Exercise 5.3

Question 1:

Find $\frac{dy}{dx}$:

$2x+3y=\sin x$

Answer

The given relationship is $2x+3y=\sin x$

Differentiating this relationship with respect to *x*, we obtain

$$\frac{d}{dx}(2x+3y)=\frac{d}{dx}(\sin x)$$

$$\Rightarrow \frac{d}{dx}(2x)+\frac{d}{dx}(3y)=\cos x$$

$$\Rightarrow 2+3\frac{dy}{dx}=\cos x$$

$$\Rightarrow 3\frac{dy}{dx}=\cos x-2$$

$$\therefore \frac{dy}{dx}=\frac{\cos x-2}{3}$$

Question 2:

Find $\frac{dy}{dx}$:

$2x+3y=\sin y$

Answer

The given relationship is $2x+3y=\sin y$

Differentiating this relationship with respect to *x*, we obtain

$$\frac{d}{dx}(2x)+\frac{d}{dx}(3y)=\frac{d}{dx}(\sin y)$$

$$\Rightarrow 2+3\frac{dy}{dx}=\cos y\frac{dy}{dx} \qquad \text{[By using chain rule]}$$

$$\Rightarrow 2=(\cos y-3)\frac{dy}{dx}$$

$$\therefore \frac{dy}{dx}=\frac{2}{\cos y-3}$$

Question 3:

Find $\frac{dy}{dx}$:

$ax+by^2=\cos y$

Answer

The given relationship is $ax+by^2=\cos y$

Differentiating this relationship with respect to *x*, we obtain

$$\frac{d}{dx}(ax)+\frac{d}{dx}(by^2)=\frac{d}{dx}(\cos y)$$

$$\Rightarrow a+b\frac{d}{dx}(y^2)=\frac{d}{dx}(\cos y) \qquad ...(1)$$

Using chain rule, we obtain $\frac{d}{dx}(y^2)=2y\frac{dy}{dx}$ and $\frac{d}{dx}(\cos y)=-\sin y\frac{dy}{dx}$ $\qquad ...(2)$

From (1) and (2), we obtain

$$a+b\times 2y\frac{dy}{dx}=-\sin y\frac{dy}{dx}$$

$$\Rightarrow (2by+\sin y)\frac{dy}{dx}=-a$$

$$\therefore \frac{dy}{dx}=\frac{-a}{2by+\sin y}$$

Question 4:

Find $\frac{dy}{dx}$:

$xy+y^2=\tan x+y$

Answer

The given relationship is $xy + y^2 = \tan x + y$

Differentiating this relationship with respect to *x*, we obtain

$$\frac{d}{dx}\left(xy + y^2\right) = \frac{d}{dx}\left(\tan x + y\right)$$

$$\Rightarrow \frac{d}{dx}(xy) + \frac{d}{dx}\left(y^2\right) = \frac{d}{dx}(\tan x) + \frac{dy}{dx}$$

$$\Rightarrow \left[y \cdot \frac{d}{dx}(x) + x.\frac{dy}{dx}\right] + 2y\frac{dy}{dx} = \sec^2 x + \frac{dy}{dx} \qquad \text{[Using product rule and chain rule]}$$

$$\Rightarrow y \cdot 1 + x.\frac{dy}{dx} + 2y\frac{dy}{dx} = \sec^2 x + \frac{dy}{dx}$$

$$\Rightarrow (x + 2y - 1)\frac{dy}{dx} = \sec^2 x - y$$

$$\therefore \frac{dy}{dx} = \frac{\sec^2 x - y}{(x + 2y - 1)}$$

Question 5:

Find $\frac{dy}{dx}$:

