix} 1 & -2 \\ -1 & 3 \end{bmatrix} = \begin{bmatrix} 1 & -2 \\ -1 & 3 \end{bmatrix}$$

We have:

$$\begin{bmatrix} 1 & -2 \\ -1 & 3 \end{bmatrix} = -\frac{1}{b} \left( \begin{bmatrix} 3 & 2 \\ 1 & 1 \end{bmatrix} + \begin{bmatrix} a & 0 \\ 0 & a \end{bmatrix} \right) = -\frac{1}{b} \begin{bmatrix} 3+a & 2 \\ 1 & 1+a \end{bmatrix} = \begin{bmatrix} \frac{-3-a}{b} & -\frac{2}{b} \\ -\frac{1}{b} & \frac{-1-a}{b} \end{bmatrix}$$

Comparing the corresponding elements of the two matrices, we have:

$$-\frac{1}{b} = -1 \Rightarrow b = 1$$

$$\frac{-3-a}{b} = 1 \Rightarrow -3-a = 1 \Rightarrow a = -4$$

Hence,  $-4$  and  $1$  are the required values of  $a$  and  $b$  respectively.

**Question 15:**

$$A = \begin{bmatrix} 1 & 1 & 1 \\ 1 & 2 & -3 \\ 2 & -1 & 3 \end{bmatrix}$$

For the matrix  $A$  show that  $A^3 - 6A^2 + 5A + 11I = O$ . Hence, find  $A^{-1}$ .

**Answer**

$$A = \begin{bmatrix} 1 & 1 & 1 \\ 1 & 2 & -3 \\ 2 & -1 & 3 \end{bmatrix}$$

$$A^2 = \begin{bmatrix} 1 & 1 & 1 \\ 1 & 2 & -3 \\ 2 & -1 & 3 \end{bmatrix} \begin{bmatrix} 1 & 1 & 1 \\ 1 & 2 & -3 \\ 2 & -1 & 3 \end{bmatrix}$$

$$= \begin{bmatrix} 1+1+2 & 1+2-1 & 1-3+3 \\ 1+2-6 & 1+4+3 & 1-6-9 \\ 2-1+6 & 2-2-3 & 2+3+9 \end{bmatrix} = \begin{bmatrix} 4 & 2 & 1 \\ -3 & 8 & -14 \\ 7 & -3 & 14 \end{bmatrix}$$

$$\begin{aligned} A^3 &= A^2 \cdot A = \begin{bmatrix} 4 & 2 & 1 \\ -3 & 8 & -14 \\ 7 & -3 & 14 \end{bmatrix} \begin{bmatrix} 1 & 1 & 1 \\ 1 & 2 & -3 \\ 2 & -1 & 3 \end{bmatrix} \\ &= \begin{bmatrix} 4+2+2 & 4+4-1 & 4-6+3 \\ -3+8-28 & -3+16+14 & -3-24-42 \\ 7-3+28 & 7-6-14 & 7+9+42 \end{bmatrix} \\ &= \begin{bmatrix} 8 & 7 & 1 \\ -23 & 27 & -69 \\ 32 & -13 & 58 \end{bmatrix} \end{aligned}$$

$$\therefore A^3 - 6A^2 + 5A + 11I$$

$$= \begin{bmatrix} 8 & 7 & 1 \\ -23 & 27 & -69 \\ 32 & -13 & 58 \end{bmatrix} - 6 \begin{bmatrix} 4 & 2 & 1 \\ -3 & 8 & -14 \\ 7 & -3 & 14 \end{bmatrix} + 5 \begin{bmatrix} 1 & 1 & 1 \\ 1 & 2 & -3 \\ 2 & -1 & 3 \end{bmatrix} + 11 \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}$$

$$= \begin{bmatrix} 8 & 7 & 1 \\ -23 & 27 & -69 \\ 32 & -13 & 58 \end{bmatrix} - \begin{bmatrix} 24 & 12 & 6 \\ -18 & 48 & -84 \\ 42 & -18 & 84 \end{bmatrix} + \begin{bmatrix} 5 & 5 & 5 \\ 5 & 10 & -15 \\ 10 & -5 & 15 \end{bmatrix} + \begin{bmatrix} 11 & 0 & 0 \\ 0 & 11 & 0 \\ 0 & 0 & 11 \end{bmatrix}$$

$$= \begin{bmatrix} 24 & 12 & 6 \\ -18 & 48 & -84 \\ 42 & -18 & 84 \end{bmatrix} - \begin{bmatrix} 24 & 12 & 6 \\ -18 & 48 & -84 \\ 42 & -18 & 84 \end{bmatrix}$$

$$= \begin{bmatrix} 0 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix} = O$$

$$\text{Thus, } A^3 - 6A^2 + 5A + 11I = O.$$

Now,

$$A^3 - 6A^2 + 5A + 11I = O$$

$$\Rightarrow (AAA)A^{-1} - 6(AA)A^{-1} + 5AA^{-1} + 11IA^{-1} = 0 \quad [\text{Post-multiplying by } A^{-1} \text{ as } |A| \neq 0]$$

$$\Rightarrow AA(AA^{-1}) - 6A(AA^{-1}) + 5(AA^{-1}) = -11(A^{-1})$$

$$\Rightarrow A^2 - 6A + 5I = -11A^{-1}$$

$$\Rightarrow A^{-1} = -\frac{1}{11}(A^2 - 6A + 5I) \quad \dots(1)$$

Now,

$$A^2 - 6A + 5I$$

$$\begin{aligned}
 &= \begin{bmatrix} 4 & 2 & 1 \\ -3 & 8 & -14 \\ 7 & -3 & 14 \end{bmatrix} - 6 \begin{bmatrix} 1 & 1 & 1 \\ 1 & 2 & -3 \\ 2 & -1 & 3 \end{bmatrix} + 5 \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix} \\
 &= \begin{bmatrix} 4 & 2 & 1 \\ -3 & 8 & -14 \\ 7 & -3 & 14 \end{bmatrix} - \begin{bmatrix} 6 & 6 & 6 \\ 6 & 12 & -18 \\ 12 & -6 & 18 \end{bmatrix} + \begin{bmatrix} 5 & 0 & 0 \\ 0 & 5 & 0 \\ 0 & 0 & 5 \end{bmatrix} \\
 &= \begin{bmatrix} 9 & 2 & 1 \\ -3 & 13 & -14 \\ 7 & -3 & 19 \end{bmatrix} - \begin{bmatrix} 6 & 6 & 6 \\ 6 & 12 & -18 \\ 12 & -6 & 18 \end{bmatrix} \\
 &= \begin{bmatrix} 3 & -4 & -5 \\ -9 & 1 & 4 \\ -5 & 3 & 1 \end{bmatrix}
 \end{aligned}$$

From equation (1), we have:

$$A^{-1} = -\frac{1}{11} \begin{bmatrix} 3 & -4 & -5 \\ -9 & 1 & 4 \\ -5 & 3 & 1 \end{bmatrix} = \frac{1}{11} \begin{bmatrix} -3 & 4 & 5 \\ 9 & -1 & -4 \\ 5 & -3 & -1 \end{bmatrix}$$

**Question 16:**

$$A = \begin{bmatrix} 2 & -1 & 1 \\ -1 & 2 & -1 \\ 1 & -1 & 2 \end{bmatrix}$$

If  $\begin{bmatrix} 2 & -1 & 1 \\ -1 & 2 & -1 \\ 1 & -1 & 2 \end{bmatrix}$  verify that  $A^3 - 6A^2 + 9A - 4I = O$  and hence find  $A^{-1}$

Answer

$$A = \begin{bmatrix} 2 & -1 & 1 \\ -1 & 2 & -1 \\ 1 & -1 & 2 \end{bmatrix}$$

$$A^2 = \begin{bmatrix} 2 & -1 & 1 \\ -1 & 2 & -1 \\ 1 & -1 & 2 \end{bmatrix} \begin{bmatrix} 2 & -1 & 1 \\ -1 & 2 & -1 \\ 1 & -1 & 2 \end{bmatrix}$$

$$= \begin{bmatrix} 4+1+1 & -2-2-1 & 2+1+2 \\ -2-2-1 & 1+4+1 & -1-2-2 \\ 2+1+2 & -1-2-2 & 1+1+4 \end{bmatrix}$$

$$= \begin{bmatrix} 6 & -5 & 5 \\ -5 & 6 & -5 \\ 5 & -5 & 6 \end{bmatrix}$$

$$A^3 = A^2 A = \begin{bmatrix} 6 & -5 & 5 \\ -5 & 6 & -5 \\ 5 & -5 & 6 \end{bmatrix} \begin{bmatrix} 2 & -1 & 1 \\ -1 & 2 & -1 \\ 1 & -1 & 2 \end{bmatrix}$$

$$= \begin{bmatrix} 12+5+5 & -6-10-5 & 6+5+10 \\ -10-6-5 & 5+12+5 & -5-6-10 \\ 10+5+6 & -5-10-6 & 5+5+12 \end{bmatrix}$$

$$= \begin{bmatrix} 22 & -21 & 21 \\ -21 & 22 & -21 \\ 21 & -21 & 22 \end{bmatrix}$$

