Exercise 5.1

Question 1:

Prove that the function $f(x)=5x-3$ is continuous at $x=0$, at $x=-3$ and at $x=5$.

Answer

The given function is $f(x)=5x-3$

At $x=0$, $f(0)=5\times0-3=3$

$\lim\limits_{x\to0} f(x)=\lim\limits_{x\to0}(5x-3)=5\times0-3=-3$

$\therefore \lim\limits_{x\to0} f(x)=f(0)$

Therefore, *f* is continuous at *x* = 0

At $x=-3$, $f(-3)=5\times(-3)-3=-18$

$\lim\limits_{x\to-3} f(x)=\lim\limits_{x\to-3}(5x-3)=5\times(-3)-3=-18$

$\therefore \lim\limits_{x\to-3} f(x)=f(-3)$

Therefore, *f* is continuous at *x* = −3

At $x=5$, $f(x)=f(5)=5\times5-3=25-3=22$

$\lim\limits_{x\to5} f(x)=\lim\limits_{x\to5}(5x-3)=5\times5-3=22$

$\therefore \lim\limits_{x\to5} f(x)=f(5)$

Therefore, *f* is continuous at *x* = 5

Question 2:

Examine the continuity of the function $f(x)=2x^2-1$ at $x=3$.

Answer

The given function is $f(x)=2x^2-1$

At $x=3$, $f(x)=f(3)=2\times3^2-1=17$

$\lim\limits_{x\to3} f(x)=\lim\limits_{x\to3}(2x^2-1)=2\times3^2-1=17$

$\therefore \lim\limits_{x\to3} f(x)=f(3)$

Thus, *f* is continuous at *x* = 3

Question 3:

Examine the following functions for continuity.

(a) $f(x) = x - 5$ (b) $f(x) = \frac{1}{x-5}, x \neq 5$

(c) $f(x) = \frac{x^2 - 25}{x+5}, x \neq -5$ (d) $f(x) = |x - 5|$

Answer

(a) The given function is $f(x) = x - 5$

It is evident that *f* is defined at every real number *k* and its value at *k* is *k* − 5.

It is also observed that, $\lim_{x \to k} f(x) = \lim_{x \to k} (x - 5) = k - 5 = f(k)$

$\therefore \lim_{x \to k} f(x) = f(k)$

Hence, *f* is continuous at every real number and therefore, it is a continuous function.

(b) The given function is $f(x) = \frac{1}{x-5}, x \neq 5$

For any real number *k* ≠ 5, we obtain

$$\lim_{x \to k} f(x) = \lim_{x \to k} \frac{1}{x-5} = \frac{1}{k-5}$$

$$\text{Also, } f(k) = \frac{1}{k-5} \qquad (\text{As } k \neq 5)$$

$$\therefore \lim_{x \to k} f(x) = f(k)$$

Hence, *f* is continuous at every point in the domain of *f* and therefore, it is a continuous function.

(c) The given function is $f(x) = \frac{x^2 - 25}{x+5}, x \neq -5$

For any real number *c* ≠ −5, we obtain

$$\lim_{x\to c} f(x) = \lim_{x\to c}\frac{x^2-25}{x+5} = \lim_{x\to c}\frac{(x+5)(x-5)}{x+5} = \lim_{x\to c}(x-5) = (c-5)$$

$$\text{Also, } f(c) = \frac{(c+5)(c-5)}{c+5} = (c-5) \quad (\text{as } c \neq -5)$$

$$\therefore \lim_{x\to c} f(x) = f(c)$$

Hence, *f* is continuous at every point in the domain of *f* and therefore, it is a continuous function.

(d) The given function is $f(x) = |x-5| = \begin{cases} 5-x, \text{ if } x < 5 \\ x-5, \text{ if } x \geq 5 \end{cases}$

This function *f* is defined at all points of the real line.

Let c be a point on a real line. Then, $c < 5$ or $c = 5$ or $c > 5$

Case I: $c < 5$

Then, $f(c) = 5 - c$

$$\lim_{x\to c} f(x) = \lim_{x\to c}(5-x) = 5-c$$

$$\therefore \lim_{x\to c} f(x) = f(c)$$

Therefore, *f* is continuous at all real numbers less than 5.

Case II : $c = 5$

Then, $f(c) = f(5) = (5-5) = 0$

$$\lim_{x\to 5^-} f(x) = \lim_{x\to 5}(5-x) = (5-5) = 0$$

$$\lim_{x\to 5^+} f(x) = \lim_{x\to 5}(x-5) = 0$$

$$\therefore \lim_{x\to c^-} f(x) = \lim_{x\to c^+} f(x) = f(c)$$

Therefore, *f* is continuous at $x = 5$

Case III: $c > 5$

$$\text{Then, } f(c) = f(5) = c-5$$

$$\lim_{x\to c} f(x) = \lim_{x\to c}(x-5) = c-5$$

$$\therefore \lim_{x\to c} f(x) = f(c)$$

Therefore, *f* is continuous at all real numbers greater than 5.

Hence, *f* is continuous at every real number and therefore, it is a continuous function.

Question 4:

Prove that the function $f(x) = x^n$ is continuous at *x* = *n*, where *n* is a positive integer.

Answer

The given function is *f* (*x*) = x^n

It is evident that *f* is defined at all positive integers, *n*, and its value at *n* is n^n.

$$\text{Then, } \lim_{x \to n} f(n) = \lim_{x \to n} (x^n) = n^n$$

$$\therefore \lim_{x \to n} f(x) = f(n)$$

Therefore, *f* is continuous at *n*, where *n* is a positive integer.

Question 5:

Is the function *f* defined by

$$f(x) = \begin{cases} x, & \text{if } x \le 1 \\ 5, & \text{if } x > 1 \end{cases}$$

continuous at *x* = 0? At *x* = 1? At *x* = 2?

Answer

The given function *f* is $f(x) = \begin{cases} x, & \text{if } x \le 1 \\ 5, & \text{if } x > 1 \end{cases}$

At *x* = 0,

It is evident that *f* is defined at 0 and its value at 0 is 0.

$$\text{Then, } \lim_{x \to 0} f(x) = \lim_{x \to 0} x = 0$$

$$\therefore \lim_{x \to 0} f(x) = f(0)$$

Therefore, *f* is continuous at *x* = 0

At *x* = 1,

f is defined at 1 and its value at 1 is 1.

The left hand limit of *f* at *x* = 1 is,

$$\lim_{x \to 1^-} f(x) = \lim_{x \to 1^-} x = 1$$

The right hand limit of *f* at *x* = 1 is,

$$\lim_{x\to 1^+} f(x) = \lim_{x\to 1^+}(5) = 5$$

$$\therefore \lim_{x\to 1^-} f(x) \neq \lim_{x\to 1^+} f(x)$$

Therefore, *f* is not continuous at $x = 1$

At $x = 2$,

f is defined at 2 and its value at 2 is 5.

$$\text{Then, } \lim_{x\to 2} f(x) = \lim_{x\to 2}(5) = 5$$

$$\therefore \lim_{x\to 2} f(x) = f(2)$$

Therefore, *f* is continuous at $x = 2$

Question 6:

Find all points of discontinuity of *f*, where *f* is defined by

$$f(x) = \begin{cases} 2x+3, & \text{if } x \le 2 \\ 2x-3, & \text{if } x > 2 \end{cases}$$

Answer

The given function *f* is $f(x) = \begin{cases} 2x+3, & \text{if } x \le 2 \\ 2x-3, & \text{if } x > 2 \end{cases}$

It is evident that the given function *f* is defined at all the points of the real line.

Let *c* be a point on the real line. Then, three cases arise.

(i) $c < 2$

(ii) $c > 2$

(iii) $c = 2$

Case (i) $c < 2$

$$\text{Then, } f(c) = 2c + 3$$

$$\lim_{x\to c} f(x) = \lim_{x\to c}(2x+3) = 2c+3$$

$$\therefore \lim_{x\to c} f(x) = f(c)$$

Therefore, *f* is continuous at all points *x*, such that $x < 2$

Case (ii) $c > 2$

Then, $f(c) = 2c - 3$

$\lim_{x \to c} f(x) = \lim_{x \to c}(2x - 3) = 2c - 3$

$\therefore \lim_{x \to c} f(x) = f(c)$

Therefore, *f* is continuous at all points *x*, such that *x* > 2

Case (iii) *c* = 2

Then, the left hand limit of *f* at *x* = 2 is,

$\lim_{x \to 2^-} f(x) = \lim_{x \to 2^-}(2x + 3) = 2 \times 2 + 3 = 7$

The right hand limit of *f* at *x* = 2 is,

$\lim_{x \to 2^+} f(x) = \lim_{x \to 2^+}(2x - 3) = 2 \times 2 - 3 = 1$

It is observed that the left and right hand limit of *f* at *x* = 2 do not coincide.