$x^2 + xy + y^2 = 100$

Answer

The given relationship is $x^2 + xy + y^2 = 100$

Differentiating this relationship with respect to *x*, we obtain

$$\frac{d}{dx}\left(x^2 + xy + y^2\right) = \frac{d}{dx}(100)$$

$$\Rightarrow \frac{d}{dx}\left(x^2\right) + \frac{d}{dx}(xy) + \frac{d}{dx}\left(y^2\right) = 0$$ [Derivative of constant function is 0]

$$\Rightarrow 2x + \left[y \cdot \frac{d}{dx}(x) + x \cdot \frac{dy}{dx} \right] + 2y\frac{dy}{dx} = 0 \qquad \text{[Using product rule and chain rule]}$$

$$\Rightarrow 2x + y \cdot 1 + x \cdot \frac{dy}{dx} + 2y\frac{dy}{dx} = 0$$

$$\Rightarrow 2x + y + (x + 2y)\frac{dy}{dx} = 0$$

$$\therefore \frac{dy}{dx} = -\frac{2x + y}{x + 2y}$$

Question 6:

Find $\frac{dy}{dx}$:

$x^3 + x^2y + xy^2 + y^3 = 81$

Answer

The given relationship is $x^3 + x^2y + xy^2 + y^3 = 81$

Differentiating this relationship with respect to *x*, we obtain

$$\frac{d}{dx}\left(x^3 + x^2y + xy^2 + y^3\right) = \frac{d}{dx}(81)$$

$$\Rightarrow \frac{d}{dx}\left(x^3\right) + \frac{d}{dx}\left(x^2y\right) + \frac{d}{dx}\left(xy^2\right) + \frac{d}{dx}\left(y^3\right) = 0$$

$$\Rightarrow 3x^2 + \left[y\frac{d}{dx}\left(x^2\right) + x^2\frac{dy}{dx} \right] + \left[y^2\frac{d}{dx}(x) + x\frac{d}{dx}\left(y^2\right) \right] + 3y^2\frac{dy}{dx} = 0$$

$$\Rightarrow 3x^2 + \left[y \cdot 2x + x^2\frac{dy}{dx} \right] + \left[y^2 \cdot 1 + x \cdot 2y \cdot \frac{dy}{dx} \right] + 3y^2\frac{dy}{dx} = 0$$

$$\Rightarrow \left(x^2 + 2xy + 3y^2\right)\frac{dy}{dx} + \left(3x^2 + 2xy + y^2\right) = 0$$

$$\therefore \frac{dy}{dx} = \frac{-\left(3x^2 + 2xy + y^2\right)}{\left(x^2 + 2xy + 3y^2\right)}$$

Question 7:

Find $\frac{dy}{dx}$:

$\sin^2 y + \cos xy = \pi$

Answer

The given relationship is $\sin^2 y + \cos xy = \pi$

Differentiating this relationship with respect to *x*, we obtain

$$\frac{d}{dx}\left(\sin^2 y + \cos xy\right) = \frac{d}{dx}(\pi)$$

$$\Rightarrow \frac{d}{dx}\left(\sin^2 y\right) + \frac{d}{dx}(\cos xy) = 0 \qquad ...(1)$$

Using chain rule, we obtain

$$\frac{d}{dx}\left(\sin^2 y\right) = 2\sin y \frac{d}{dx}(\sin y) = 2\sin y \cos y \frac{dy}{dx} \qquad ...(2)$$

$$\frac{d}{dx}(\cos xy) = -\sin xy \frac{d}{dx}(xy) = -\sin xy\left[y\frac{d}{dx}(x) + x\frac{dy}{dx}\right]$$

$$= -\sin xy\left[y.1 + x\frac{dy}{dx}\right] = -y\sin xy - x\sin xy\frac{dy}{dx} \qquad ...(3)$$

From (1), (2), and (3), we obtain

$$2\sin y\cos y\frac{dy}{dx} - y\sin xy - x\sin xy\frac{dy}{dx} = 0$$

$$\Rightarrow (2\sin y\cos y - x\sin xy)\frac{dy}{dx} = y\sin xy$$

$$\Rightarrow (\sin 2y - x\sin xy)\frac{dy}{dx} = y\sin xy$$

$$\therefore \frac{dy}{dx} = \frac{y\sin xy}{\sin 2y - x\sin xy}$$

Question 8:

Find $\frac{dy}{dx}$:

$\sin^2 x + \cos^2 y = 1$

Answer

The given relationship is $\sin^2 x + \cos^2 y = 1$

Differentiating this relationship with respect to *x*, we obtain

$$\frac{d}{dx}\left(\sin^2 x+\cos^2 y\right)=\frac{d}{dx}(1)$$
$$\Rightarrow \frac{d}{dx}\left(\sin^2 x\right)+\frac{d}{dx}\left(\cos^2 y\right)=0$$
$$\Rightarrow 2\sin x\cdot\frac{d}{dx}(\sin x)+2\cos y\cdot\frac{d}{dx}(\cos y)=0$$
$$\Rightarrow 2\sin x\cos x+2\cos y(-\sin y)\cdot\frac{dy}{dx}=0$$
$$\Rightarrow \sin 2x-\sin 2y\frac{dy}{dx}=0$$
$$\therefore \frac{dy}{dx}=\frac{\sin 2x}{\sin 2y}$$

Question 9:

Find $\frac{dy}{dx}$:

$$y=\sin^{-1}\left(\frac{2x}{1+x^2}\right)$$

Answer

The given relationship is $y=\sin^{-1}\left(\frac{2x}{1+x^2}\right)$

$$y=\sin^{-1}\left(\frac{2x}{1+x^2}\right)$$
$$\Rightarrow \sin y=\frac{2x}{1+x^2}$$

Differentiating this relationship with respect to *x*, we obtain

$$\frac{d}{dx}(\sin y)=\frac{d}{dx}\left(\frac{2x}{1+x^2}\right)$$
$$\Rightarrow \cos y\frac{dy}{dx}=\frac{d}{dx}\left(\frac{2x}{1+x^2}\right) \qquad ...(1)$$

The function, $\frac{2x}{1+x^2}$, is of the form of $\frac{u}{v}$.

Therefore, by quotient rule, we obtain

$$\frac{d}{dx}\left(\frac{2x}{1+x^2}\right)=\frac{\left(1+x^2\right).\frac{d}{dx}(2x)-2x\cdot\frac{d}{dx}\left(1+x^2\right)}{\left(1+x^2\right)^2}$$

$$=\frac{\left(1+x^2\right).2-2x\cdot[0+2x]}{\left(1+x^2\right)^2}=\frac{2+2x^2-4x^2}{\left(1+x^2\right)^2}=\frac{2\left(1-x^2\right)}{\left(1+x^2\right)^2} \qquad ...(2)$$

Also, $\sin y=\frac{2x}{1+x^2}$

$$\Rightarrow \cos y=\sqrt{1-\sin^2 y}=\sqrt{1-\left(\frac{2x}{1+x^2}\right)^2}=\sqrt{\frac{\left(1+x^2\right)^2-4x^2}{\left(1+x^2\right)^2}}$$

$$=\sqrt{\frac{\left(1-x^2\right)^2}{\left(1+x^2\right)^2}}=\frac{1-x^2}{1+x^2} \qquad ...(3)$$

From (1), (2), and (3), we obtain

$$\frac{1-x^2}{1+x^2}\times\frac{dy}{dx}=\frac{2\left(1-x^2\right)}{\left(1+x^2\right)^2}$$

$$\Rightarrow \frac{dy}{dx}=\frac{2}{1+x^2}$$

Question 10:

Find $\frac{dy}{dx}$:

$$y=\tan^{-1}\left(\frac{3x-x^3}{1-3x^2}\right),-\frac{1}{\sqrt{3}}<x<\frac{1}{\sqrt{3}}$$

Answer

The given relationship is $y=\tan^{-1}\left(\frac{3x-x^3}{1-3x^2}\right)$

$$y = \tan^{-1}\left(\frac{3x - x^3}{1 - 3x^2}\right)$$

$$\Rightarrow \tan y = \frac{3x - x^3}{1 - 3x^2} \qquad ...(1)$$

It is known that,

$$\tan y = \frac{3\tan\frac{y}{3} - \tan^3\frac{y}{3}}{1 - 3\tan^2\frac{y}{3}} \qquad ...(2)$$