Now,

$$A^3 - 6A^2 + 9A - 4I$$

$$= \begin{bmatrix} 22 & -21 & 21 \\ -21 & 22 & -21 \\ 21 & -21 & 22 \end{bmatrix} - 6 \begin{bmatrix} 6 & -5 & 5 \\ -5 & 6 & -5 \\ 5 & -5 & 6 \end{bmatrix} + 9 \begin{bmatrix} 2 & -1 & 1 \\ -1 & 2 & -1 \\ 1 & -1 & 2 \end{bmatrix} - 4 \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}$$

$$= \begin{bmatrix} 22 & -21 & 21 \\ -21 & 22 & -21 \\ 21 & -21 & 22 \end{bmatrix} - \begin{bmatrix} 36 & -30 & 30 \\ -30 & 36 & -30 \\ 30 & -30 & 36 \end{bmatrix} + \begin{bmatrix} 18 & -9 & 9 \\ -9 & 18 & -9 \\ 9 & -9 & 18 \end{bmatrix} - \begin{bmatrix} 4 & 0 & 0 \\ 0 & 4 & 0 \\ 0 & 0 & 4 \end{bmatrix}$$

$$= \begin{bmatrix} 40 & -30 & 30 \\ -30 & 40 & -30 \\ 30 & -30 & 40 \end{bmatrix} - \begin{bmatrix} 40 & -30 & 30 \\ -30 & 40 & -30 \\ 30 & -30 & 40 \end{bmatrix} = \begin{bmatrix} 0 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix}$$

$$\therefore A^3 - 6A^2 + 9A - 4I = O$$

Now,

$$A^3 - 6A^2 + 9A - 4I = O$$

$$\Rightarrow (AAA)A^{-1} - 6(AA)A^{-1} + 9AA^{-1} - 4IA^{-1} = O$$

[Post-multiplying by  $A^{-1}$  as  $|A| \neq 0$ ]

$$\Rightarrow AA(AA^{-1}) - 6A(AA^{-1}) + 9(AA^{-1}) = 4(AA^{-1})$$

$$\Rightarrow AAI - 6AI + 9I = 4A^{-1}$$

$$\Rightarrow A^2 - 6A + 9I = 4A^{-1}$$

$$\Rightarrow A^{-1} = \frac{1}{4}(A^2 - 6A + 9I) \quad \dots(1)$$

$$A^2 - 6A + 9I$$

$$= \begin{bmatrix} 6 & -5 & 5 \\ -5 & 6 & -5 \\ 5 & -5 & 6 \end{bmatrix} - 6 \begin{bmatrix} 2 & -1 & 1 \\ -1 & 2 & -1 \\ 1 & -1 & 2 \end{bmatrix} + 9 \begin{bmatrix} 0 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix}$$

$$= \begin{bmatrix} 6 & -5 & 5 \\ -5 & 6 & -5 \\ 5 & -5 & 6 \end{bmatrix} - \begin{bmatrix} 12 & -6 & 6 \\ -6 & 12 & -6 \\ 6 & -6 & 12 \end{bmatrix} + \begin{bmatrix} 9 & 0 & 0 \\ 0 & 9 & 0 \\ 0 & 0 & 9 \end{bmatrix}$$

$$= \begin{bmatrix} 3 & 1 & -1 \\ 1 & 3 & 1 \\ -1 & 1 & 3 \end{bmatrix}$$

From equation (1), we have:

$$A^{-1} = \frac{1}{4} \begin{bmatrix} 3 & 1 & -1 \\ 1 & 3 & 1 \\ -1 & 1 & 3 \end{bmatrix}$$

**Question 17:**

Let  $A$  be a nonsingular square matrix of order  $3 \times 3$ . Then  $|adj A|$  is equal to

- A.**  $|A|$  **B.**  $|A|^2$  **C.**  $|A|^3$  **D.**  $3|A|$

Answer **B**

We know that,

$$\begin{aligned} (adj A) A &= |A| I = \begin{bmatrix} |A| & 0 & 0 \\ 0 & |A| & 0 \\ 0 & 0 & |A| \end{bmatrix} \\ \Rightarrow |(adj A) A| &= \begin{vmatrix} |A| & 0 & 0 \\ 0 & |A| & 0 \\ 0 & 0 & |A| \end{vmatrix} \\ \Rightarrow |adj A| |A| &= |A|^3 \begin{vmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{vmatrix} = |A|^3 (I) \\ \therefore |adj A| &= |A|^2 \end{aligned}$$

Hence, the correct answer is B.

**Question 18:**

If  $A$  is an invertible matrix of order 2, then  $\det(A^{-1})$  is equal to

- A.**  $\det(A)$  **B.**  $\frac{1}{\det(A)}$  **C.** 1 **D.** 0

Answer

$$A^{-1} \text{ exists and } A^{-1} = \frac{1}{|A|} adj A.$$

Since  $A$  is an invertible matrix,

As matrix  $A$  is of order 2, let  $A = \begin{bmatrix} a & b \\ c & d \end{bmatrix}$ .

Then,  $|A| = ad - bc$  and  $\text{adj}A = \begin{bmatrix} d & -b \\ -c & a \end{bmatrix}$ .

Now,

$$A^{-1} = \frac{1}{|A|} \text{adj}A = \begin{bmatrix} \frac{d}{|A|} & \frac{-b}{|A|} \\ \frac{-c}{|A|} & \frac{a}{|A|} \end{bmatrix}$$

$$\therefore |A^{-1}| = \begin{vmatrix} \frac{d}{|A|} & \frac{-b}{|A|} \\ \frac{-c}{|A|} & \frac{a}{|A|} \end{vmatrix} = \frac{1}{|A|^2} \begin{vmatrix} d & -b \\ -c & a \end{vmatrix} = \frac{1}{|A|^2} (ad - bc) = \frac{1}{|A|^2} \cdot |A| = \frac{1}{|A|}$$

$$\therefore \det(A^{-1}) = \frac{1}{\det(A)}$$

Hence, the correct answer is B.

**Exercise 4.6****Question 1:**

Examine the consistency of the system of equations.

$$x + 2y = 2$$

$$2x + 3y = 3$$

Answer

The given system of equations is:

$$x + 2y = 2$$

$$2x + 3y = 3$$

The given system of equations can be written in the form of  $AX = B$ , where

$$A = \begin{bmatrix} 1 & 2 \\ 2 & 3 \end{bmatrix}, X = \begin{bmatrix} x \\ y \end{bmatrix} \text{ and } B = \begin{bmatrix} 2 \\ 3 \end{bmatrix}.$$

Now,

$$|A| = 1(3) - 2(2) = 3 - 4 = -1 \neq 0$$

$\therefore A$  is non-singular.

Therefore,  $A^{-1}$  exists.

Hence, the given system of equations is consistent.

**Question 2:**

Examine the consistency of the system of equations.

$$2x - y = 5$$

$$x + y = 4$$

Answer

The given system of equations is:

$$2x - y = 5$$

$$x + y = 4$$

The given system of equations can be written in the form of  $AX = B$ , where

$$A = \begin{bmatrix} 2 & -1 \\ 1 & 1 \end{bmatrix}, X = \begin{bmatrix} x \\ z \end{bmatrix} \text{ and } B = \begin{bmatrix} 5 \\ 4 \end{bmatrix}.$$

Now,

$$|A| = 2(1) - (-1)(1) = 2 + 1 = 3 \neq 0$$

$\therefore A$  is non-singular.

Therefore,  $A^{-1}$  exists.

Hence, the given system of equations is consistent.

**Question 3:**

Examine the consistency of the system of equations.

$$x + 3y = 5$$

$$2x + 6y = 8$$

Answer

The given system of equations is:

$$x + 3y = 5$$

$$2x + 6y = 8$$

The given system of equations can be written in the form of  $AX = B$ , where

$$A = \begin{bmatrix} 1 & 3 \\ 2 & 6 \end{bmatrix}, X = \begin{bmatrix} x \\ y \end{bmatrix} \text{ and } B = \begin{bmatrix} 5 \\ 8 \end{bmatrix}.$$

Now,

$$|A| = 1(6) - 3(2) = 6 - 6 = 0$$

$\therefore A$  is a singular matrix.

Now,

$$(\text{adj}A) = \begin{bmatrix} 6 & -3 \\ -2 & 1 \end{bmatrix}$$

$$(\text{adj}A)B = \begin{bmatrix} 6 & -3 \\ -2 & 1 \end{bmatrix} \begin{bmatrix} 5 \\ 8 \end{bmatrix} = \begin{bmatrix} 30-24 \\ -10+8 \end{bmatrix} = \begin{bmatrix} 6 \\ -2 \end{bmatrix} \neq O$$

Thus, the solution of the given system of equations does not exist. Hence, the system of equations is inconsistent.

#### Question 4:

Examine the consistency of the system of equations.

$$x + y + z = 1$$

$$2x + 3y + 2z = 2$$

$$ax + ay + 2az = 4$$

Answer

The given system of equations is:

$$x + y + z = 1$$

$$2x + 3y + 2z = 2$$

$$ax + ay + 2az = 4$$

This system of equations can be written in the form  $AX = B$ , where

$$A = \begin{bmatrix} 1 & 1 & 1 \\ 2 & 3 & 2 \\ a & a & 2a \end{bmatrix}, X = \begin{bmatrix} x \\ y \\ z \end{bmatrix} \text{ and } B = \begin{bmatrix} 1 \\ 2 \\ 4 \end{bmatrix}.$$

Now,

$$\begin{aligned} |A| &= 1(6a-2a) - 1(4a-2a) + 1(2a-3a) \\ &= 4a - 2a - a = 4a - 3a = a \neq 0 \end{aligned}$$

$\therefore A$  is non-singular.

Therefore,  $A^{-1}$  exists.

Hence, the given system of equations is consistent.