Therefore, *f* is not continuous at *x* = 2

Hence, *x* = 2 is the only point of discontinuity of *f*.

Question 7:

Find all points of discontinuity of *f*, where *f* is defined by

$$f(x) = \begin{cases} |x| + 3, \text{if } x \le -3 \\ -2x, \text{if } -3 < x < 3 \\ 6x + 2, \text{if } x \ge 3 \end{cases}$$

Answer

$$f(x) = \begin{cases} |x| + 3 = -x + 3, \text{ if } x \le -3 \\ -2x, \text{ if } -3 < x < 3 \\ 6x + 2, \text{ if } x \ge 3 \end{cases}$$

The given function *f* is

The given function *f* is defined at all the points of the real line.

Let *c* be a point on the real line.

Case I:

If $c < -3$, then $f(c) = -c + 3$

$\lim_{x \to c} f(x) = \lim_{x \to c}(-x + 3) = -c + 3$

$\therefore \lim_{x \to c} f(x) = f(c)$

Therefore, *f* is continuous at all points *x*, such that $x < -3$

Case II:

$$\text{If } c = -3, \text{ then } f(-3) = -(-3) + 3 = 6$$

$$\lim_{x \to -3^-} f(x) = \lim_{x \to -3^-} (-x+3) = -(-3) + 3 = 6$$

$$\lim_{x \to -3^+} f(x) = \lim_{x \to -3^+} (-2x) = -2 \times (-3) = 6$$

$$\therefore \lim_{x \to -3} f(x) = f(-3)$$

Therefore, *f* is continuous at $x = -3$

Case III:

$$\text{If } -3 < c < 3, \text{ then } f(c) = -2c \text{ and } \lim_{x \to c} f(x) = \lim_{x \to c}(-2x) = -2c$$

$$\therefore \lim_{x \to c} f(x) = f(c)$$

Therefore, *f* is continuous in $(-3, 3)$.

Case IV:

If $c = 3$, then the left hand limit of *f* at $x = 3$ is,

$$\lim_{x \to 3^-} f(x) = \lim_{x \to 3^-}(-2x) = -2 \times 3 = -6$$

The right hand limit of *f* at $x = 3$ is,

$$\lim_{x \to 3^+} f(x) = \lim_{x \to 3^+}(6x+2) = 6 \times 3 + 2 = 20$$

It is observed that the left and right hand limit of *f* at $x = 3$ do not coincide.

Therefore, *f* is not continuous at $x = 3$

Case V:

$$\text{If } c > 3, \text{ then } f(c) = 6c + 2 \text{ and } \lim_{x \to c} f(x) = \lim_{x \to c}(6x+2) = 6c + 2$$

$$\therefore \lim_{x \to c} f(x) = f(c)$$

Therefore, *f* is continuous at all points *x*, such that $x > 3$

Hence, $x = 3$ is the only point of discontinuity of *f*.

Question 8:

Find all points of discontinuity of *f*, where *f* is defined by

$$f(x)=\begin{cases}\dfrac{|x|}{x} \text{ if } x\neq 0\\ 0, \text{ if } x=0\end{cases}$$

Answer

The given function *f* is $f(x)=\begin{cases}\dfrac{|x|}{x} \text{ if } x\neq 0\\ 0, \text{ if } x=0\end{cases}$

It is known that, $x<0\Rightarrow |x|=-x$ and $x>0\Rightarrow |x|=x$

Therefore, the given function can be rewritten as

$$f(x)=\begin{cases}\dfrac{|x|}{x}=\dfrac{-x}{x}=-1 \text{ if } x<0\\ 0, \text{ if } x=0\\ \dfrac{|x|}{x}=\dfrac{x}{x}=1, \text{ if } x>0\end{cases}$$

The given function *f* is defined at all the points of the real line.

Let *c* be a point on the real line.

Case I:

$$\text{If } c<0, \text{ then } f(c)=-1$$

$$\lim_{x\to c} f(x)=\lim_{x\to c}(-1)=-1$$

$$\therefore \lim_{x\to c} f(x)=f(c)$$

Therefore, *f* is continuous at all points $x < 0$

Case II:

If $c = 0$, then the left hand limit of *f* at $x = 0$ is,

$$\lim_{x\to 0^-} f(x)=\lim_{x\to 0^-}(-1)=-1$$

The right hand limit of *f* at $x = 0$ is,

$$\lim_{x\to 0^+} f(x)=\lim_{x\to 0^+}(1)=1$$

It is observed that the left and right hand limit of *f* at $x = 0$ do not coincide.

Therefore, *f* is not continuous at $x = 0$

Case III:

$\text{If } c > 0, \text{ then } f(c) = 1$

$\lim_{x \to c} f(x) = \lim_{x \to c} (1) = 1$

$\therefore \lim_{x \to c} f(x) = f(c)$

Therefore, *f* is continuous at all points *x*, such that $x > 0$

Hence, $x = 0$ is the only point of discontinuity of *f*.

Question 9:

Find all points of discontinuity of *f*, where *f* is defined by

$$f(x) = \begin{cases} \dfrac{x}{|x|}, \text{ if } x < 0 \\ -1, \text{ if } x \geq 0 \end{cases}$$

Answer

The given function *f* is $f(x) = \begin{cases} \dfrac{x}{|x|}, \text{ if } x < 0 \\ -1, \text{ if } x \geq 0 \end{cases}$

It is known that, $x < 0 \Rightarrow |x| = -x$

Therefore, the given function can be rewritten as

$$f(x) = \begin{cases} \dfrac{x}{|x|} = \dfrac{x}{-x} = -1, \text{ if } x < 0 \\ -1, \text{ if } x \geq 0 \end{cases}$$

$\Rightarrow f(x) = -1 \text{ for all } x \in \mathbf{R}$

Let *c* be any real number. Then, $\lim_{x \to c} f(x) = \lim_{x \to c} (-1) = -1$

Also, $f(c) = -1 = \lim_{x \to c} f(x)$

Therefore, the given function is a continuous function.

Hence, the given function has no point of discontinuity.

Question 10:

Find all points of discontinuity of *f*, where *f* is defined by

$$f(x)=\begin{cases}x+1, & \text{if } x\geq 1\\ x^2+1, & \text{if } x<1\end{cases}$$

Answer

The given function *f* is $f(x)=\begin{cases}x+1, & \text{if } x\geq 1\\ x^2+1, & \text{if } x<1\end{cases}$

The given function *f* is defined at all the points of the real line.

Let *c* be a point on the real line.

Case I:

$$\text{If } c<1, \text{ then } f(c)=c^2+1 \text{ and } \lim_{x\to c} f(x)=\lim_{x\to c}\left(x^2+1\right)=c^2+1$$

$$\therefore \lim_{x\to c} f(x)=f(c)$$

Therefore, *f* is continuous at all points *x*, such that *x* < 1

Case II:

$$\text{If } c=1, \text{ then } f(c)=f(1)=1+1=2$$

The left hand limit of *f* at *x* = 1 is,

$$\lim_{x\to 1^-} f(x)=\lim_{x\to 1^-}\left(x^2+1\right)=1^2+1=2$$

The right hand limit of *f* at *x* = 1 is,

$$\lim_{x\to 1^+} f(x)=\lim_{x\to 1^+}(x+1)=1+1=2$$

$$\therefore \lim_{x\to 1} f(x)=f(1)$$

Therefore, *f* is continuous at *x* = 1

Case III:

$$\text{If } c>1, \text{ then } f(c)=c+1$$

$$\lim_{x\to c} f(x)=\lim_{x\to c}(x+1)=c+1$$

$$\therefore \lim_{x\to c} f(x)=f(c)$$

Therefore, *f* is continuous at all points *x*, such that *x* > 1

Hence, the given function *f* has no point of discontinuity.