Comparing equations (1) and (2), we obtain

$$x = \tan\frac{y}{3}$$

Differentiating this relationship with respect to *x*, we obtain

$$\frac{d}{dx}(x) = \frac{d}{dx}\left(\tan\frac{y}{3}\right)$$

$$\Rightarrow 1 = \sec^2\frac{y}{3}\cdot\frac{d}{dx}\left(\frac{y}{3}\right)$$

$$\Rightarrow 1 = \sec^2\frac{y}{3}\cdot\frac{1}{3}\cdot\frac{dy}{dx}$$

$$\Rightarrow \frac{dy}{dx} = \frac{3}{\sec^2\frac{y}{3}} = \frac{3}{1 + \tan^2\frac{y}{3}}$$

$$\therefore \frac{dy}{dx} = \frac{3}{1 + x^2}$$

Question 11:

Find $\frac{dy}{dx}$:

$$y = \cos^{-1}\left(\frac{1 - x^2}{1 + x^2}\right), 0 < x < 1$$

x

Answer

The given relationship is,

$$y=\cos^{-1}\left(\frac{1-x^2}{1+x^2}\right)$$

$$\Rightarrow \cos y=\frac{1-x^2}{1+x^2}$$

$$\Rightarrow \frac{1-\tan^2\frac{y}{2}}{1+\tan^2\frac{y}{2}}=\frac{1-x^2}{1+x^2}$$

On comparing L.H.S. and R.H.S. of the above relationship, we obtain

$$\tan\frac{y}{2}=x$$

Differentiating this relationship with respect to *x*, we obtain

$$\sec^2\frac{y}{2}\cdot\frac{d}{dx}\left(\frac{y}{2}\right)=\frac{d}{dx}(x)$$

$$\Rightarrow \sec^2\frac{y}{2}\times\frac{1}{2}\frac{dy}{dx}=1$$

$$\Rightarrow \frac{dy}{dx}=\frac{2}{\sec^2\frac{y}{2}}$$

$$\Rightarrow \frac{dy}{dx}=\frac{2}{1+\tan^2\frac{y}{2}}$$

$$\therefore \frac{dy}{dx}=\frac{1}{1+x^2}$$

Question 12:

Find $\frac{dy}{dx}$:

$$y=\sin^{-1}\left(\frac{1-x^2}{1+x^2}\right),\ 0<x<1$$

Answer

The given relationship is $y=\sin^{-1}\left(\frac{1-x^2}{1+x^2}\right)$

$$y = \sin^{-1}\left(\frac{1-x^2}{1+x^2}\right)$$

$$\Rightarrow \sin y = \frac{1-x^2}{1+x^2}$$

Differentiating this relationship with respect to *x*, we obtain

$$\frac{d}{dx}(\sin y) = \frac{d}{dx}\left(\frac{1-x^2}{1+x^2}\right) \qquad ...(1)$$

Using chain rule, we obtain

$$\frac{d}{dx}(\sin y) = \cos y \cdot \frac{dy}{dx}$$

$$\cos y = \sqrt{1-\sin^2 y} = \sqrt{1-\left(\frac{1-x^2}{1+x^2}\right)^2}$$

$$= \sqrt{\frac{\left(1+x^2\right)^2-\left(1-x^2\right)^2}{\left(1+x^2\right)^2}} = \sqrt{\frac{4x^2}{\left(1+x^2\right)^2}} = \frac{2x}{1+x^2}$$

$$\therefore \frac{d}{dx}(\sin y) = \frac{2x}{1+x^2}\frac{dy}{dx} \qquad ...(2)$$

$$\frac{d}{dx}\left(\frac{1-x^2}{1+x^2}\right) = \frac{\left(1+x^2\right)\cdot\left(1-x^2\right)' - \left(1-x^2\right)\cdot\left(1+x^2\right)'}{\left(1+x^2\right)^2} \qquad \left[\text{Using quotient rule}\right]$$

$$= \frac{\left(1+x^2\right)(-2x)-\left(1-x^2\right)\cdot(2x)}{\left(1+x^2\right)^2}$$