**Question 5:**

Examine the consistency of the system of equations.

$$3x - y - 2z = 2$$

$$2y - z = -1$$

$$3x - 5y = 3$$

Answer

The given system of equations is:

$$3x - y - 2z = 2$$

$$2y - z = -1$$

$$3x - 5y = 3$$

This system of equations can be written in the form of  $AX = B$ , where

$$A = \begin{bmatrix} 3 & -1 & -2 \\ 0 & 2 & -1 \\ 3 & -5 & 0 \end{bmatrix}, X = \begin{bmatrix} x \\ y \\ z \end{bmatrix} \text{ and } B = \begin{bmatrix} 2 \\ -1 \\ 3 \end{bmatrix}.$$

Now,

$$|A| = 3(0 - 5) - 0 + 3(1 + 4) = -15 + 15 = 0$$

$\therefore A$  is a singular matrix.

Now,

$$(\text{adj}A) = \begin{bmatrix} -5 & 10 & 5 \\ -3 & 6 & 3 \\ -6 & 12 & 6 \end{bmatrix}$$

$$\therefore (\text{adj}A)B = \begin{bmatrix} -5 & 10 & 5 \\ -3 & 6 & 3 \\ -6 & 12 & 6 \end{bmatrix} \begin{bmatrix} 2 \\ -1 \\ 3 \end{bmatrix} = \begin{bmatrix} -10 - 10 + 15 \\ -6 - 6 + 9 \\ -12 - 12 + 18 \end{bmatrix} = \begin{bmatrix} -5 \\ -3 \\ -6 \end{bmatrix} \neq O$$

Thus, the solution of the given system of equations does not exist. Hence, the system of equations is inconsistent.

**Question 6:**

Examine the consistency of the system of equations.

$$5x - y + 4z = 5$$

$$2x + 3y + 5z = 2$$

$$5x - 2y + 6z = -1$$

Answer

The given system of equations is:

$$5x - y + 4z = 5$$

$$2x + 3y + 5z = 2$$

$$5x - 2y + 6z = -1$$

This system of equations can be written in the form of  $AX = B$ , where

$$A = \begin{bmatrix} 5 & -1 & 4 \\ 2 & 3 & 5 \\ 5 & -2 & 6 \end{bmatrix}, X = \begin{bmatrix} x \\ y \\ z \end{bmatrix} \text{ and } B = \begin{bmatrix} 5 \\ 2 \\ -1 \end{bmatrix}.$$

Now,

$$\begin{aligned} |A| &= 5(18+10) + 1(12-25) + 4(-4-15) \\ &= 5(28) + 1(-13) + 4(-19) \\ &= 140 - 13 - 76 \\ &= 51 \neq 0 \end{aligned}$$

$\therefore A$  is non-singular.

Therefore,  $A^{-1}$  exists.

Hence, the given system of equations is consistent.

**Question 7:**

Solve system of linear equations, using matrix method.

$$5x + 2y = 4$$

$$7x + 3y = 5$$

Answer

The given system of equations can be written in the form of  $AX = B$ , where

$$A = \begin{bmatrix} 5 & 2 \\ 7 & 3 \end{bmatrix}, X = \begin{bmatrix} x \\ y \end{bmatrix} \text{ and } B = \begin{bmatrix} 4 \\ 5 \end{bmatrix}.$$

$$\text{Now, } |A| = 15 - 14 = 1 \neq 0.$$

Thus,  $A$  is non-singular. Therefore, its inverse exists.

Now,

$$A^{-1} = \frac{1}{|A|}(\text{adj}A)$$

$$\therefore A^{-1} = \begin{bmatrix} 3 & -2 \\ -7 & 5 \end{bmatrix}$$

$$\therefore X = A^{-1}B = \begin{bmatrix} 3 & -2 \\ -7 & 5 \end{bmatrix} \begin{bmatrix} 4 \\ 5 \end{bmatrix}$$

$$\Rightarrow \begin{bmatrix} x \\ y \end{bmatrix} = \begin{bmatrix} 12 - 10 \\ -28 + 25 \end{bmatrix} = \begin{bmatrix} 2 \\ -3 \end{bmatrix}$$

Hence,  $x = 2$  and  $y = -3$ .

**Question 8:**

Solve system of linear equations, using matrix method.

$$2x - y = -2$$

$$3x + 4y = 3$$

Answer

The given system of equations can be written in the form of  $AX = B$ , where

$$A = \begin{bmatrix} 2 & -1 \\ 3 & 4 \end{bmatrix}, X = \begin{bmatrix} x \\ y \end{bmatrix} \text{ and } B = \begin{bmatrix} -2 \\ 3 \end{bmatrix}.$$

Now,

$$|A| = 8 + 3 = 11 \neq 0$$

Thus,  $A$  is non-singular. Therefore, its inverse exists.

Now,

$$A^{-1} = \frac{1}{|A|} \text{adj}A = \frac{1}{11} \begin{bmatrix} 4 & 1 \\ -3 & 2 \end{bmatrix}$$

$$\therefore X = A^{-1}B = \frac{1}{11} \begin{bmatrix} 4 & 1 \\ -3 & 2 \end{bmatrix} \begin{bmatrix} -2 \\ 3 \end{bmatrix}$$

$$\Rightarrow \begin{bmatrix} x \\ y \end{bmatrix} = \frac{1}{11} \begin{bmatrix} -8+3 \\ 6+6 \end{bmatrix} = \frac{1}{11} \begin{bmatrix} -5 \\ 12 \end{bmatrix} = \begin{bmatrix} -\frac{5}{11} \\ \frac{12}{11} \end{bmatrix}$$

$$\text{Hence, } x = -\frac{5}{11} \text{ and } y = \frac{12}{11}.$$

**Question 9:**

Solve system of linear equations, using matrix method.

$$4x - 3y = 3$$

$$3x - 5y = 7$$

Answer

The given system of equations can be written in the form of  $AX = B$ , where

$$A = \begin{bmatrix} 4 & -3 \\ 3 & -5 \end{bmatrix}, X = \begin{bmatrix} x \\ y \end{bmatrix} \text{ and } B = \begin{bmatrix} 3 \\ 7 \end{bmatrix}.$$

Now,

$$|A| = -20 + 9 = -11 \neq 0$$

Thus,  $A$  is non-singular. Therefore, its inverse exists.

Now,

$$A^{-1} = \frac{1}{|A|} (\text{adj} A) = -\frac{1}{11} \begin{bmatrix} -5 & 3 \\ -3 & 4 \end{bmatrix} = \frac{1}{11} \begin{bmatrix} 5 & -3 \\ 3 & -4 \end{bmatrix}$$

$$\therefore X = A^{-1}B = \frac{1}{11} \begin{bmatrix} 5 & -3 \\ 3 & -4 \end{bmatrix} \begin{bmatrix} 3 \\ 7 \end{bmatrix}$$

$$\Rightarrow \begin{bmatrix} x \\ y \end{bmatrix} = \frac{1}{11} \begin{bmatrix} 5 & -3 \\ 3 & -4 \end{bmatrix} \begin{bmatrix} 3 \\ 7 \end{bmatrix} = \frac{1}{11} \begin{bmatrix} 15 - 21 \\ 9 - 28 \end{bmatrix} = \frac{1}{11} \begin{bmatrix} -6 \\ -19 \end{bmatrix} = \begin{bmatrix} -\frac{6}{11} \\ -\frac{19}{11} \end{bmatrix}$$

$$\text{Hence, } x = -\frac{6}{11} \text{ and } y = -\frac{19}{11}.$$

**Question 10:**

Solve system of linear equations, using matrix method.

$$5x + 2y = 3$$

$$3x + 2y = 5$$

Answer

The given system of equations can be written in the form of  $AX = B$ , where

$$A = \begin{bmatrix} 5 & 2 \\ 3 & 2 \end{bmatrix}, X = \begin{bmatrix} x \\ y \end{bmatrix} \text{ and } B = \begin{bmatrix} 3 \\ 5 \end{bmatrix}.$$

Now,

$$|A| = 10 - 6 = 4 \neq 0$$

Thus,  $A$  is non-singular. Therefore, its inverse exists.

**Question 11:**

Solve system of linear equations, using matrix method.

$$2x + y + z = 1$$

$$x - 2y - z = \frac{3}{2}$$

$$3y - 5z = 9$$

Answer

The given system of equations can be written in the form of  $AX = B$ , where

$$A = \begin{bmatrix} 2 & 1 & 1 \\ 1 & -2 & -1 \\ 0 & 3 & -5 \end{bmatrix}, X = \begin{bmatrix} x \\ y \\ z \end{bmatrix} \text{ and } B = \begin{bmatrix} 1 \\ 3 \\ 2 \\ 9 \end{bmatrix}.$$

Now,

$$|A| = 2(10 + 3) - 1(-5 - 3) + 0 = 2(13) - 1(-8) = 26 + 8 = 34 \neq 0$$

Thus,  $A$  is non-singular. Therefore, its inverse exists.

$$\text{Now, } A_{11} = 13, A_{12} = 5, A_{13} = 3$$

$$A_{21} = 8, A_{22} = -10, A_{23} = -6$$

$$A_{31} = 1, A_{32} = 3, A_{33} = -5$$

$$\therefore A^{-1} = \frac{1}{|A|} (\text{adj} A) = \frac{1}{34} \begin{bmatrix} 13 & 8 & 1 \\ 5 & -10 & 3 \\ 3 & -6 & -5 \end{bmatrix}$$

$$\therefore X = A^{-1}B = \frac{1}{34} \begin{bmatrix} 13 & 8 & 1 \\ 5 & -10 & 3 \\ 3 & -6 & -5 \end{bmatrix} \begin{bmatrix} 1 \\ 3 \\ 2 \\ 9 \end{bmatrix}$$

$$\Rightarrow \begin{bmatrix} x \\ y \\ z \end{bmatrix} = \frac{1}{34} \begin{bmatrix} 13 + 12 + 9 \\ 5 - 15 + 27 \\ 3 - 9 - 45 \end{bmatrix}$$

$$= \frac{1}{34} \begin{bmatrix} 34 \\ 17 \\ -51 \end{bmatrix} = \begin{bmatrix} 1 \\ \frac{1}{2} \\ -\frac{3}{2} \end{bmatrix}$$

$$\text{Hence, } x = 1, y = \frac{1}{2}, \text{ and } z = -\frac{3}{2}.$$

#### Question 12:

Solve system of linear equations, using matrix method.