Question 11:

Find all points of discontinuity of *f*, where *f* is defined by

$$f(x)=\begin{cases}x^3-3, & \text{if } x\le 2\\ x^2+1, & \text{if } x>2\end{cases}$$

Answer

The given function *f* is $f(x)=\begin{cases}x^3-3, & \text{if } x\le 2\\ x^2+1, & \text{if } x>2\end{cases}$

The given function *f* is defined at all the points of the real line.

Let *c* be a point on the real line.

Case I:

If $c<2$, then $f(c)=c^3-3$ and $\lim\limits_{x\to c} f(x)=\lim\limits_{x\to c}\left(x^3-3\right)=c^3-3$

$\therefore \lim\limits_{x\to c} f(x)=f(c)$

Therefore, *f* is continuous at all points *x*, such that *x* < 2

Case II:

If $c=2$, then $f(c)=f(2)=2^3-3=5$

$\lim\limits_{x\to 2^-} f(x)=\lim\limits_{x\to 2^-}\left(x^3-3\right)=2^3-3=5$

$\lim\limits_{x\to 2^+} f(x)=\lim\limits_{x\to 2^+}\left(x^2+1\right)=2^2+1=5$

$\therefore \lim\limits_{x\to 2} f(x)=f(2)$

Therefore, *f* is continuous at *x* = 2

Case III:

If $c>2$, then $f(c)=c^2+1$

$\lim\limits_{x\to c} f(x)=\lim\limits_{x\to c}\left(x^2+1\right)=c^2+1$

$\therefore \lim\limits_{x\to c} f(x)=f(c)$

Therefore, *f* is continuous at all points *x*, such that *x* > 2

Thus, the given function *f* is continuous at every point on the real line.

Hence, *f* has no point of discontinuity.

Question 12:

Find all points of discontinuity of *f*, where *f* is defined by

$$f(x)=\begin{cases}x^{10}-1, & \text{if } x\le 1\\ x^2, & \text{if } x>1\end{cases}$$

Answer

The given function *f* is $f(x)=\begin{cases}x^{10}-1, & \text{if } x\le 1\\ x^2, & \text{if } x>1\end{cases}$

The given function *f* is defined at all the points of the real line.

Let *c* be a point on the real line.

Case I:

If $c<1$, then $f(c)=c^{10}-1$ and $\lim\limits_{x\to c} f(x)=\lim\limits_{x\to c}\left(x^{10}-1\right)=c^{10}-1$

$\therefore \lim\limits_{x\to c} f(x)=f(c)$

Therefore, *f* is continuous at all points *x*, such that *x* < 1

Case II:

If *c* = 1, then the left hand limit of *f* at *x* = 1 is,

$$\lim_{x\to 1^-} f(x)=\lim_{x\to 1^-}\left(x^{10}-1\right)=1^{10}-1=1-1=0$$

The right hand limit of *f* at *x* = 1 is,

$$\lim_{x\to 1^+} f(x)=\lim_{x\to 1^+}\left(x^2\right)=1^2=1$$

It is observed that the left and right hand limit of *f* at *x* = 1 do not coincide.

Therefore, *f* is not continuous at *x* = 1

Case III:

If $c>1$, then $f(c)=c^2$

$\lim\limits_{x\to c} f(x)=\lim\limits_{x\to c}\left(x^2\right)=c^2$

$\therefore \lim\limits_{x\to c} f(x)=f(c)$

Therefore, *f* is continuous at all points *x*, such that *x* > 1

Thus, from the above observation, it can be concluded that *x* = 1 is the only point of discontinuity of *f*.

Question 13:

Is the function defined by

$$f(x)=\begin{cases} x+5, \text{ if } x\le 1 \\ x-5, \text{ if } x>1 \end{cases}$$

a continuous function?

Answer

The given function is $f(x)=\begin{cases} x+5, \text{ if } x\le 1 \\ x-5, \text{ if } x>1 \end{cases}$

The given function *f* is defined at all the points of the real line.

Let *c* be a point on the real line.

Case I:

If $c<1$, then $f(c)=c+5$ and $\lim\limits_{x\to c} f(x)=\lim\limits_{x\to c}(x+5)=c+5$

$\therefore \lim\limits_{x\to c} f(x)=f(c)$

Therefore, *f* is continuous at all points *x*, such that *x* < 1

Case II:

If $c=1$, then $f(1)=1+5=6$

The left hand limit of *f* at *x* = 1 is,

$\lim\limits_{x\to 1^-} f(x)=\lim\limits_{x\to 1^-}(x+5)=1+5=6$

The right hand limit of *f* at *x* = 1 is,

$\lim\limits_{x\to 1^+} f(x)=\lim\limits_{x\to 1^+}(x-5)=1-5=-4$

It is observed that the left and right hand limit of *f* at *x* = 1 do not coincide.

Therefore, *f* is not continuous at *x* = 1

Case III:

If $c>1$, then $f(c)=c-5$ and $\lim\limits_{x\to c} f(x)=\lim\limits_{x\to c}(x-5)=c-5$

$\therefore \lim\limits_{x\to c} f(x)=f(c)$

Therefore, *f* is continuous at all points *x*, such that *x* > 1

Thus, from the above observation, it can be concluded that *x* = 1 is the only point of discontinuity of *f*.

Question 14:

Discuss the continuity of the function *f*, where *f* is defined by

$$f(x)=\begin{cases}3, & \text{if } 0\le x\le 1\\ 4, & \text{if } 1<x<3\\ 5, & \text{if } 3\le x\le 10\end{cases}$$

Answer

The given function is $f(x)=\begin{cases}3, & \text{if } 0\le x\le 1\\ 4, & \text{if } 1<x<3\\ 5, & \text{if } 3\le x\le 10\end{cases}$

The given function is defined at all points of the interval [0, 10].

Let *c* be a point in the interval [0, 10].

Case I:

$$\text{If } 0\le c<1, \text{ then } f(c)=3 \text{ and } \lim_{x\to c} f(x)=\lim_{x\to c}(3)=3$$

$$\therefore \lim_{x\to c} f(x)=f(c)$$

Therefore, *f* is continuous in the interval [0, 1).

Case II:

$$\text{If } c=1, \text{ then } f(3)=3$$

The left hand limit of *f* at *x* = 1 is,

$$\lim_{x\to 1^-} f(x)=\lim_{x\to 1^-}(3)=3$$

The right hand limit of *f* at *x* = 1 is,

$$\lim_{x\to 1^+} f(x)=\lim_{x\to 1^+}(4)=4$$

It is observed that the left and right hand limits of *f* at *x* = 1 do not coincide.

Therefore, *f* is not continuous at *x* = 1

Case III:

$$\text{If } 1<c<3, \text{ then } f(c)=4 \text{ and } \lim_{x\to c} f(x)=\lim_{x\to c}(4)=4$$

$$\therefore \lim_{x\to c} f(x)=f(c)$$

Therefore, *f* is continuous at all points of the interval (1, 3).

Case IV:

$$\text{If } c=3, \text{ then } f(c)=5$$

The left hand limit of *f* at *x* = 3 is,

$$\lim_{x\to3^-} f(x) = \lim_{x\to3^-}(4) = 4$$

The right hand limit of *f* at *x* = 3 is,

$$\lim_{x\to3^+} f(x) = \lim_{x\to3^+}(5) = 5$$

It is observed that the left and right hand limits of *f* at *x* = 3 do not coincide.

Therefore, *f* is not continuous at *x* = 3

Case V:

$$\text{If } 3 < c \le 10, \text{ then } f(c) = 5 \text{ and } \lim_{x\to c} f(x) = \lim_{x\to c}(5) = 5$$

$$\lim_{x\to c} f(x) = f(c)$$

Therefore, *f* is continuous at all points of the interval (3, 10].

Hence, *f* is not continuous at *x* = 1 and *x* = 3

Question 15:

Discuss the continuity of the function *f*, where *f* is defined by

$$f(x) = \begin{cases} 2x, & \text{if } x < 0 \\ 0, & \text{if } 0 \le x \le 1 \\ 4x, & \text{if } x > 1 \end{cases}$$

Answer

The given function is $f(x) = \begin{cases} 2x, & \text{if } x < 0 \\ 0, & \text{if } 0 \le x \le 1 \\ 4x, & \text{if } x > 1 \end{cases}$

The given function is defined at all points of the real line.