$$= \frac{-2x-2x^3-2x+2x^3}{\left(1+x^2\right)^2}$$

$$= \frac{-4x}{\left(1+x^2\right)^2} \qquad ...(3)$$

From (1), (2), and (3), we obtain

$$\frac{2x}{1+x^2}\frac{dy}{dx}=\frac{-4x}{\left(1+x^2\right)^2}$$

$$\Rightarrow \frac{dy}{dx}=\frac{-2}{1+x^2}$$

Alternate method

$$y=\sin^{-1}\left(\frac{1-x^2}{1+x^2}\right)$$

$$\Rightarrow \sin y=\frac{1-x^2}{1+x^2}$$

$$\Rightarrow \left(1+x^2\right)\sin y=1-x^2$$

$$\Rightarrow \left(1+\sin y\right)x^2=1-\sin y$$

$$\Rightarrow x^2=\frac{1-\sin y}{1+\sin y}$$

$$\Rightarrow x^2=\frac{\left(\cos\frac{y}{2}-\sin\frac{y}{2}\right)^2}{\left(\cos\frac{y}{2}+\sin\frac{y}{2}\right)^2}$$

$$\Rightarrow x=\frac{\cos\frac{y}{2}-\sin\frac{y}{2}}{\cos\frac{y}{2}+\sin\frac{y}{2}}$$

$$\Rightarrow x=\frac{1-\tan\frac{y}{2}}{1+\tan\frac{y}{2}}$$

$$\Rightarrow x=\tan\left(\frac{\pi}{4}-\frac{y}{2}\right)$$

Differentiating this relationship with respect to *x*, we obtain

$$\frac{d}{dx}(x)=\frac{d}{dx}\cdot\left[\tan\left(\frac{\pi}{4}-\frac{y}{2}\right)\right]$$

$$\Rightarrow 1=\sec^2\left(\frac{\pi}{4}-\frac{y}{2}\right)\cdot\frac{d}{dx}\left(\frac{\pi}{4}-\frac{y}{2}\right)$$

$$\Rightarrow 1=\left[1+\tan^2\left(\frac{\pi}{4}-\frac{y}{2}\right)\right]\cdot\left(-\frac{1}{2}\frac{dy}{dx}\right)$$

$$\Rightarrow 1=\left(1+x^2\right)\left(-\frac{1}{2}\frac{dy}{dx}\right)$$

$$\Rightarrow \frac{dy}{dx}=\frac{-2}{1+x^2}$$

Question 13:

Find $\frac{dy}{dx}$:

$$y=\cos^{-1}\left(\frac{2x}{1+x^2}\right),-1<x<1$$

Answer

The given relationship is $y=\cos^{-1}\left(\frac{2x}{1+x^2}\right)$

$$y=\cos^{-1}\left(\frac{2x}{1+x^2}\right)$$

$$\Rightarrow \cos y=\frac{2x}{1+x^2}$$

Differentiating this relationship with respect to *x*, we obtain

$$\frac{d}{dx}(\cos y)=\frac{d}{dx}\cdot\left(\frac{2x}{1+x^2}\right)$$

$$\Rightarrow -\sin y\cdot\frac{dy}{dx}=\frac{\left(1+x^2\right)\cdot\frac{d}{dx}(2x)-2x\cdot\frac{d}{dx}\left(1+x^2\right)}{\left(1+x^2\right)^2}$$

$$\Rightarrow -\sqrt{1-\cos^2 y}\frac{dy}{dx} = \frac{\left(1+x^2\right)\times 2 - 2x\cdot 2x}{\left(1+x^2\right)^2}$$

$$\Rightarrow \left[\sqrt{1-\left(\frac{2x}{1+x^2}\right)^2}\right]\frac{dy}{dx} = -\left[\frac{2\left(1-x^2\right)}{\left(1+x^2\right)^2}\right]$$

$$\Rightarrow \sqrt{\frac{\left(1+x^2\right)^2-4x^2}{\left(1+x^2\right)^2}}\frac{dy}{dx} = \frac{-2\left(1-x^2\right)}{\left(1+x^2\right)^2}$$