$$x - y + z = 4$$

$$2x + y - 3z = 0$$

$$x + y + z = 2$$

Answer

The given system of equations can be written in the form of  $AX = B$ , where

$$A = \begin{bmatrix} 1 & -1 & 1 \\ 2 & 1 & -3 \\ 1 & 1 & 1 \end{bmatrix}, X = \begin{bmatrix} x \\ y \\ z \end{bmatrix} \text{ and } B = \begin{bmatrix} 4 \\ 0 \\ 2 \end{bmatrix}.$$

Now,

$$|A| = 1(1+3) + 1(2+3) + 1(2-1) = 4 + 5 + 1 = 10 \neq 0$$

Thus,  $A$  is non-singular. Therefore, its inverse exists.

$$\text{Now, } A_{11} = 4, A_{12} = -5, A_{13} = 1$$

$$A_{21} = 2, A_{22} = 0, A_{23} = -2$$

$$A_{31} = 2, A_{32} = 5, A_{33} = 3$$

$$\therefore A^{-1} = \frac{1}{|A|}(\text{adj}A) = \frac{1}{10} \begin{bmatrix} 4 & 2 & 2 \\ -5 & 0 & 5 \\ 1 & -2 & 3 \end{bmatrix}$$

$$\therefore X = A^{-1}B = \frac{1}{10} \begin{bmatrix} 4 & 2 & 2 \\ -5 & 0 & 5 \\ 1 & -2 & 3 \end{bmatrix} \begin{bmatrix} 4 \\ 0 \\ 2 \end{bmatrix}$$

$$\Rightarrow \begin{bmatrix} x \\ y \\ z \end{bmatrix} = \frac{1}{10} \begin{bmatrix} 16+0+4 \\ -20+0+10 \\ 4+0+6 \end{bmatrix}$$

$$= \frac{1}{10} \begin{bmatrix} 20 \\ -10 \\ 10 \end{bmatrix}$$

$$= \begin{bmatrix} 2 \\ -1 \\ 1 \end{bmatrix}$$

Hence,  $x = 2, y = -1$ , and  $z = 1$ .

### Question 13:

Solve system of linear equations, using matrix method.

$$2x + 3y + 3z = 5$$

$$x - 2y + z = -4$$

$$3x - y - 2z = 3$$

Answer

The given system of equations can be written in the form  $AX = B$ , where

$$A = \begin{bmatrix} 2 & 3 & 3 \\ 1 & -2 & 1 \\ 3 & -1 & -2 \end{bmatrix}, X = \begin{bmatrix} x \\ y \\ z \end{bmatrix} \text{ and } B = \begin{bmatrix} 5 \\ -4 \\ 3 \end{bmatrix}.$$

Now,

$$|A| = 2(4+1) - 3(-2-3) + 3(-1+6) = 2(5) - 3(-5) + 3(5) = 10 + 15 + 15 = 40 \neq 0$$

Thus,  $A$  is non-singular. Therefore, its inverse exists.

$$\text{Now, } A_{11} = 5, A_{12} = 5, A_{13} = 5$$

$$A_{21} = 3, A_{22} = -13, A_{23} = 11$$

$$A_{31} = 9, A_{32} = 1, A_{33} = -7$$

$$\therefore A^{-1} = \frac{1}{|A|} (\text{adj} A) = \frac{1}{40} \begin{bmatrix} 5 & 3 & 9 \\ 5 & -13 & 1 \\ 5 & 11 & -7 \end{bmatrix}$$

$$\therefore X = A^{-1}B = \frac{1}{40} \begin{bmatrix} 5 & 3 & 9 \\ 5 & -13 & 1 \\ 5 & 11 & -7 \end{bmatrix} \begin{bmatrix} 5 \\ -4 \\ 3 \end{bmatrix}$$

$$\Rightarrow \begin{bmatrix} x \\ y \\ z \end{bmatrix} = \frac{1}{40} \begin{bmatrix} 25 - 12 + 27 \\ 25 + 52 + 3 \\ 25 - 44 - 21 \end{bmatrix}$$

$$= \frac{1}{40} \begin{bmatrix} 40 \\ 80 \\ -40 \end{bmatrix}$$

$$= \begin{bmatrix} 1 \\ 2 \\ -1 \end{bmatrix}$$

Hence,  $x = 1$ ,  $y = 2$ , and  $z = -1$ .

**Question 14:**

Solve system of linear equations, using matrix method.

$$x - y + 2z = 7$$

$$3x + 4y - 5z = -5$$

$$2x - y + 3z = 12$$

Answer

The given system of equations can be written in the form of  $AX = B$ , where

$$A = \begin{bmatrix} 1 & -1 & 2 \\ 3 & 4 & -5 \\ 2 & -1 & 3 \end{bmatrix}, X = \begin{bmatrix} x \\ y \\ z \end{bmatrix} \text{ and } B = \begin{bmatrix} 7 \\ -5 \\ 12 \end{bmatrix}.$$

Now,

$$|A| = 1(12 - 5) + 1(9 + 10) + 2(-3 - 8) = 7 + 19 - 22 = 4 \neq 0$$

Thus,  $A$  is non-singular. Therefore, its inverse exists.

$$\text{Now, } A_{11} = 7, A_{12} = -19, A_{13} = -11$$

$$A_{21} = 1, A_{22} = -1, A_{23} = -1$$

$$A_{31} = -3, A_{32} = 11, A_{33} = 7$$

$$\therefore A^{-1} = \frac{1}{|A|}(\text{adj}A) = \frac{1}{4} \begin{bmatrix} 7 & 1 & -3 \\ -19 & -1 & 11 \\ -11 & -1 & 7 \end{bmatrix}$$

$$\therefore X = A^{-1}B = \frac{1}{4} \begin{bmatrix} 7 & 1 & -3 \\ -19 & -1 & 11 \\ -11 & -1 & 7 \end{bmatrix} \begin{bmatrix} 7 \\ -5 \\ 12 \end{bmatrix}$$

$$\Rightarrow \begin{bmatrix} x \\ y \\ z \end{bmatrix} = \frac{1}{4} \begin{bmatrix} 49 - 5 - 36 \\ -133 + 5 + 132 \\ -77 + 5 + 84 \end{bmatrix}$$

$$= \frac{1}{4} \begin{bmatrix} 8 \\ 4 \\ 12 \end{bmatrix} = \begin{bmatrix} 2 \\ 1 \\ 3 \end{bmatrix}$$

Hence,  $x = 2, y = 1$ , and  $z = 3$ .

**Question 15:**

If  $A = \begin{bmatrix} 2 & -3 & 5 \\ 3 & 2 & -4 \\ 1 & 1 & -2 \end{bmatrix}$ , find  $A^{-1}$ . Using  $A^{-1}$  solve the system of equations

$$2x - 3y + 5z = 11$$

$$3x + 2y - 4z = -5$$

$$x + y - 2z = -3$$

Answer

$$A = \begin{bmatrix} 2 & -3 & 5 \\ 3 & 2 & -4 \\ 1 & 1 & -2 \end{bmatrix}$$

$$\therefore |A| = 2(-4 + 4) + 3(-6 + 4) + 5(3 - 2) = 0 - 6 + 5 = -1 \neq 0$$

$$\text{Now, } A_{11} = 0, A_{12} = 2, A_{13} = 1$$

$$A_{21} = -1, A_{22} = -9, A_{23} = -5$$

$$A_{31} = 2, A_{32} = 23, A_{33} = 13$$

$$\therefore A^{-1} = \frac{1}{|A|} (\text{adj}A) = - \begin{bmatrix} 0 & -1 & 2 \\ 2 & -9 & 23 \\ 1 & -5 & 13 \end{bmatrix} = \begin{bmatrix} 0 & 1 & -2 \\ -2 & 9 & -23 \\ -1 & 5 & -13 \end{bmatrix} \quad \dots(1)$$

Now, the given system of equations can be written in the form of  $AX = B$ , where

$$A = \begin{bmatrix} 2 & -3 & 5 \\ 3 & 2 & -4 \\ 1 & 1 & -2 \end{bmatrix}, X = \begin{bmatrix} x \\ y \\ z \end{bmatrix} \text{ and } B = \begin{bmatrix} 11 \\ -5 \\ -3 \end{bmatrix}.$$

The solution of the system of equations is given by  $X = A^{-1}B$ .

$$X = A^{-1}B$$

$$\Rightarrow \begin{bmatrix} x \\ y \\ z \end{bmatrix} = \begin{bmatrix} 0 & 1 & -2 \\ -2 & 9 & -23 \\ -1 & 5 & -13 \end{bmatrix} \begin{bmatrix} 11 \\ -5 \\ -3 \end{bmatrix} \quad [\text{Using (1)}]$$

$$= \begin{bmatrix} 0 - 5 + 6 \\ -22 - 45 + 69 \\ -11 - 25 + 39 \end{bmatrix}$$

$$= \begin{bmatrix} 1 \\ 2 \\ 3 \end{bmatrix}$$

Hence,  $x = 1$ ,  $y = 2$ , and  $z = 3$ .