Let *c* be a point on the real line.

Case I:

$$\text{If } c < 0, \text{ then } f(c) = 2c$$

$$\lim_{x\to c} f(x) = \lim_{x\to c}(2x) = 2c$$

$$\therefore \lim_{x\to c} f(x) = f(c)$$

Therefore, *f* is continuous at all points *x*, such that *x* < 0

Case II:

$\text{If } c=0, \text{ then } f(c)=f(0)=0$

The left hand limit of *f* at *x* = 0 is,

$\lim_{x\to 0^-} f(x)=\lim_{x\to 0^-}(2x)=2\times 0=0$

The right hand limit of *f* at *x* = 0 is,

$\lim_{x\to 0^+} f(x)=\lim_{x\to 0^+}(0)=0$

$\therefore \lim_{x\to 0} f(x)=f(0)$

Therefore, *f* is continuous at *x* = 0

Case III:

$\text{If } 0<c<1, \text{ then } f(x)=0 \text{ and } \lim_{x\to c} f(x)=\lim_{x\to c}(0)=0$

$\therefore \lim_{x\to c} f(x)=f(c)$

Therefore, *f* is continuous at all points of the interval (0, 1).

Case IV:

$\text{If } c=1, \text{ then } f(c)=f(1)=0$

The left hand limit of *f* at *x* = 1 is,

$\lim_{x\to 1^-} f(x)=\lim_{x\to 1^-}(0)=0$

The right hand limit of *f* at *x* = 1 is,

$\lim_{x\to 1^+} f(x)=\lim_{x\to 1^+}(4x)=4\times 1=4$

It is observed that the left and right hand limits of *f* at *x* = 1 do not coincide.

Therefore, *f* is not continuous at *x* = 1

Case V:

$\text{If } c<1, \text{ then } f(c)=4c \text{ and } \lim_{x\to c} f(x)=\lim_{x\to c}(4x)=4c$

$\therefore \lim_{x\to c} f(x)=f(c)$

Therefore, *f* is continuous at all points *x*, such that *x* > 1

Hence, *f* is not continuous only at *x* = 1

Question 16:

Discuss the continuity of the function *f*, where *f* is defined by

$$f(x)=\begin{cases}-2, \text{ if } x\le -1\\ 2x, \text{ if } -1<x\le 1\\ 2, \text{ if } x>1\end{cases}$$

Answer

The given function *f* is $f(x)=\begin{cases}-2, \text{ if } x\le -1\\ 2x, \text{ if } -1<x\le 1\\ 2, \text{ if } x>1\end{cases}$

The given function is defined at all points of the real line.

Let *c* be a point on the real line.

Case I:

If $c<-1$, then $f(c)=-2$ and $\lim\limits_{x\to c} f(x)=\lim\limits_{x\to c}(-2)=-2$

$\therefore \lim\limits_{x\to c} f(x)=f(c)$

Therefore, *f* is continuous at all points *x*, such that $x < -1$

Case II:

If $c=-1$, then $f(c)=f(-1)=-2$

The left hand limit of *f* at $x = -1$ is,

$\lim\limits_{x\to -1^-} f(x)=\lim\limits_{x\to -1^-}(-2)=-2$

The right hand limit of *f* at $x = -1$ is,

$\lim\limits_{x\to -1^+} f(x)=\lim\limits_{x\to -1^+}(2x)=2\times(-1)=-2$

$\therefore \lim\limits_{x\to -1} f(x)=f(-1)$

Therefore, *f* is continuous at $x = -1$

Case III:

If $-1<c<1$, then $f(c)=2c$

$\lim\limits_{x\to c} f(x)=\lim\limits_{x\to c}(2x)=2c$

$\therefore \lim\limits_{x\to c} f(x)=f(c)$

Therefore, *f* is continuous at all points of the interval $(-1, 1)$.

Case IV:

$\text{If } c = 1, \text{ then } f(c) = f(1) = 2 \times 1 = 2$

The left hand limit of *f* at *x* = 1 is,

$$\lim_{x \to 1^-} f(x) = \lim_{x \to 1^-} (2x) = 2 \times 1 = 2$$

The right hand limit of *f* at *x* = 1 is,

$$\lim_{x \to 1^+} f(x) = \lim_{x \to 1^+} 2 = 2$$

$$\therefore \lim_{x \to 1} f(x) = f(c)$$

Therefore, *f* is continuous at *x* = 2

Case V:

$\text{If } c > 1, \text{ then } f(c) = 2 \text{ and } \lim_{x \to c} f(x) = \lim_{x \to c} (2) = 2$

$$\lim_{x \to c} f(x) = f(c)$$

Therefore, *f* is continuous at all points *x*, such that *x* > 1

Thus, from the above observations, it can be concluded that *f* is continuous at all points of the real line.

Question 17:

Find the relationship between *a* and *b* so that the function *f* defined by

$$f(x) = \begin{cases} ax + 1, & \text{if } x \le 3 \\ bx + 3, & \text{if } x > 3 \end{cases}$$

is continuous at *x* = 3.

Answer

The given function *f* is $f(x) = \begin{cases} ax + 1, & \text{if } x \le 3 \\ bx + 3, & \text{if } x > 3 \end{cases}$

If *f* is continuous at *x* = 3, then

$$\lim_{x\to 3^-} f(x) = \lim_{x\to 3^+} f(x) = f(3) \quad ...(1)$$

Also,

$$\lim_{x\to 3^-} f(x) = \lim_{x\to 3^-} (ax+1) = 3a+1$$

$$\lim_{x\to 3^+} f(x) = \lim_{x\to 3^+} (bx+3) = 3b+3$$

$$f(3) = 3a+1$$

Therefore, from (1), we obtain

$$3a+1 = 3b+3 = 3a+1$$

$$\Rightarrow 3a+1 = 3b+3$$

$$\Rightarrow 3a = 3b+2$$

$$\Rightarrow a = b+\frac{2}{3}$$

Therefore, the required relationship is given by, $a = b+\frac{2}{3}$

Question 18:

For what value of λ is the function defined by

$$f(x) = \begin{cases} \lambda(x^2-2x), & \text{if } x \le 0 \\ 4x+1, & \text{if } x > 0 \end{cases}$$

continuous at x = 0? What about continuity at x = 1?

Answer

The given function f is $f(x) = \begin{cases} \lambda(x^2-2x), & \text{if } x \le 0 \\ 4x+1, & \text{if } x > 0 \end{cases}$

If f is continuous at x = 0, then

$$\lim_{x\to 0^-} f(x) = \lim_{x\to 0^+} f(x) = f(0)$$

$$\Rightarrow \lim_{x\to 0^-} \lambda(x^2-2x) = \lim_{x\to 0^+} (4x+1) = \lambda(0^2-2\times 0)$$

$$\Rightarrow \lambda(0^2-2\times 0) = 4\times 0+1 = 0$$

$$\Rightarrow 0 = 1 = 0, \text{ which is not possible}$$

Therefore, there is no value of λ for which f is continuous at x = 0

At $x = 1$,

$f(1) = 4x + 1 = 4 \times 1 + 1 = 5$

$$\lim_{x\to 1}(4x+1) = 4\times 1+1 = 5$$

$$\therefore \lim_{x\to 1} f(x) = f(1)$$

Therefore, for any values of λ, f is continuous at $x = 1$

Question 19:

Show that the function defined by $g(x) = x - [x]$ is discontinuous at all integral point.

Here $[x]$ denotes the greatest integer less than or equal to x.

Answer

The given function is $g(x) = x - [x]$

It is evident that g is defined at all integral points.

Let n be an integer.

Then,

$$g(n) = n - [n] = n - n = 0$$

The left hand limit of f at $x = n$ is,

$$\lim_{x\to n^-} g(x) = \lim_{x\to n^-}(x - [x]) = \lim_{x\to n^-}(x) - \lim_{x\to n^-}[x] = n - (n-1) = 1$$

The right hand limit of f at $x = n$ is,

$$\lim_{x\to n^+} g(x) = \lim_{x\to n^+}(x - [x]) = \lim_{x\to n^+}(x) - \lim_{x\to n^+}[x] = n - n = 0$$

It is observed that the left and right hand limits of f at $x = n$ do not coincide.

