Exercise 5.4

Question 1:

Differentiate the following w.r.t. x:

$$\frac{e^x}{\sin x}$$

Answer

Let $y = \frac{e^x}{\sin x}$

By using the quotient rule, we obtain

$$\frac{dy}{dx} = \frac{\sin x \frac{d}{dx}\left(e^x\right) - e^x \frac{d}{dx}\left(\sin x\right)}{\sin^2 x}$$

$$= \frac{\sin x.\left(e^x\right) - e^x \cdot \left(\cos x\right)}{\sin^2 x}$$

$$= \frac{e^x\left(\sin x - \cos x\right)}{\sin^2 x}, x \neq n\pi, n \in \mathbf{Z}$$

Question 2:

Differentiate the following w.r.t. x:

$$e^{\sin^{-1} x}$$

Answer

Let $y = e^{\sin^{-1} x}$

By using the chain rule, we obtain

$$\frac{dy}{dx}=\frac{d}{dx}\left(e^{\sin^{-1}x}\right)$$

$$\Rightarrow \frac{dy}{dx}=e^{\sin^{-1}x}\cdot\frac{d}{dx}\left(\sin^{-1}x\right)$$

$$=e^{\sin^{-1}x}\cdot\frac{1}{\sqrt{1-x^2}}$$

$$=\frac{e^{\sin^{-1}x}}{\sqrt{1-x^2}}$$

$$\therefore \frac{dy}{dx}=\frac{e^{\sin^{-1}x}}{\sqrt{1-x^2}}, x\in(-1,1)$$

Question 2:

Show that the function given by $f(x) = e^{2x}$ is strictly increasing on **R**.

Answer

Let x_1 and x_2 be any two numbers in **R**.

Then, we have:

$$x_1<x_2 \Rightarrow 2x_1<2x_2 \Rightarrow e^{2x_1}<e^{2x_2} \Rightarrow f(x_1)<f(x_2)$$

Hence, *f* is strictly increasing on **R**.

Question 3:

Differentiate the following w.r.t. *x*:

e^{x^3}

Answer

Let $y=e^{x^3}$

By using the chain rule, we obtain

$$\frac{dy}{dx}=\frac{d}{dx}\left(e^{x^3}\right)=e^{x^3}\cdot\frac{d}{dx}\left(x^3\right)=e^{x^3}\cdot 3x^2=3x^2e^{x^3}$$

Question 4:

Differentiate the following w.r.t. *x*:

$\sin\left(\tan^{-1}e^{-x}\right)$

Answer

Let $y = \sin\left(\tan^{-1} e^{-x}\right)$

By using the chain rule, we obtain

$$\frac{dy}{dx} = \frac{d}{dx}\left[\sin\left(\tan^{-1} e^{-x}\right)\right]$$

$$= \cos\left(\tan^{-1} e^{-x}\right)\cdot\frac{d}{dx}\left(\tan^{-1} e^{-x}\right)$$

$$= \cos\left(\tan^{-1} e^{-x}\right)\cdot\frac{1}{1+\left(e^{-x}\right)^2}\cdot\frac{d}{dx}\left(e^{-x}\right)$$

$$= \frac{\cos\left(\tan^{-1} e^{-x}\right)}{1+e^{-2x}}\cdot e^{-x}\cdot\frac{d}{dx}(-x)$$

$$= \frac{e^{-x}\cos\left(\tan^{-1} e^{-x}\right)}{1+e^{-2x}}\times(-1)$$

$$= \frac{-e^{-x}\cos\left(\tan^{-1} e^{-x}\right)}{1+e^{-2x}}$$

Question 5:

Differentiate the following w.r.t. *x*:

$\log\left(\cos e^{x}\right)$

Answer

Let $y = \log\left(\cos e^{x}\right)$

By using the chain rule, we obtain

$$\frac{dy}{dx} = \frac{d}{dx}\left[\log\left(\cos e^{x}\right)\right]$$

$$= \frac{1}{\cos e^{x}}\cdot\frac{d}{dx}\left(\cos e^{x}\right)$$

$$= \frac{1}{\cos e^{x}}\cdot\left(-\sin e^{x}\right)\cdot\frac{d}{dx}\left(e^{x}\right)$$

$$= \frac{-\sin e^{x}}{\cos e^{x}}\cdot e^{x}$$

$$= -e^{x}\tan e^{x}, e^{x} \neq (2n+1)\frac{\pi}{2}, n\in\mathbf{N}$$

Question 6:

Differentiate the following w.r.t. *x*:

$e^x + e^{x^2} + \ldots + e^{x^5}$

Answer

$$\frac{d}{dx}\left(e^x + e^{x^2} + \ldots + e^{x^5}\right)$$

$$= \frac{d}{dx}\left(e^x\right) + \frac{d}{dx}\left(e^{x^2}\right) + \frac{d}{dx}\left(e^{x^3}\right) + \frac{d}{dx}\left(e^{x^4}\right) + \frac{d}{dx}\left(e^{x^5}\right)$$

$$= e^x + \left[e^{x^2} \times \frac{d}{dx}\left(x^2\right)\right] + \left[e^{x^3} \cdot \frac{d}{dx}\left(x^3\right)\right] + \left[e^{x^4} \cdot \frac{d}{dx}\left(x^4\right)\right] + \left[e^{x^5} \cdot \frac{d}{dx}\left(x^5\right)\right]$$

$$= e^x + \left(e^{x^2} \times 2x\right) + \left(e^{x^3} \times 3x^2\right) + \left(e^{x^4} \times 4x^3\right) + \left(e^{x^5} \times 5x^4\right)$$

$$= e^x + 2xe^{x^2} + 3x^2e^{x^3} + 4x^3e^{x^4} + 5x^4e^{x^5}$$

Question 7:

Differentiate the following w.r.t. *x*:

$\sqrt{e^{\sqrt{x}}}, x > 0$

Answer

Let $y = \sqrt{e^{\sqrt{x}}}$

Then, $y^2 = e^{\sqrt{x}}$

By differentiating this relationship with respect to *x*, we obtain

$y^2 = e^{\sqrt{x}}$

$\Rightarrow 2y\frac{dy}{dx} = e^{\sqrt{x}}\frac{d}{dx}\left(\sqrt{x}\right)$ [By applying the chain rule]

$\Rightarrow 2y\frac{dy}{dx} = e^{\sqrt{x}}\frac{1}{2}.\frac{1}{\sqrt{x}}$

$\Rightarrow \frac{dy}{dx} = \frac{e^{\sqrt{x}}}{4y\sqrt{x}}$

$\Rightarrow \frac{dy}{dx} = \frac{e^{\sqrt{x}}}{4\sqrt{e^{\sqrt{x}}}\sqrt{x}}$

$\Rightarrow \frac{dy}{dx} = \frac{e^{\sqrt{x}}}{4\sqrt{xe^{\sqrt{x}}}},\ x > 0$

Question 8:

Differentiate the following w.r.t. *x*:

$\log\left(\log x\right), x > 1$

Answer

Let $y = \log\left(\log x\right)$

By using the chain rule, we obtain

$\frac{dy}{dx} = \frac{d}{dx}\left[\log\left(\log x\right)\right]$

$= \frac{1}{\log x}\cdot\frac{d}{dx}\left(\log x\right)$

$= \frac{1}{\log x}\cdot\frac{1}{x}$

$= \frac{1}{x\log x}$, $x > 1$

Question 9:

Differentiate the following w.r.t. *x*:

$\frac{\cos x}{\log x}, x > 0$

Answer

Let $y=\frac{\cos x}{\log x}$

By using the quotient rule, we obtain

$$\frac{dy}{dx}=\frac{\frac{d}{dx}(\cos x)\times\log x-\cos x\times\frac{d}{dx}(\log x)}{(\log x)^2}$$

$$=\frac{-\sin x\log x-\cos x\times\frac{1}{x}}{(\log x)^2}$$

$$=\frac{-[x\log x.\sin x+\cos x]}{x(\log x)^2}, x>0$$

Question 10:

Differentiate the following w.r.t. *x*:

$\cos\left(\log x+e^x\right), x>0$

Answer

Let $y=\cos\left(\log x+e^x\right)$

By using the chain rule, we obtain

$$\frac{dy}{dx}=-\sin\left(\log x+e^x\right)\cdot\frac{d}{dx}\left(\log x+e^x\right)$$

$$=-\sin\left(\log x+e^x\right)\cdot\left[\frac{d}{dx}(\log x)+\frac{d}{dx}\left(e^x\right)\right]$$

$$=-\sin\left(\log x+e^x\right)\cdot\left(\frac{1}{x}+e^x\right)$$

$$=-\left(\frac{1}{x}+e^x\right)\sin\left(\log x+e^x\right), x>0$$