#### Question 16:

The cost of 4 kg onion, 3 kg wheat and 2 kg rice is Rs 60. The cost of 2 kg onion, 4 kg wheat and 6 kg rice is Rs 90. The cost of 6 kg onion 2 kg wheat and 3 kg rice is Rs 70. Find cost of each item per kg by matrix method.

Answer

Let the cost of onions, wheat, and rice per kg be Rs  $x$ , Rs  $y$ , and Rs  $z$  respectively.

Then, the given situation can be represented by a system of equations as:

$$4x + 3y + 2z = 60$$

$$2x + 4y + 6z = 90$$

$$6x + 2y + 3z = 70$$

This system of equations can be written in the form of  $AX = B$ , where

$$A = \begin{bmatrix} 4 & 3 & 2 \\ 2 & 4 & 6 \\ 6 & 2 & 3 \end{bmatrix}, X = \begin{bmatrix} x \\ y \\ z \end{bmatrix} \text{ and } B = \begin{bmatrix} 60 \\ 90 \\ 70 \end{bmatrix}.$$

$$|A| = 4(12 - 12) - 3(6 - 36) + 2(4 - 24) = 0 + 90 - 40 = 50 \neq 0$$

$$\text{Now, } A_{11} = 0, A_{12} = 30, A_{13} = -20$$

$$A_{21} = -5, A_{22} = 0, A_{23} = 10$$

$$A_{31} = 10, A_{32} = -20, A_{33} = 10$$

$$\therefore \text{adj}A = \begin{bmatrix} 0 & -5 & 10 \\ 30 & 0 & -20 \\ -20 & 10 & 10 \end{bmatrix}$$

$$\therefore A^{-1} = \frac{1}{|A|} \text{adj}A = \frac{1}{50} \begin{bmatrix} 0 & -5 & 10 \\ 30 & 0 & -20 \\ -20 & 10 & 10 \end{bmatrix}$$

Now,

$$X = A^{-1} B$$

$$\Rightarrow X = \frac{1}{50} \begin{bmatrix} 0 & -5 & 10 \\ 30 & 0 & -20 \\ -20 & 10 & 10 \end{bmatrix} \begin{bmatrix} 60 \\ 90 \\ 70 \end{bmatrix}$$

$$\Rightarrow \begin{bmatrix} x \\ y \\ z \end{bmatrix} = \frac{1}{50} \begin{bmatrix} 0 - 450 + 700 \\ 1800 + 0 - 1400 \\ -1200 + 900 + 700 \end{bmatrix}$$

$$= \frac{1}{50} \begin{bmatrix} 250 \\ 400 \\ 400 \end{bmatrix}$$

$$= \begin{bmatrix} 5 \\ 8 \\ 8 \end{bmatrix}$$

$$\therefore x = 5, y = 8, \text{ and } z = 8.$$

Hence, the cost of onions is Rs 5 per kg, the cost of wheat is Rs 8 per kg, and the cost of rice is Rs 8 per kg.

## Miscellaneous Solutions

## Question 1:

Prove that the determinant  $\begin{vmatrix} x & \sin \theta & \cos \theta \\ -\sin \theta & -x & 1 \\ \cos \theta & 1 & x \end{vmatrix}$  is independent of  $\theta$ .

Answer

$$\begin{aligned}
 \Delta &= \begin{vmatrix} x & \sin \theta & \cos \theta \\ -\sin \theta & -x & 1 \\ \cos \theta & 1 & x \end{vmatrix} \\
 &= x(x^2 - 1) - \sin \theta(-x \sin \theta - \cos \theta) + \cos \theta(-\sin \theta + x \cos \theta) \\
 &= x^3 - x + x \sin^2 \theta + \sin \theta \cos \theta - \sin \theta \cos \theta + x \cos^2 \theta \\
 &= x^3 - x + x(\sin^2 \theta + \cos^2 \theta) \\
 &= x^3 - x + x \\
 &= x^3 \text{ (Independent of } \theta)
 \end{aligned}$$

Hence,  $\Delta$  is independent of  $\theta$ .

## Question 2:

Without expanding the determinant, prove that

$$\begin{vmatrix} a & a^2 & bc \\ b & b^2 & ca \\ c & c^2 & ab \end{vmatrix} = \begin{vmatrix} 1 & a^2 & a^3 \\ 1 & b^2 & b^3 \\ 1 & c^2 & c^3 \end{vmatrix}$$

Answer

$$\begin{aligned}
 \text{L.H.S.} &= \begin{vmatrix} a & a^2 & bc \\ b & b^2 & ca \\ c & c^2 & ab \end{vmatrix} \\
 &= \frac{1}{abc} \begin{vmatrix} a^2 & a^3 & abc \\ b^2 & b^3 & abc \\ c^2 & c^3 & abc \end{vmatrix} && [R_1 \rightarrow aR_1, R_2 \rightarrow bR_2, \text{ and } R_3 \rightarrow cR_3] \\
 &= \frac{1}{abc} \cdot abc \begin{vmatrix} a^2 & a^3 & 1 \\ b^2 & b^3 & 1 \\ c^2 & c^3 & 1 \end{vmatrix} && [\text{Taking out factor } abc \text{ from } C_3] \\
 &= \begin{vmatrix} a^2 & a^3 & 1 \\ b^2 & b^3 & 1 \\ c^2 & c^3 & 1 \end{vmatrix} \\
 &= \begin{vmatrix} 1 & a^2 & a^3 \\ 1 & b^2 & b^3 \\ 1 & c^2 & c^3 \end{vmatrix} && [\text{Applying } C_1 \leftrightarrow C_3 \text{ and } C_2 \leftrightarrow C_3] \\
 &= \text{R.H.S.}
 \end{aligned}$$

Hence, the given result is proved.

**Question 3:**

Evaluate  $\begin{vmatrix} \cos \alpha \cos \beta & \cos \alpha \sin \beta & -\sin \alpha \\ -\sin \beta & \cos \beta & 0 \\ \sin \alpha \cos \beta & \sin \alpha \sin \beta & \cos \alpha \end{vmatrix}$

Answer

$$\Delta = \begin{vmatrix} \cos \alpha \cos \beta & \cos \alpha \sin \beta & -\sin \alpha \\ -\sin \beta & \cos \beta & 0 \\ \sin \alpha \cos \beta & \sin \alpha \sin \beta & \cos \alpha \end{vmatrix}$$

Expanding along  $C_3$ , we have:

$$\begin{aligned}
 \Delta &= -\sin \alpha (-\sin \alpha \sin^2 \beta - \cos^2 \beta \sin \alpha) + \cos \alpha (\cos \alpha \cos^2 \beta + \cos \alpha \sin^2 \beta) \\
 &= \sin^2 \alpha (\sin^2 \beta + \cos^2 \beta) + \cos^2 \alpha (\cos^2 \beta + \sin^2 \beta) \\
 &= \sin^2 \alpha (1) + \cos^2 \alpha (1) \\
 &= 1
 \end{aligned}$$

**Question 4:**

$$\Delta = \begin{vmatrix} b+c & c+a & a+b \\ c+a & a+b & b+c \\ a+b & b+c & c+a \end{vmatrix} = 0$$

If  $a$ ,  $b$  and  $c$  are real numbers, and

Show that either  $a + b + c = 0$  or  $a = b = c$ .

Answer

$$\Delta = \begin{vmatrix} b+c & c+a & a+b \\ c+a & a+b & b+c \\ a+b & b+c & c+a \end{vmatrix}$$

Applying  $R_1 \rightarrow R_1 + R_2 + R_3$ , we have:

$$\begin{aligned}
 \Delta &= \begin{vmatrix} 2(a+b+c) & 2(a+b+c) & 2(a+b+c) \\ c+a & a+b & b+c \\ a+b & b+c & c+a \end{vmatrix} \\
 &= 2(a+b+c) \begin{vmatrix} 1 & 1 & 1 \\ c+a & a+b & b+c \\ a+b & b+c & c+a \end{vmatrix}
 \end{aligned}$$

Applying  $C_2 \rightarrow C_2 - C_1$  and  $C_3 \rightarrow C_3 - C_1$ , we have:

$$\Delta = 2(a+b+c) \begin{vmatrix} 1 & 0 & 0 \\ c+a & b-c & b-a \\ a+b & c-a & c-b \end{vmatrix}$$

Expanding along  $R_1$ , we have:

$$\begin{aligned}\Delta &= 2(a+b+c)(1)[(b-c)(c-b)-(b-a)(c-a)] \\ &= 2(a+b+c)[-b^2-c^2+2bc-bc+ba+ac-a^2] \\ &= 2(a+b+c)[ab+bc+ca-a^2-b^2-c^2]\end{aligned}$$

It is given that  $\Delta = 0$ .

$$(a+b+c)[ab+bc+ca-a^2-b^2-c^2] = 0$$

$\Rightarrow$  Either  $a+b+c=0$ , or  $ab+bc+ca-a^2-b^2-c^2=0$ .

Now,

$$ab+bc+ca-a^2-b^2-c^2=0$$

$$\Rightarrow -2ab-2bc-2ca+2a^2+2b^2+2c^2=0$$

$$\Rightarrow (a-b)^2+(b-c)^2+(c-a)^2=0$$

$$\Rightarrow (a-b)^2=(b-c)^2=(c-a)^2=0 \quad \left[ (a-b)^2, (b-c)^2, (c-a)^2 \text{ are non-negative} \right]$$

$$\Rightarrow (a-b)=(b-c)=(c-a)=0$$

$$\Rightarrow a=b=c$$

Hence, if  $\Delta = 0$ , then either  $a+b+c=0$  or  $a=b=c$ .