Therefore, f is not continuous at $x = n$

Hence, g is discontinuous at all integral points.

Question 20:

Is the function defined by $f(x) = x^2 - \sin x + 5$ continuous at x = p?

Answer

The given function is $f(x) = x^2 - \sin x + 5$

It is evident that *f* is defined at *x* = p

At $x=\pi$, $f(x)=f(\pi)=\pi^2-\sin\pi+5=\pi^2-0+5=\pi^2+5$

Consider $\lim\limits_{x\to\pi} f(x)=\lim\limits_{x\to\pi}\left(x^2-\sin x+5\right)$

Put $x=\pi+h$

If $x\to\pi$, *then it is evident that* $h\to 0$

$$\begin{aligned}
\therefore \lim_{x\to\pi} f(x) &= \lim_{x\to\pi}\left(x^2-\sin x+5\right)\\
&=\lim_{h\to 0}\left[(\pi+h)^2-\sin(\pi+h)+5\right]\\
&=\lim_{h\to 0}(\pi+h)^2-\lim_{h\to 0}\sin(\pi+h)+\lim_{h\to 0}5\\
&=(\pi+0)^2-\lim_{h\to 0}\left[\sin\pi\cosh+\cos\pi\sinh\right]+5\\
&=\pi^2-\lim_{h\to 0}\sin\pi\cosh-\lim_{h\to 0}\cos\pi\sinh+5\\
&=\pi^2-\sin\pi\cos 0-\cos\pi\sin 0+5\\
&=\pi^2-0\times 1-(-1)\times 0+5\\
&=\pi^2+5
\end{aligned}$$

$\therefore \lim\limits_{x\to\pi} f(x)=f(\pi)$

Therefore, the given function *f* is continuous at *x* = π

Question 21:

Discuss the continuity of the following functions.

(a) *f* (*x*) = sin *x* + cos *x*

(b) *f* (*x*) = sin *x* − cos *x*

(c) *f* (*x*) = sin *x* × cos x

Answer

It is known that if *g* and *h* are two continuous functions, then

$g+h,\ g-h,\ \text{and } g.h$ are also continuous.

It has to proved first that *g* (*x*) = sin *x* and *h* (*x*) = cos *x* are continuous functions.

Let *g* (*x*) = sin *x*

It is evident that *g* (*x*) = sin *x* is defined for every real number.

Let *c* be a real number. Put *x* = *c* + *h*

If *x* → *c*, then *h* → 0

$$g(c) = \sin c$$

$$\lim_{x \to c} g(x) = \lim_{x \to c} \sin x$$
$$= \lim_{h \to 0} \sin(c+h)$$
$$= \lim_{h \to 0} [\sin c \cos h + \cos c \sin h]$$
$$= \lim_{h \to 0} (\sin c \cos h) + \lim_{h \to 0} (\cos c \sin h)$$
$$= \sin c \cos 0 + \cos c \sin 0$$
$$= \sin c + 0$$
$$= \sin c$$

$$\therefore \lim_{x \to c} g(x) = g(c)$$

Therefore, *g* is a continuous function.

Let *h* (*x*) = cos *x*

It is evident that *h* (*x*) = cos *x* is defined for every real number.

Let *c* be a real number. Put *x* = *c* + *h*

If *x* → *c*, then *h* → 0

h (*c*) = cos *c*

$$\lim_{x \to c} h(x) = \lim_{x \to c} \cos x$$
$$= \lim_{h \to 0} \cos(c+h)$$
$$= \lim_{h \to 0} [\cos c \cos h - \sin c \sin h]$$
$$= \lim_{h \to 0} \cos c \cos h - \lim_{h \to 0} \sin c \sin h$$
$$= \cos c \cos 0 - \sin c \sin 0$$
$$= \cos c \times 1 - \sin c \times 0$$
$$= \cos c$$

$$\therefore \lim_{x \to c} h(x) = h(c)$$

Therefore, *h* is a continuous function.

Therefore, it can be concluded that

(a) *f* (*x*) = *g* (*x*) + *h* (*x*) = sin *x* + cos *x* is a continuous function

(b) *f* (*x*) = *g* (*x*) − *h* (*x*) = sin *x* − cos *x* is a continuous function

(c) *f* (*x*) = *g* (*x*) × *h* (*x*) = sin *x* × cos *x* is a continuous function

Question 22:

Discuss the continuity of the cosine, cosecant, secant and cotangent functions,

Answer

It is known that if *g* and *h* are two continuous functions, then

$(i)\ \dfrac{h(x)}{g(x)},\ g(x) \neq 0$ is continuous

$(ii)\ \dfrac{1}{g(x)},\ g(x) \neq 0$ is continuous

$(iii)\ \dfrac{1}{h(x)},\ h(x) \neq 0$ is continuous

It has to be proved first that $g(x) = \sin x$ and $h(x) = \cos x$ are continuous functions.

Let $g(x) = \sin x$

It is evident that $g(x) = \sin x$ is defined for every real number.

Let c be a real number. Put $x = c + h$

If $x \to c$, then $h \to 0$

$$g(c) = \sin c$$

$$\begin{aligned}\lim_{x\to c} g(x) &= \lim_{x\to c} \sin x\\ &= \lim_{h\to 0} \sin(c+h)\\ &= \lim_{h\to 0}[\sin c\cos h + \cos c \sin h]\\ &= \lim_{h\to 0}(\sin c\cos h) + \lim_{h\to 0}(\cos c\sin h)\\ &= \sin c\cos 0 + \cos c\sin 0\\ &= \sin c + 0\\ &= \sin c\end{aligned}$$

$$\therefore\ \lim_{x\to c} g(x) = g(c)$$

Therefore, *g* is a continuous function.

Let $h(x) = \cos x$

It is evident that $h(x) = \cos x$ is defined for every real number.

Let c be a real number. Put $x = c + h$

If *x* ® *c*, then *h* ® 0

$h(c) = \cos c$

$$\lim_{x\to c} h(x) = \lim_{x\to c} \cos x$$
$$= \lim_{h\to 0} \cos(c+h)$$
$$= \lim_{h\to 0} [\cos c \cos h - \sin c \sin h]$$
$$= \lim_{h\to 0} \cos c \cos h - \lim_{h\to 0} \sin c \sin h$$
$$= \cos c \cos 0 - \sin c \sin 0$$
$$= \cos c \times 1 - \sin c \times 0$$
$$= \cos c$$
$$\therefore \lim_{x\to c} h(x) = h(c)$$

Therefore, *h* (*x*) = cos *x* is continuous function.

It can be concluded that,

$$\operatorname{cosec} x = \frac{1}{\sin x},\ \sin x \neq 0 \text{ is continuous}$$
$$\Rightarrow \operatorname{cosec} x,\ x \neq n\pi\ (n \in Z) \text{ is continuous}$$

Therefore, cosecant is continuous except at *x* = *n*p, *n* Î **Z**

$$\sec x = \frac{1}{\cos x},\ \cos x \neq 0 \text{ is continuous}$$
$$\Rightarrow \sec x,\ x \neq (2n+1)\frac{\pi}{2}\ (n \in \mathbf{Z}) \text{ is continuous}$$

Therefore, secant is continuous except at $x = (2n+1)\frac{\pi}{2}\ (n \in \mathbf{Z})$

$$\cot x = \frac{\cos x}{\sin x},\ \sin x \neq 0 \text{ is continuous}$$
$$\Rightarrow \cot x,\ x \neq n\pi\ (n \in Z) \text{ is continuous}$$

Therefore, cotangent is continuous except at *x* = *n*p, *n* Î **Z**

Question 23:

Find the points of discontinuity of *f*, where

$$f(x) = \begin{cases} \frac{\sin x}{x}, \text{ if } x < 0 \\ x+1, \text{ if } x \geq 0 \end{cases}$$

Answer

The given function *f* is $f(x)=\begin{cases}\dfrac{\sin x}{x}, & \text{if } x<0\\ x+1, & \text{if } x\geq 0\end{cases}$

It is evident that *f* is defined at all points of the real line.

Let *c* be a real number.