$$\Rightarrow \sqrt{\frac{\left(1-x^2\right)^2}{\left(1+x^2\right)^2}}\frac{dy}{dx} = \frac{-2\left(1-x^2\right)}{\left(1+x^2\right)^2}$$

$$\Rightarrow \frac{1-x^2}{1+x^2}\cdot\frac{dy}{dx} = \frac{-2\left(1-x^2\right)}{\left(1+x^2\right)^2}$$

$$\Rightarrow \frac{dy}{dx} = \frac{-2}{1+x^2}$$

Question 14:

Find $\frac{dy}{dx}$:

$y = \sin^{-1}\left(2x\sqrt{1-x^2}\right),\ -\frac{1}{\sqrt{2}} < x < \frac{1}{\sqrt{2}}$

Answer

lationship is $y = \sin^{-1}\left(2x\sqrt{1-x^2}\right)$

$$y = \sin^{-1}\left(2x\sqrt{1-x^2}\right)$$

$$\Rightarrow \sin y = 2x\sqrt{1-x^2}$$

Differentiating this relationship with respect to *x*, we obtain

$$\cos y\frac{dy}{dx}=2\left[x\frac{d}{dx}\left(\sqrt{1-x^2}\right)+\sqrt{1-x^2}\frac{dx}{dx}\right]$$

$$\Rightarrow\sqrt{1-\sin^2 y}\frac{dy}{dx}=2\left[\frac{x}{2}.\frac{-2x}{\sqrt{1-x^2}}+\sqrt{1-x^2}\right]$$

$$\Rightarrow\sqrt{1-\left(2x\sqrt{1-x^2}\right)^2}\frac{dy}{dx}=2\left[\frac{-x^2+1-x^2}{\sqrt{1-x^2}}\right]$$

$$\Rightarrow\sqrt{1-4x^2\left(1-x^2\right)}\frac{dy}{dx}=2\left[\frac{1-2x^2}{\sqrt{1-x^2}}\right]$$

$$\Rightarrow\sqrt{\left(1-2x^2\right)^2}\frac{dy}{dx}=2\left[\frac{1-2x^2}{\sqrt{1-x^2}}\right]$$

$$\Rightarrow\left(1-2x^2\right)\frac{dy}{dx}=2\left[\frac{1-2x^2}{\sqrt{1-x^2}}\right]$$

$$\Rightarrow\frac{dy}{dx}=\frac{2}{\sqrt{1-x^2}}$$

Question 15:

Find $\frac{dy}{dx}$:

$$y=\sec^{-1}\left(\frac{1}{2x^2-1}\right),0<x<\frac{1}{\sqrt{2}}$$

Answer

The given relationship is $y=\sec^{-1}\left(\frac{1}{2x^2-1}\right)$

$$y=\sec^{-1}\left(\frac{1}{2x^2-1}\right)$$

$$\Rightarrow \sec y = \frac{1}{2x^2 - 1}$$

$$\Rightarrow \cos y = 2x^2 - 1$$

$$\Rightarrow 2x^2 = 1 + \cos y$$

$$\Rightarrow 2x^2 = 2\cos^2 \frac{y}{2}$$

$$\Rightarrow x = \cos \frac{y}{2}$$

Differentiating this relationship with respect to *x*, we obtain

$$\frac{d}{dx}(x) = \frac{d}{dx}\left(\cos \frac{y}{2}\right)$$

$$\Rightarrow 1 = -\sin \frac{y}{2} \cdot \frac{d}{dx}\left(\frac{y}{2}\right)$$

$$\Rightarrow \frac{-1}{\sin \frac{y}{2}} = \frac{1}{2}\ \frac{dy}{dx}$$

$$\Rightarrow \frac{dy}{dx} = \frac{-2}{\sin \frac{y}{2}} = \frac{-2}{\sqrt{1 - \cos^2 \frac{y}{2}}}$$

$$\Rightarrow \frac{dy}{dx} = \frac{-2}{\sqrt{1 - x^2}}$$