#### Question 5:

$$\begin{vmatrix} x+a & x & x \\ x & x+a & x \\ x & x & x+a \end{vmatrix} = 0, a \neq 0$$

Solve the equations

Answer

$$\begin{vmatrix} x+a & x & x \\ x & x+a & x \\ x & x & x+a \end{vmatrix} = 0$$

Applying  $R_1 \rightarrow R_1 + R_2 + R_3$ , we get:

$$\begin{vmatrix} 3x+a & 3x+a & 3x+a \\ x & x+a & x \\ x & x & x+a \end{vmatrix} = 0$$

$$\Rightarrow (3x+a) \begin{vmatrix} 1 & 1 & 1 \\ x & x+a & x \\ x & x & x+a \end{vmatrix} = 0$$

Applying  $C_2 \rightarrow C_2 - C_1$  and  $C_3 \rightarrow C_3 - C_1$ , we have:

$$(3x+a) \begin{vmatrix} 1 & 0 & 0 \\ x & a & 0 \\ x & 0 & a \end{vmatrix} = 0$$

Expanding along  $R_1$ , we have:

$$(3x+a)[1 \times a^2] = 0$$

$$\Rightarrow a^2(3x+a) = 0$$

But  $a \neq 0$ .

Therefore, we have:

$$3x+a=0$$

$$\Rightarrow x = -\frac{a}{3}$$

**Question 6:**

$$\begin{vmatrix} a^2 & bc & ac+c^2 \\ a^2+ab & b^2 & ac \\ ab & b^2+bc & c^2 \end{vmatrix} = 4a^2b^2c^2$$

Prove that

Answer

$$\Delta = \begin{vmatrix} a^2 & bc & ac+c^2 \\ a^2+ab & b^2 & ac \\ ab & b^2+bc & c^2 \end{vmatrix}$$

Taking out common factors  $a$ ,  $b$ , and  $c$  from  $C_1$ ,  $C_2$ , and  $C_3$ , we have:

$$\Delta = abc \begin{vmatrix} a & c & a+c \\ a+b & b & a \\ b & b+c & c \end{vmatrix}$$

Applying  $R_2 \rightarrow R_2 - R_1$  and  $R_3 \rightarrow R_3 - R_1$ , we have:

$$\Delta = abc \begin{vmatrix} a & c & a+c \\ b & b-c & -c \\ b-a & b & -a \end{vmatrix}$$

Applying  $R_2 \rightarrow R_2 + R_1$ , we have:

$$\Delta = abc \begin{vmatrix} a & c & a+c \\ a+b & b & a \\ b-a & b & -a \end{vmatrix}$$

Applying  $R_3 \rightarrow R_3 + R_2$ , we have:

$$\begin{aligned} \Delta &= abc \begin{vmatrix} a & c & a+c \\ a+b & b & a \\ 2b & 2b & 0 \end{vmatrix} \\ &= 2ab^2c \begin{vmatrix} a & c & a+c \\ a+b & b & a \\ 1 & 1 & 0 \end{vmatrix} \end{aligned}$$

Applying  $C_2 \rightarrow C_2 - C_1$ , we have:

$$\Delta = 2ab^2c \begin{vmatrix} a & c-a & a+c \\ a+b & -a & a \\ 1 & 0 & 0 \end{vmatrix}$$

Expanding along  $R_3$ , we have:

$$\begin{aligned} \Delta &= 2ab^2c [a(c-a) + a(a+c)] \\ &= 2ab^2c [ac - a^2 + a^2 + ac] \\ &= 2ab^2c (2ac) \\ &= 4a^2b^2c^2 \end{aligned}$$

Hence, the given result is proved.

**Question 8:**

$$A = \begin{bmatrix} 1 & -2 & 1 \\ -2 & 3 & 1 \\ 1 & 1 & 5 \end{bmatrix}$$

Let  $\quad$  verify that

(i)  $[adj A]^{-1} = adj(A^{-1})$

(ii)  $(A^{-1})^{-1} = A$

Answer

$$A = \begin{bmatrix} 1 & -2 & 1 \\ -2 & 3 & 1 \\ 1 & 1 & 5 \end{bmatrix}$$

$$\therefore |A| = 1(15 - 1) + 2(-10 - 1) + 1(-2 - 3) = 14 - 22 - 5 = -13$$

$$\text{Now, } A_{11} = 14, A_{12} = 11, A_{13} = -5$$

$$A_{21} = 11, A_{22} = 4, A_{23} = -3$$

$$A_{31} = -5, A_{32} = -3, A_{33} = -1$$

$$\therefore adj A = \begin{bmatrix} 14 & 11 & -5 \\ 11 & 4 & -3 \\ -5 & -3 & -1 \end{bmatrix}$$

$$\therefore A^{-1} = \frac{1}{|A|} (adj A)$$

$$= -\frac{1}{13} \begin{bmatrix} 14 & 11 & -5 \\ 11 & 4 & -3 \\ -5 & -3 & -1 \end{bmatrix} = \frac{1}{13} \begin{bmatrix} -14 & -11 & 5 \\ -11 & -4 & 3 \\ 5 & 3 & 1 \end{bmatrix}$$

(i)

$$\begin{aligned} |adj A| &= 14(-4 - 9) - 11(-11 - 15) - 5(-33 + 20) \\ &= 14(-13) - 11(-26) - 5(-13) \\ &= -182 + 286 + 65 = 169 \end{aligned}$$

We have,

$$\text{adj}(\text{adj}A) = \begin{bmatrix} -13 & 26 & -13 \\ 26 & -39 & -13 \\ -13 & -13 & -65 \end{bmatrix}$$

$$\begin{aligned} \therefore [\text{adj}A]^{-1} &= \frac{1}{|\text{adj}A|} (\text{adj}(\text{adj}A)) \\ &= \frac{1}{169} \begin{bmatrix} -13 & 26 & -13 \\ 26 & -39 & -13 \\ -13 & -13 & -65 \end{bmatrix} \\ &= \frac{1}{13} \begin{bmatrix} -1 & 2 & -1 \\ 2 & -3 & -1 \\ -1 & -1 & -5 \end{bmatrix} \end{aligned}$$

$$\text{Now, } A^{-1} = \frac{1}{13} \begin{bmatrix} -14 & -11 & 5 \\ -11 & -4 & 3 \\ 5 & 3 & 1 \end{bmatrix} = \begin{bmatrix} -\frac{14}{13} & -\frac{11}{13} & \frac{5}{13} \\ -\frac{11}{13} & -\frac{4}{13} & \frac{3}{13} \\ \frac{5}{13} & \frac{3}{13} & \frac{1}{13} \end{bmatrix}$$

$$\begin{aligned} \therefore \text{adj}(A^{-1}) &= \begin{bmatrix} -\frac{4}{169} - \frac{9}{169} & -\left(-\frac{11}{169} - \frac{15}{169}\right) & -\frac{33}{169} + \frac{20}{169} \\ -\left(-\frac{11}{169} - \frac{15}{169}\right) & -\frac{14}{169} - \frac{25}{169} & -\left(-\frac{42}{169} + \frac{55}{169}\right) \\ -\frac{33}{169} + \frac{20}{169} & -\left(-\frac{42}{169} + \frac{55}{169}\right) & \frac{56}{169} - \frac{121}{169} \end{bmatrix} \\ &= \frac{1}{169} \begin{bmatrix} -13 & 26 & -13 \\ 26 & -39 & -13 \\ -13 & -13 & -65 \end{bmatrix} = \frac{1}{13} \begin{bmatrix} -1 & 2 & -1 \\ 2 & -3 & -1 \\ -1 & -1 & -5 \end{bmatrix} \end{aligned}$$

$$\text{Hence, } [\text{adj}A]^{-1} = \text{adj}(A^{-1}).$$

(ii)

We have shown that:

$$A^{-1} = \frac{1}{13} \begin{bmatrix} -14 & -11 & 5 \\ -11 & -4 & 3 \\ 5 & 3 & 1 \end{bmatrix}$$

$$\text{And, } \text{adj}A^{-1} = \frac{1}{13} \begin{bmatrix} -1 & 2 & -1 \\ 2 & -3 & -1 \\ -1 & -1 & -5 \end{bmatrix}$$

Now,

$$|A^{-1}| = \left(\frac{1}{13}\right)^3 [-14 \times (-13) + 11 \times (-26) + 5 \times (-13)] = \left(\frac{1}{13}\right)^3 \times (-169) = -\frac{1}{13}$$

$$\therefore (A^{-1})^{-1} = \frac{\text{adj}A^{-1}}{|A^{-1}|} = \frac{1}{\left(-\frac{1}{13}\right)} \times \frac{1}{13} \begin{bmatrix} -1 & 2 & -1 \\ 2 & -3 & -1 \\ -1 & -1 & -5 \end{bmatrix} = \begin{bmatrix} 1 & -2 & 1 \\ -2 & 3 & 1 \\ 1 & 1 & 5 \end{bmatrix} = A$$

$$\therefore (A^{-1})^{-1} = A$$

**Question 9:**

$$\text{Evaluate } \begin{vmatrix} x & y & x+y \\ y & x+y & x \\ x+y & x & y \end{vmatrix}$$

Answer

$$\Delta = \begin{vmatrix} x & y & x+y \\ y & x+y & x \\ x+y & x & y \end{vmatrix}$$

Applying  $R_1 \rightarrow R_1 + R_2 + R_3$ , we have:

$$\Delta = \begin{vmatrix} 2(x+y) & 2(x+y) & 2(x+y) \\ y & x+y & x \\ x+y & x & y \end{vmatrix}$$

$$= 2(x+y) \begin{vmatrix} 1 & 1 & 1 \\ y & x+y & x \\ x+y & x & y \end{vmatrix}$$