Case I:

$$\text{If } c<0, \text{ then } f(c)=\frac{\sin c}{c} \text{ and } \lim_{x\to c} f(x)=\lim_{x\to c}\left(\frac{\sin x}{x}\right)=\frac{\sin c}{c}$$

$$\therefore \lim_{x\to c} f(x)=f(c)$$

Therefore, *f* is continuous at all points *x*, such that $x < 0$

Case II:

$$\text{If } c>0, \text{ then } f(c)=c+1 \text{ and } \lim_{x\to c} f(x)=\lim_{x\to c}(x+1)=c+1$$

$$\therefore \lim_{x\to c} f(x)=f(c)$$

Therefore, *f* is continuous at all points *x*, such that $x > 0$

Case III:

$$\text{If } c=0, \text{ then } f(c)=f(0)=0+1=1$$

The left hand limit of *f* at $x = 0$ is,

$$\lim_{x\to 0^-} f(x)=\lim_{x\to 0}\frac{\sin x}{x}=1$$

The right hand limit of *f* at $x = 0$ is,

$$\lim_{x\to 0^+} f(x)=\lim_{x\to 0^+}(x+1)=1$$

$$\therefore \lim_{x\to 0^-} f(x)=\lim_{x\to 0^+} f(x)=f(0)$$

Therefore, *f* is continuous at $x = 0$

From the above observations, it can be concluded that *f* is continuous at all points of the real line.

Thus, *f* has no point of discontinuity.

Question 24:

Determine if *f* defined by

$$f(x)=\begin{cases} x^2\sin\frac{1}{x}, & \text{if } x\neq 0 \\ 0, & \text{if } x=0 \end{cases}$$

is a continuous function?

Answer

The given function *f* is $$f(x)=\begin{cases} x^2\sin\frac{1}{x}, & \text{if } x\neq 0 \\ 0, & \text{if } x=0 \end{cases}$$

It is evident that *f* is defined at all points of the real line.

Let *c* be a real number.

Case I:

$$\text{If } c\neq 0,\ \text{then } f(c)=c^2\sin\frac{1}{c}$$

$$\lim_{x\to c} f(x)=\lim_{x\to c}\left(x^2\sin\frac{1}{x}\right)=\left(\lim_{x\to c}x^2\right)\left(\lim_{x\to c}\sin\frac{1}{x}\right)=c^2\sin\frac{1}{c}$$

$$\therefore \lim_{x\to c} f(x)=f(c)$$

Therefore, *f* is continuous at all points $x \neq 0$

Case II:

$$\text{If } c=0,\ \text{then } f(0)=0$$

$$\lim_{x\to 0^-} f(x) = \lim_{x\to 0^-}\left(x^2 \sin\frac{1}{x}\right) = \lim_{x\to 0}\left(x^2 \sin\frac{1}{x}\right)$$

It is known that, $-1 \le \sin\frac{1}{x} \le 1,\ x \ne 0$

$$\Rightarrow -x^2 \le \sin\frac{1}{x} \le x^2$$

$$\Rightarrow \lim_{x\to 0}\left(-x^2\right) \le \lim_{x\to 0}\left(x^2 \sin\frac{1}{x}\right) \le \lim_{x\to 0} x^2$$

$$\Rightarrow 0 \le \lim_{x\to 0}\left(x^2 \sin\frac{1}{x}\right) \le 0$$

$$\Rightarrow \lim_{x\to 0}\left(x^2 \sin\frac{1}{x}\right) = 0$$

$$\therefore \lim_{x\to 0^-} f(x) = 0$$

Similarly, $\lim_{x\to 0^+} f(x) = \lim_{x\to 0^+}\left(x^2 \sin\frac{1}{x}\right) = \lim_{x\to 0}\left(x^2 \sin\frac{1}{x}\right) = 0$

$$\therefore \lim_{x\to 0^-} f(x) = f(0) = \lim_{x\to 0^+} f(x)$$

Therefore, *f* is continuous at $x = 0$

From the above observations, it can be concluded that *f* is continuous at every point of the real line.

Thus, *f* is a continuous function.

Question 25:

Examine the continuity of *f*, where *f* is defined by

$$f(x) = \begin{cases} \sin x - \cos x, & \text{if } x \ne 0 \\ -1 & \text{if } x = 0 \end{cases}$$

Answer

The given function *f* is $f(x) = \begin{cases} \sin x - \cos x, & \text{if } x \ne 0 \\ -1 & \text{if } x = 0 \end{cases}$

It is evident that *f* is defined at all points of the real line.

Let *c* be a real number.

Case I:

$\text{If } c \neq 0, \text{ then } f(c) = \sin c - \cos c$

$\lim\limits_{x \to c} f(x) = \lim\limits_{x \to c}(\sin x - \cos x) = \sin c - \cos c$

$\therefore \lim\limits_{x \to c} f(x) = f(c)$

Therefore, *f* is continuous at all points *x*, such that $x \neq 0$

Case II:

$\text{If } c = 0, \text{ then } f(0) = -1$

$\lim\limits_{x \to 0^-} f(x) = \lim\limits_{x \to 0}(\sin x - \cos x) = \sin 0 - \cos 0 = 0 - 1 = -1$

$\lim\limits_{x \to 0^+} f(x) = \lim\limits_{x \to 0}(\sin x - \cos x) = \sin 0 - \cos 0 = 0 - 1 = -1$

$\therefore \lim\limits_{x \to 0^-} f(x) = \lim\limits_{x \to 0^+} f(x) = f(0)$

Therefore, *f* is continuous at $x = 0$

From the above observations, it can be concluded that *f* is continuous at every point of the real line.

Thus, *f* is a continuous function.

Question 26:

Find the values of *k* so that the function *f* is continuous at the indicated point.

$$f(x) = \begin{cases} \dfrac{k \cos x}{\pi - 2x}, & \text{if } x \neq \dfrac{\pi}{2} \\ 3, & \text{if } x = \dfrac{\pi}{2} \end{cases} \quad \text{at } x = \frac{\pi}{2}$$

Answer

The given function *f* is $f(x) = \begin{cases} \dfrac{k \cos x}{\pi - 2x}, & \text{if } x \neq \dfrac{\pi}{2} \\ 3, & \text{if } x = \dfrac{\pi}{2} \end{cases}$

The given function *f* is continuous at $x = \frac{\pi}{2}$, if *f* is defined at $x = \frac{\pi}{2}$ and if the value of the *f* at $x = \frac{\pi}{2}$ equals the limit of *f* at $x = \frac{\pi}{2}$.

It is evident that *f* is defined at $x = \frac{\pi}{2}$ and $f\left(\frac{\pi}{2}\right) = 3$

$$\lim_{x \to \frac{\pi}{2}} f(x) = \lim_{x \to \frac{\pi}{2}} \frac{k \cos x}{\pi - 2x}$$

$$\text{Put } x = \frac{\pi}{2} + h$$

$$\text{Then, } x \to \frac{\pi}{2} \Rightarrow h \to 0$$

$$\therefore \lim_{x \to \frac{\pi}{2}} f(x) = \lim_{x \to \frac{\pi}{2}} \frac{k \cos x}{\pi - 2x} = \lim_{h \to 0} \frac{k \cos\left(\frac{\pi}{2} + h\right)}{\pi - 2\left(\frac{\pi}{2} + h\right)}$$

$$= k \lim_{h \to 0} \frac{-\sin h}{-2h} = \frac{k}{2} \lim_{h \to 0} \frac{\sin h}{h} = \frac{k}{2}.1 = \frac{k}{2}$$

$$\therefore \lim_{x \to \frac{\pi}{2}} f(x) = f\left(\frac{\pi}{2}\right)$$

$$\Rightarrow \frac{k}{2} = 3$$

$$\Rightarrow k = 6$$

Therefore, the required value of *k* is 6.

Question 27:

Find the values of *k* so that the function *f* is continuous at the indicated point.

$$f(x) = \begin{cases} kx^2, & \text{if } x \le 2 \\ 3, & \text{if } x > 2 \end{cases} \qquad \text{at } x = 2$$

Answer

The given function is $f(x) = \begin{cases} kx^2, & \text{if } x \le 2 \\ 3, & \text{if } x > 2 \end{cases}$

The given function *f* is continuous at $x = 2$, if *f* is defined at $x = 2$ and if the value of *f* at $x = 2$ equals the limit of *f* at $x = 2$

It is evident that *f* is defined at $x = 2$ and $f(2) = k(2)^2 = 4k$

$$\lim_{x\to 2^-} f(x) = \lim_{x\to 2^+} f(x) = f(2)$$
$$\Rightarrow \lim_{x\to 2^-}\left(kx^2\right) = \lim_{x\to 2^+}(3) = 4k$$
$$\Rightarrow k\times 2^2 = 3 = 4k$$
$$\Rightarrow 4k = 3 = 4k$$
$$\Rightarrow 4k = 3$$
$$\Rightarrow k = \frac{3}{4}$$

Therefore, the required value of k is $\frac{3}{4}$.