Applying  $C_2 \rightarrow C_2 - C_1$  and  $C_3 \rightarrow C_3 - C_1$ , we have:

$$\Delta = 2(x+y) \begin{vmatrix} 1 & 0 & 0 \\ y & x & x-y \\ x+y & -y & -x \end{vmatrix}$$

Expanding along  $R_1$ , we have:

$$\begin{aligned} \Delta &= 2(x+y) [-x^2 + y(x-y)] \\ &= -2(x+y)(x^2 + y^2 - yx) \\ &= -2(x^3 + y^3) \end{aligned}$$

**Question 10:**

$$\text{Evaluate } \begin{vmatrix} 1 & x & y \\ 1 & x+y & y \\ 1 & x & x+y \end{vmatrix}$$

Answer

$$\Delta = \begin{vmatrix} 1 & x & y \\ 1 & x+y & y \\ 1 & x & x+y \end{vmatrix}$$

Applying  $R_2 \rightarrow R_2 - R_1$  and  $R_3 \rightarrow R_3 - R_1$ , we have:

$$\Delta = \begin{vmatrix} 1 & x & y \\ 0 & y & 0 \\ 0 & 0 & x \end{vmatrix}$$

Expanding along  $C_1$ , we have:

$$\Delta = 1(xy - 0) = xy$$

#### Question 11:

Using properties of determinants, prove that:

$$\begin{vmatrix} \alpha & \alpha^2 & \beta + \gamma \\ \beta & \beta^2 & \gamma + \alpha \\ \gamma & \gamma^2 & \alpha + \beta \end{vmatrix} = (\beta - \gamma)(\gamma - \alpha)(\alpha - \beta)(\alpha + \beta + \gamma)$$

Answer

$$\Delta = \begin{vmatrix} \alpha & \alpha^2 & \beta + \gamma \\ \beta & \beta^2 & \gamma + \alpha \\ \gamma & \gamma^2 & \alpha + \beta \end{vmatrix}$$

Applying  $R_2 \rightarrow R_2 - R_1$  and  $R_3 \rightarrow R_3 - R_1$ , we have:

$$\begin{aligned} \Delta &= \begin{vmatrix} \alpha & \alpha^2 & \beta + \gamma \\ \beta - \alpha & \beta^2 - \alpha^2 & \alpha - \beta \\ \gamma - \alpha & \gamma^2 - \alpha^2 & \alpha - \gamma \end{vmatrix} \\ &= (\beta - \alpha)(\gamma - \alpha) \begin{vmatrix} \alpha & \alpha^2 & \beta + \gamma \\ 1 & \beta + \alpha & -1 \\ 1 & \gamma + \alpha & -1 \end{vmatrix} \end{aligned}$$

Applying  $R_3 \rightarrow R_3 - R_2$ , we have:

$$\Delta = (\beta - \alpha)(\gamma - \alpha) \begin{vmatrix} \alpha & \alpha^2 & \beta + \gamma \\ 1 & \beta + \alpha & -1 \\ 0 & \gamma - \beta & 0 \end{vmatrix}$$

Expanding along  $R_3$ , we have:

$$\begin{aligned} \Delta &= (\beta - \alpha)(\gamma - \alpha) [-(\gamma - \beta)(-\alpha - \beta - \gamma)] \\ &= (\beta - \alpha)(\gamma - \alpha)(\gamma - \beta)(\alpha + \beta + \gamma) \\ &= (\alpha - \beta)(\beta - \gamma)(\gamma - \alpha)(\alpha + \beta + \gamma) \end{aligned}$$

Hence, the given result is proved.

### Question 12:

Using properties of determinants, prove that:

$$\begin{vmatrix} x & x^2 & 1 + px^3 \\ y & y^2 & 1 + py^3 \\ z & z^2 & 1 + pz^3 \end{vmatrix} = (1 + pxyz)(x - y)(y - z)(z - x)$$

Answer

$$\Delta = \begin{vmatrix} x & x^2 & 1 + px^3 \\ y & y^2 & 1 + py^3 \\ z & z^2 & 1 + pz^3 \end{vmatrix}$$

Applying  $R_2 \rightarrow R_2 - R_1$  and  $R_3 \rightarrow R_3 - R_1$ , we have:

$$\begin{aligned} \Delta &= \begin{vmatrix} x & x^2 & 1 + px^3 \\ y - x & y^2 - x^2 & p(y^3 - x^3) \\ z - x & z^2 - x^2 & p(z^3 - x^3) \end{vmatrix} \\ &= (y - x)(z - x) \begin{vmatrix} x & x^2 & 1 + px^3 \\ 1 & y + x & p(y^2 + x^2 + xy) \\ 1 & z + x & p(z^2 + x^2 + xz) \end{vmatrix} \end{aligned}$$

Applying  $R_3 \rightarrow R_3 - R_2$ , we have:

$$\Delta = (y-x)(z-x) \begin{vmatrix} x & x^2 & 1+px^3 \\ 1 & y+x & p(y^2+x^2+xy) \\ 0 & z-y & p(z-y)(x+y+z) \end{vmatrix}$$

$$= (y-x)(z-x)(z-y) \begin{vmatrix} x & x^2 & 1+px^3 \\ 1 & y+x & p(y^2+x^2+xy) \\ 0 & 1 & p(x+y+z) \end{vmatrix}$$

Expanding along  $R_3$ , we have:

$$\Delta = (x-y)(y-z)(z-x) \left[ (-1)(p)(xy^2+x^3+x^2y) + 1 + px^3 + p(x+y+z)(xy) \right]$$

$$= (x-y)(y-z)(z-x) \left[ -pxy^2 - px^3 - px^2y + 1 + px^3 + px^2y + pxy^2 + pxyz \right]$$

$$= (x-y)(y-z)(z-x)(1+pxyz)$$

Hence, the given result is proved.

### Question 13:

Using properties of determinants, prove that:

$$\begin{vmatrix} 3a & -a+b & -a+c \\ -b+a & 3b & -b+c \\ -c+a & -c+b & 3c \end{vmatrix} = 3(a+b+c)(ab+bc+ca)$$

Answer

$$\Delta = \begin{vmatrix} 3a & -a+b & -a+c \\ -b+a & 3b & -b+c \\ -c+a & -c+b & 3c \end{vmatrix}$$

Applying  $C_1 \rightarrow C_1 + C_2 + C_3$ , we have:

$$\Delta = \begin{vmatrix} a+b+c & -a+b & -a+c \\ a+b+c & 3b & -b+c \\ a+b+c & -c+b & 3c \end{vmatrix}$$

$$= (a+b+c) \begin{vmatrix} 1 & -a+b & -a+c \\ 1 & 3b & -b+c \\ 1 & -c+b & 3c \end{vmatrix}$$

Applying  $R_2 \rightarrow R_2 - R_1$  and  $R_3 \rightarrow R_3 - R_1$ , we have:

$$\Delta = (a+b+c) \begin{vmatrix} 1 & -a+b & -a+c \\ 0 & 2b+a & a-b \\ 0 & a-c & 2c+a \end{vmatrix}$$

Expanding along  $C_1$ , we have:

$$\begin{aligned} \Delta &= (a+b+c) [(2b+a)(2c+a) - (a-b)(a-c)] \\ &= (a+b+c) [4bc + 2ab + 2ac + a^2 - a^2 + ac + ba - bc] \\ &= (a+b+c)(3ab + 3bc + 3ac) \\ &= 3(a+b+c)(ab + bc + ca) \end{aligned}$$

Hence, the given result is proved.

#### Question 14:

Using properties of determinants, prove that:

$$\begin{vmatrix} 1 & 1+p & 1+p+q \\ 2 & 3+2p & 4+3p+2q \\ 3 & 6+3p & 10+6p+3q \end{vmatrix} = 1$$

Answer

$$\Delta = \begin{vmatrix} 1 & 1+p & 1+p+q \\ 2 & 3+2p & 4+3p+2q \\ 3 & 6+3p & 10+6p+3q \end{vmatrix}$$

Applying  $R_2 \rightarrow R_2 - 2R_1$  and  $R_3 \rightarrow R_3 - 3R_1$ , we have:

$$\Delta = \begin{vmatrix} 1 & 1+p & 1+p+q \\ 0 & 1 & 2+p \\ 0 & 3 & 7+3p \end{vmatrix}$$

Applying  $R_3 \rightarrow R_3 - 3R_2$ , we have:

$$\Delta = \begin{vmatrix} 1 & 1+p & 1+p+q \\ 0 & 1 & 2+p \\ 0 & 0 & 1 \end{vmatrix}$$

Expanding along  $C_1$ , we have:

$$\Delta = 1 \begin{vmatrix} 2+p & 1 \\ 0 & 1 \end{vmatrix} = 1(1-0) = 1$$

Hence, the given result is proved.