Question 28:

Find the values of *k* so that the function *f* is continuous at the indicated point.

$$f(x) = \begin{cases} kx+1, & \text{if } x \le \pi \\ \cos x, & \text{if } x > \pi \end{cases} \qquad \text{at } x = \pi$$

Answer

The given function is $f(x) = \begin{cases} kx+1, & \text{if } x \le \pi \\ \cos x, & \text{if } x > \pi \end{cases}$

The given function *f* is continuous at *x* = p, if *f* is defined at *x* = p and if the value of *f* at *x* = p equals the limit of *f* at *x* = p

It is evident that *f* is defined at *x* = p and $f(\pi) = k\pi + 1$

$$\lim_{x\to \pi^-} f(x) = \lim_{x\to \pi^+} f(x) = f(\pi)$$
$$\Rightarrow \lim_{x\to \pi^-}(kx+1) = \lim_{x\to \pi^+}\cos x = k\pi + 1$$
$$\Rightarrow k\pi + 1 = \cos\pi = k\pi + 1$$
$$\Rightarrow k\pi + 1 = -1 = k\pi + 1$$
$$\Rightarrow k = -\frac{2}{\pi}$$

Therefore, the required value of k is $-\frac{2}{\pi}$.

Question 29:

Find the values of *k* so that the function *f* is continuous at the indicated point.

$$f(x)=\begin{cases} kx+1, & \text{if } x\le 5 \\ 3x-5, & \text{if } x>5 \end{cases} \qquad \text{at } x=5$$

Answer

The given function *f* is $f(x)=\begin{cases} kx+1, & \text{if } x\le 5 \\ 3x-5, & \text{if } x>5 \end{cases}$

The given function *f* is continuous at *x* = 5, if *f* is defined at *x* = 5 and if the value of *f* at *x* = 5 equals the limit of *f* at *x* = 5

It is evident that *f* is defined at *x* = 5 and $f(5)=kx+1=5k+1$

$$\lim_{x\to 5^-} f(x)=\lim_{x\to 5^+} f(x)=f(5)$$
$$\Rightarrow \lim_{x\to 5^-}(kx+1)=\lim_{x\to 5^+}(3x-5)=5k+1$$
$$\Rightarrow 5k+1=15-5=5k+1$$
$$\Rightarrow 5k+1=10$$
$$\Rightarrow 5k=9$$
$$\Rightarrow k=\frac{9}{5}$$

Therefore, the required value of k is $\frac{9}{5}$.

Question 30:

Find the values of *a* and *b* such that the function defined by

$$f(x)=\begin{cases} 5, & \text{if } x\le 2 \\ ax+b, & \text{if } 2<x<10 \\ 21, & \text{if } x\ge 10 \end{cases}$$

is a continuous function.

Answer

The given function *f* is
$$f(x)=\begin{cases}5, & \text{if } x\le 2\\ ax+b, & \text{if } 2<x<10\\ 21, & \text{if } x\ge 10\end{cases}$$

It is evident that the given function *f* is defined at all points of the real line.

If *f* is a continuous function, then *f* is continuous at all real numbers.

In particular, *f* is continuous at *x* = 2 and *x* = 10

Since *f* is continuous at *x* = 2, we obtain

$$\lim_{x\to 2^-} f(x)=\lim_{x\to 2^+} f(x)=f(2)$$
$$\Rightarrow \lim_{x\to 2^-}(5)=\lim_{x\to 2^+}(ax+b)=5$$
$$\Rightarrow 5=2a+b=5$$
$$\Rightarrow 2a+b=5 \qquad \ldots(1)$$

Since *f* is continuous at *x* = 10, we obtain

$$\lim_{x\to 10^-} f(x)=\lim_{x\to 10^+} f(x)=f(10)$$
$$\Rightarrow \lim_{x\to 10^-}(ax+b)=\lim_{x\to 10^+}(21)=21$$
$$\Rightarrow 10a+b=21=21$$
$$\Rightarrow 10a+b=21 \qquad \ldots(2)$$

On subtracting equation (1) from equation (2), we obtain

8*a* = 16

⇒ *a* = 2

By putting *a* = 2 in equation (1), we obtain

2 × 2 + *b* = 5

⇒ 4 + *b* = 5

$\Rightarrow b = 1$

Therefore, the values of *a* and *b* for which *f* is a continuous function are 2 and 1 respectively.

Question 31:

Show that the function defined by $f(x) = \cos(x^2)$ is a continuous function.

Answer

The given function is $f(x) = \cos(x^2)$

This function *f* is defined for every real number and *f* can be written as the composition of two functions as,

$f = g \circ h$, where $g(x) = \cos x$ and $h(x) = x^2$

$$\left[\because (goh)(x) = g\left(h(x)\right) = g\left(x^2\right) = \cos\left(x^2\right) = f(x)\right]$$

It has to be first proved that $g(x) = \cos x$ and $h(x) = x^2$ are continuous functions.

It is evident that *g* is defined for every real number.

Let *c* be a real number.

Then, $g(c) = \cos c$

Put $x = c + h$

If $x \to c$, then $h \to 0$

$$\begin{aligned}
\lim_{x\to c} g(x) &= \lim_{x\to c} \cos x \\
&= \lim_{h\to 0} \cos(c+h) \\
&= \lim_{h\to 0}\left[\cos c \cos h - \sin c \sin h\right] \\
&= \lim_{h\to 0} \cos c \cos h - \lim_{h\to 0} \sin c \sin h \\
&= \cos c \cos 0 - \sin c \sin 0 \\
&= \cos c \times 1 - \sin c \times 0 \\
&= \cos c
\end{aligned}$$

$$\therefore \lim_{x\to c} g(x) = g(c)$$

Therefore, $g(x) = \cos x$ is continuous function.

h (*x*) = x^2

Clearly, *h* is defined for every real number.

Let *k* be a real number, then *h* (*k*) = k^2

$$\lim_{x\to k} h(x) = \lim_{x\to k} x^2 = k^2$$
$$\therefore \lim_{x\to k} h(x) = h(k)$$

Therefore, *h* is a continuous function.

It is known that for real valued functions *g* and *h*,such that (*g* o *h*) is defined at *c*, if *g* is continuous at *c* and if *f* is continuous at *g* (*c*), then (*f* o *g*) is continuous at *c*.

Therefore, $f(x) = (goh)(x) = \cos(x^2)$ is a continuous function.

Question 32:

Show that the function defined by $f(x) = |\cos x|$ is a continuous function.

Answer

The given function is $f(x) = |\cos x|$

This function *f* is defined for every real number and *f* can be written as the composition of two functions as,

f = *g o h*, where $g(x) = |x|$ and $h(x) = \cos x$

$$\left[\because (goh)(x) = g(h(x)) = g(\cos x) = |\cos x| = f(x)\right]$$

It has to be first proved that $g(x) = |x|$ and $h(x) = \cos x$ are continuous functions.

$g(x) = |x|$ can be written as

$$g(x) = \begin{cases} -x, & \text{if } x < 0 \\ x, & \text{if } x \geq 0 \end{cases}$$

Clearly, *g* is defined for all real numbers.

Let *c* be a real number.

Case I:

If $c < 0$, then $g(c) = -c$ and $\lim_{x\to c} g(x) = \lim_{x\to c}(-x) = -c$

$$\therefore \lim_{x\to c} g(x) = g(c)$$

Therefore, *g* is continuous at all points *x*, such that $x < 0$

Case II:

$$\text{If } c > 0, \text{ then } g(c) = c \text{ and } \lim_{x \to c} g(x) = \lim_{x \to c} x = c$$

$$\therefore \lim_{x \to c} g(x) = g(c)$$

Therefore, *g* is continuous at all points *x*, such that $x > 0$

Case III:

$$\text{If } c = 0, \text{ then } g(c) = g(0) = 0$$

$$\lim_{x \to 0^-} g(x) = \lim_{x \to 0^-} (-x) = 0$$
$$\lim_{x \to 0^+} g(x) = \lim_{x \to 0^+} (x) = 0$$
$$\therefore \lim_{x \to 0^-} g(x) = \lim_{x \to 0^+} (x) = g(0)$$

Therefore, *g* is continuous at $x = 0$

From the above three observations, it can be concluded that *g* is continuous at all points.

$h(x) = \cos x$

It is evident that $h(x) = \cos x$ is defined for every real number.

Let *c* be a real number. Put $x = c + h$

If $x \to c$, then $h \to 0$

$h(c) = \cos c$

$$\begin{aligned}
\lim_{x \to c} h(x) &= \lim_{x \to c} \cos x \\
&= \lim_{h \to 0} \cos(c + h) \\
&= \lim_{h \to 0} [\cos c \cos h - \sin c \sin h] \\
&= \lim_{h \to 0} \cos c \cos h - \lim_{h \to 0} \sin c \sin h \\
&= \cos c \cos 0 - \sin c \sin 0 \\
&= \cos c \times 1 - \sin c \times 0 \\
&= \cos c
\end{aligned}$$

$$\therefore \lim_{x \to c} h(x) = h(c)$$

Therefore, $h(x) = \cos x$ is a continuous function.

It is known that for real valued functions *g* and *h*,such that (*g* o *h*) is defined at *c*, if *g* is continuous at *c* and if *f* is continuous at *g* (*c*), then (*f* o *g*) is continuous at *c*.

Therefore, $f(x)=(goh)(x)=g(h(x))=g(\cos x)=|\cos x|$ is a continuous function.

Question 33:

Examine that $\sin|x|$ is a continuous function.

Answer

Let $f(x)=\sin|x|$

This function *f* is defined for every real number and *f* can be written as the composition of two functions as,

f = *g o h*, where $g(x)=|x|$ and $h(x)=\sin x$

$\left[\because (goh)(x)=g(h(x))=g(\sin x)=|\sin x|=f(x)\right]$

It has to be proved first that $g(x)=|x|$ and $h(x)=\sin x$ are continuous functions.

$g(x)=|x|$ can be written as

$$g(x)=\begin{cases}-x, & \text{if } x<0\\ x, & \text{if } x\geq 0\end{cases}$$

Clearly, *g* is defined for all real numbers.

Let *c* be a real number.

Case I:

If $c<0$, then $g(c)=-c$ and $\lim\limits_{x\to c} g(x)=\lim\limits_{x\to c}(-x)=-c$

$\therefore \lim\limits_{x\to c} g(x)=g(c)$

Therefore, *g* is continuous at all points *x*, such that *x* < 0

Case II:

If $c>0$, then $g(c)=c$ and $\lim\limits_{x\to c} g(x)=\lim\limits_{x\to c} x=c$

$\therefore \lim\limits_{x\to c} g(x)=g(c)$

Therefore, *g* is continuous at all points *x*, such that *x* > 0

Case III:

If $c=0$, then $g(c)=g(0)=0$

$$\lim_{x \to 0^-} g(x) = \lim_{x \to 0^-} (-x) = 0$$
$$\lim_{x \to 0^+} g(x) = \lim_{x \to 0^+} (x) = 0$$
$$\therefore \lim_{x \to 0^-} g(x) = \lim_{x \to 0^+} (x) = g(0)$$

Therefore, *g* is continuous at *x* = 0

From the above three observations, it can be concluded that *g* is continuous at all points.

h (*x*) = sin *x*

It is evident that *h* (*x*) = sin *x* is defined for every real number.

Let *c* be a real number. Put *x* = *c* + *k*

If *x* → *c*, then *k* → 0

h (*c*) = sin *c*

$$h(c) = \sin c$$
$$\begin{aligned}\lim_{x \to c} h(x) &= \lim_{x \to c} \sin x \\ &= \lim_{k \to 0} \sin(c+k) \\ &= \lim_{k \to 0} [\sin c \cos k + \cos c \sin k] \\ &= \lim_{k \to 0} (\sin c \cos k) + \lim_{h \to 0} (\cos c \sin k) \\ &= \sin c \cos 0 + \cos c \sin 0 \\ &= \sin c + 0 \\ &= \sin c\end{aligned}$$
$$\therefore \lim_{x \to c} h(x) = g(c)$$

Therefore, *h* is a continuous function.

It is known that for real valued functions *g* and *h*,such that (*g* o *h*) is defined at *c*, if *g* is continuous at *c* and if *f* is continuous at *g* (*c*), then (*f* o *g*) is continuous at *c*.

Therefore, $f(x) = (goh)(x) = g(h(x)) = g(\sin x) = |\sin x|$ is a continuous function.

Question 34:

Find all the points of discontinuity of *f* defined by $f(x) = |x| - |x+1|$.

Answer

The given function is $f(x) = |x| - |x+1|$

The two functions, *g* and *h*, are defined as

$g(x)=|x|$ and $h(x)=|x+1|$

Then, $f = g - h$

The continuity of *g* and *h* is examined first.

$g(x)=|x|$ can be written as

$$g(x)=\begin{cases}-x, & \text{if } x<0\\ x, & \text{if } x\geq 0\end{cases}$$

Clearly, *g* is defined for all real numbers.

Let *c* be a real number.

Case I:

If $c<0$, then $g(c)=-c$ and $\lim\limits_{x\to c} g(x)=\lim\limits_{x\to c}(-x)=-c$

$\therefore \lim\limits_{x\to c} g(x)=g(c)$

Therefore, *g* is continuous at all points *x*, such that $x < 0$

Case II:

If $c>0$, then $g(c)=c$ and $\lim\limits_{x\to c} g(x)=\lim\limits_{x\to c} x=c$

$\therefore \lim\limits_{x\to c} g(x)=g(c)$

Therefore, *g* is continuous at all points *x*, such that $x > 0$

Case III:

If $c=0$, then $g(c)=g(0)=0$

$\lim\limits_{x\to 0^-} g(x)=\lim\limits_{x\to 0^-}(-x)=0$

$\lim\limits_{x\to 0^+} g(x)=\lim\limits_{x\to 0^+}(x)=0$

$\therefore \lim\limits_{x\to 0^-} g(x)=\lim\limits_{x\to 0^+}(x)=g(0)$

Therefore, *g* is continuous at $x = 0$

From the above three observations, it can be concluded that *g* is continuous at all points.

$h(x)=|x+1|$ can be written as

$$h(x)=\begin{cases}-(x+1), & \text{if, } x<-1\\ x+1, & \text{if } x\geq -1\end{cases}$$

Clearly, *h* is defined for every real number.

Let *c* be a real number.

Case I:

$$\text{If } c < -1, \text{ then } h(c) = -(c+1) \text{ and } \lim_{x \to c} h(x) = \lim_{x \to c}\left[-(x+1)\right] = -(c+1)$$

$$\therefore \lim_{x \to c} h(x) = h(c)$$

Therefore, *h* is continuous at all points *x*, such that *x* < −1

Case II:

$$\text{If } c > -1, \text{ then } h(c) = c+1 \text{ and } \lim_{x \to c} h(x) = \lim_{x \to c}(x+1) = c+1$$

$$\therefore \lim_{x \to c} h(x) = h(c)$$

Therefore, *h* is continuous at all points *x*, such that *x* > −1

Case III:

$$\text{If } c = -1, \text{ then } h(c) = h(-1) = -1+1 = 0$$

$$\lim_{x \to -1^-} h(x) = \lim_{x \to -1^-}\left[-(x+1)\right] = -(-1+1) = 0$$

$$\lim_{x \to -1^+} h(x) = \lim_{x \to -1^+}(x+1) = (-1+1) = 0$$

$$\therefore \lim_{x \to -1^-} h(x) = \lim_{h \to -1^+} h(x) = h(-1)$$

Therefore, *h* is continuous at *x* = −1

From the above three observations, it can be concluded that *h* is continuous at all points of the real line.

g and *h* are continuous functions. Therefore, *f* = *g* − *h* is also a continuous function.

Therefore, *f* has no point of discontinuity.