#### Question 15:

Using properties of determinants, prove that:

$$\begin{vmatrix} \sin \alpha & \cos \alpha & \cos(\alpha + \delta) \\ \sin \beta & \cos \beta & \cos(\beta + \delta) \\ \sin \gamma & \cos \gamma & \cos(\gamma + \delta) \end{vmatrix} = 0$$

Answer

$$\Delta = \begin{vmatrix} \sin \alpha & \cos \alpha & \cos(\alpha + \delta) \\ \sin \beta & \cos \beta & \cos(\beta + \delta) \\ \sin \gamma & \cos \gamma & \cos(\gamma + \delta) \end{vmatrix}$$

$$= \frac{1}{\sin \delta \cos \delta} \begin{vmatrix} \sin \alpha \sin \delta & \cos \alpha \cos \delta & \cos \alpha \cos \delta - \sin \alpha \sin \delta \\ \sin \beta \sin \delta & \cos \beta \cos \delta & \cos \beta \cos \delta - \sin \beta \sin \delta \\ \sin \gamma \sin \delta & \cos \gamma \cos \delta & \cos \gamma \cos \delta - \sin \gamma \sin \delta \end{vmatrix}$$

Applying  $C_1 \rightarrow C_1 + C_3$ , we have:

$$\Delta = \frac{1}{\sin \delta \cos \delta} \begin{vmatrix} \cos \alpha \cos \delta & \cos \alpha \cos \delta & \cos \alpha \cos \delta - \sin \alpha \sin \delta \\ \cos \beta \cos \delta & \cos \beta \cos \delta & \cos \beta \cos \delta - \sin \beta \sin \delta \\ \cos \gamma \cos \delta & \cos \gamma \cos \delta & \cos \gamma \cos \delta - \sin \gamma \sin \delta \end{vmatrix}$$

Here, two columns  $C_1$  and  $C_2$  are identical.

$$\therefore \Delta = 0.$$

Hence, the given result is proved.

#### Question 16:

Solve the system of the following equations

$$\frac{2}{x} + \frac{3}{y} + \frac{10}{z} = 4$$

$$\frac{4}{x} - \frac{6}{y} + \frac{5}{z} = 1$$

$$\frac{6}{x} + \frac{9}{y} - \frac{20}{z} = 2$$

Answer

$$\text{Let } \frac{1}{x} = p, \frac{1}{y} = q, \frac{1}{z} = r.$$

Then the given system of equations is as follows:

$$2p + 3q + 10r = 4$$

$$4p - 6q + 5r = 1$$

$$6p + 9q - 20r = 2$$

This system can be written in the form of  $AX = B$ , where

$$A = \begin{bmatrix} 2 & 3 & 10 \\ 4 & -6 & 5 \\ 6 & 9 & -20 \end{bmatrix}, X = \begin{bmatrix} p \\ q \\ r \end{bmatrix} \text{ and } B = \begin{bmatrix} 4 \\ 1 \\ 2 \end{bmatrix}.$$

Now,

$$\begin{aligned} |A| &= 2(120 - 45) - 3(-80 - 30) + 10(36 + 36) \\ &= 150 + 330 + 720 \\ &= 1200 \end{aligned}$$

A

Thus, A is non-singular. Therefore, its inverse exists.

Now,

$$A_{11} = 75, A_{12} = 110, A_{13} = 72$$

$$A_{21} = 150, A_{22} = -100, A_{23} = 0$$

$$A_{31} = 75, A_{32} = 30, A_{33} = -24$$

$$\begin{aligned} \therefore A^{-1} &= \frac{1}{|A|} \text{adj}A \\ &= \frac{1}{1200} \begin{bmatrix} 75 & 150 & 75 \\ 110 & -100 & 30 \\ 72 & 0 & -24 \end{bmatrix} \end{aligned}$$

Now,

$$X = A^{-1}B$$

$$\begin{aligned} \Rightarrow \begin{bmatrix} p \\ q \\ r \end{bmatrix} &= \frac{1}{1200} \begin{bmatrix} 75 & 150 & 75 \\ 110 & -100 & 30 \\ 72 & 0 & -24 \end{bmatrix} \begin{bmatrix} 4 \\ 1 \\ 2 \end{bmatrix} \\ &= \frac{1}{1200} \begin{bmatrix} 300 + 150 + 150 \\ 440 - 100 + 60 \\ 288 + 0 - 48 \end{bmatrix} \\ &= \frac{1}{1200} \begin{bmatrix} 600 \\ 400 \\ 240 \end{bmatrix} = \begin{bmatrix} \frac{1}{2} \\ \frac{1}{3} \\ \frac{1}{5} \end{bmatrix} \end{aligned}$$

$$\therefore p = \frac{1}{2}, q = \frac{1}{3}, \text{ and } r = \frac{1}{5}$$

Hence,  $x = 2, y = 3, \text{ and } z = 5$ .

**Question 17:**

Choose the correct answer.

If  $a, b, c$ , are in A.P., then the determinant

$$\begin{vmatrix} x+2 & x+3 & x+2a \\ x+3 & x+4 & x+2b \\ x+4 & x+5 & x+2c \end{vmatrix}$$

**A. 0 B. 1 C.  $x$  D.  $2x$**

Answer

**Answer: A**

$$\begin{aligned} \Delta &= \begin{vmatrix} x+2 & x+3 & x+2a \\ x+3 & x+4 & x+2b \\ x+4 & x+5 & x+2c \end{vmatrix} \\ &= \begin{vmatrix} x+2 & x+3 & x+2a \\ x+3 & x+4 & x+(a+c) \\ x+4 & x+5 & x+2c \end{vmatrix} \quad (2b = a + c \text{ as } a, b, \text{ and } c \text{ are in A.P.}) \end{aligned}$$

Applying  $R_1 \rightarrow R_1 - R_2$  and  $R_3 \rightarrow R_3 - R_2$ , we have:

$$\Delta = \begin{vmatrix} -1 & -1 & a-c \\ x+3 & x+4 & x+(a+c) \\ 1 & 1 & c-a \end{vmatrix}$$

Applying  $R_1 \rightarrow R_1 + R_3$ , we have:

$$\Delta = \begin{vmatrix} 0 & 0 & 0 \\ x+3 & x+4 & x+a+c \\ 1 & 1 & c-a \end{vmatrix}$$

Here, all the elements of the first row ( $R_1$ ) are zero.

Hence, we have  $\Delta = 0$ .

The correct answer is A.

**Question 18:**

Choose the correct answer.

$$A = \begin{bmatrix} x & 0 & 0 \\ 0 & y & 0 \\ 0 & 0 & z \end{bmatrix} \text{ is}$$

If  $x, y, z$  are nonzero real numbers, then the inverse of matrix

**A.**  $\begin{bmatrix} x^{-1} & 0 & 0 \\ 0 & y^{-1} & 0 \\ 0 & 0 & z^{-1} \end{bmatrix}$  **B.**  $xyz \begin{bmatrix} x^{-1} & 0 & 0 \\ 0 & y^{-1} & 0 \\ 0 & 0 & z^{-1} \end{bmatrix}$

**C.**  $\frac{1}{xyz} \begin{bmatrix} x & 0 & 0 \\ 0 & y & 0 \\ 0 & 0 & z \end{bmatrix}$  **D.**  $\frac{1}{xyz} \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}$

Answer

**Answer: A**

$$A = \begin{bmatrix} x & 0 & 0 \\ 0 & y & 0 \\ 0 & 0 & z \end{bmatrix}$$

$$\therefore |A| = x(yz - 0) = xyz \neq 0$$

$$\text{Now, } A_{11} = yz, A_{12} = 0, A_{13} = 0$$

$$A_{21} = 0, A_{22} = xz, A_{23} = 0$$

$$A_{31} = 0, A_{32} = 0, A_{33} = xy$$

$$\therefore \text{adj}A = \begin{bmatrix} yz & 0 & 0 \\ 0 & xz & 0 \\ 0 & 0 & xy \end{bmatrix}$$

$$\therefore A^{-1} = \frac{1}{|A|} \text{adj}A$$

$$\begin{aligned}
 &= \frac{1}{xyz} \begin{bmatrix} yz & 0 & 0 \\ 0 & xz & 0 \\ 0 & 0 & xy \end{bmatrix} \\
 &= \begin{bmatrix} \frac{yz}{xyz} & 0 & 0 \\ 0 & \frac{xz}{xyz} & 0 \\ 0 & 0 & \frac{xy}{xyz} \end{bmatrix} \\
 &= \begin{bmatrix} \frac{1}{x} & 0 & 0 \\ 0 & \frac{1}{y} & 0 \\ 0 & 0 & \frac{1}{z} \end{bmatrix} = \begin{bmatrix} x^{-1} & 0 & 0 \\ 0 & y^{-1} & 0 \\ 0 & 0 & z^{-1} \end{bmatrix}
 \end{aligned}$$

The correct answer is A.

**Question 19:**

Choose the correct answer.

$$A = \begin{bmatrix} 1 & \sin \theta & 1 \\ -\sin \theta & 1 & \sin \theta \\ -1 & -\sin \theta & 1 \end{bmatrix}, \text{ where } 0 \leq \theta \leq 2\pi, \text{ then}$$

**A.**  $\text{Det}(A) = 0$

**B.**  $\text{Det}(A) \in (2, \infty)$

**C.**  $\text{Det}(A) \in (2, 4)$

**D.**  $\text{Det}(A) \in [2, 4]$

Answer

**sAnswer: D**

$$A = \begin{bmatrix} 1 & \sin \theta & 1 \\ -\sin \theta & 1 & \sin \theta \\ -1 & -\sin \theta & 1 \end{bmatrix}$$

$$\begin{aligned} \therefore |A| &= 1(1 + \sin^2 \theta) - \sin \theta(-\sin \theta + \sin \theta) + 1(\sin^2 \theta + 1) \\ &= 1 + \sin^2 \theta + \sin^2 \theta + 1 \\ &= 2 + 2\sin^2 \theta \\ &= 2(1 + \sin^2 \theta) \end{aligned}$$

Now,  $0 \leq \theta \leq 2\pi$

$$\Rightarrow 0 \leq \sin \theta \leq 1$$

$$\Rightarrow 0 \leq \sin^2 \theta \leq 1$$

$$\Rightarrow 1 \leq 1 + \sin^2 \theta \leq 2$$

$$\Rightarrow 2 \leq 2(1 + \sin^2 \theta) \leq 4$$

$$\therefore \text{Det}(A) \in [2, 4]$$

The correct answer is D.