Exercise 5.6

Question 1:

If *x* and *y* are connected parametrically by the equation, without eliminating the parameter, find $\frac{dy}{dx}$.

$x = 2at^2,\ y = at^4$

Answer

The given equations are $x = 2at^2$ and $y = at^4$

$$\text{Then, } \frac{dx}{dt} = \frac{d}{dt}\left(2at^2\right) = 2a \cdot \frac{d}{dt}\left(t^2\right) = 2a \cdot 2t = 4at$$

$$\frac{dy}{dt} = \frac{d}{dt}\left(at^4\right) = a \cdot \frac{d}{dt}\left(t^4\right) = a \cdot 4 \cdot t^3 = 4at^3$$

$$\therefore \frac{dy}{dx} = \frac{\left(\frac{dy}{dt}\right)}{\left(\frac{dx}{dt}\right)} = \frac{4at^3}{4at} = t^2$$

Question 2:

If *x* and *y* are connected parametrically by the equation, without eliminating the parameter, find $\frac{dy}{dx}$.

$x = a \cos \theta$, $y = b \cos \theta$

Answer

The given equations are $x = a \cos \theta$ and $y = b \cos \theta$

$$\text{Then, } \frac{dx}{d\theta} = \frac{d}{d\theta}(a\cos\theta) = a(-\sin\theta) = -a\sin\theta$$

$$\frac{dy}{d\theta} = \frac{d}{d\theta}(b\cos\theta) = b(-\sin\theta) = -b\sin\theta$$

$$\therefore \frac{dy}{dx} = \frac{\left(\frac{dy}{d\theta}\right)}{\left(\frac{dx}{d\theta}\right)} = \frac{-b\sin\theta}{-a\sin\theta} = \frac{b}{a}$$

Question 3:

If *x* and *y* are connected parametrically by the equation, without eliminating the parameter, find $\frac{dy}{dx}$.

$x = \sin t$, $y = \cos 2t$

Answer

The given equations are $x = \sin t$ and $y = \cos 2t$

$$\text{Then, } \frac{dx}{dt} = \frac{d}{dt}(\sin t) = \cos t$$

$$\frac{dy}{dt} = \frac{d}{dt}(\cos 2t) = -\sin 2t \cdot \frac{d}{dt}(2t) = -2\sin 2t$$

$$\therefore \frac{dy}{dx} = \frac{\left(\frac{dy}{dt}\right)}{\left(\frac{dx}{dt}\right)} = \frac{-2\sin 2t}{\cos t} = \frac{-2 \cdot 2\sin t\cos t}{\cos t} = -4\sin t$$

Question 4:

If *x* and *y* are connected parametrically by the equation, without eliminating the parameter, find $\frac{dy}{dx}$.

$$x = 4t,\ y = \frac{4}{t}$$

Answer

The given equations are $x = 4t \text{ and } y = \frac{4}{t}$

$$\frac{dx}{dt}=\frac{d}{dt}(4t)=4$$

$$\frac{dy}{dt}=\frac{d}{dt}\left(\frac{4}{t}\right)=4\cdot\frac{d}{dt}\left(\frac{1}{t}\right)=4\cdot\left(\frac{-1}{t^2}\right)=\frac{-4}{t^2}$$

$$\therefore \frac{dy}{dx}=\frac{\left(\frac{dy}{dt}\right)}{\left(\frac{dx}{dt}\right)}=\frac{\left(\frac{-4}{t^2}\right)}{4}=\frac{-1}{t^2}$$

Question 5:

If *x* and *y* are connected parametrically by the equation, without eliminating the parameter, find $\frac{dy}{dx}$.

$x=\cos\theta-\cos 2\theta,\ y=\sin\theta-\sin 2\theta$

Answer

The given equations are $x=\cos\theta-\cos 2\theta$ and $y=\sin\theta-\sin 2\theta$

$$\text{Then, } \frac{dx}{d\theta}=\frac{d}{d\theta}(\cos\theta-\cos 2\theta)=\frac{d}{d\theta}(\cos\theta)-\frac{d}{d\theta}(\cos 2\theta)$$
$$=-\sin\theta-(-2\sin 2\theta)=2\sin 2\theta-\sin\theta$$

$$\frac{dy}{d\theta}=\frac{d}{d\theta}(\sin\theta-\sin 2\theta)=\frac{d}{d\theta}(\sin\theta)-\frac{d}{d\theta}(\sin 2\theta)$$
$$=\cos\theta-2\cos 2\theta$$

$$\therefore \frac{dy}{dx}=\frac{\left(\frac{dy}{d\theta}\right)}{\left(\frac{dx}{d\theta}\right)}=\frac{\cos\theta-2\cos 2\theta}{2\sin 2\theta-\sin\theta}$$

Question 6:

If *x* and *y* are connected parametrically by the equation, without eliminating the parameter, find $\frac{dy}{dx}$.

$x=a(\theta-\sin\theta),\ y=a(1+\cos\theta)$

Answer

The given equations are $x = a(\theta - \sin\theta)$ and $y = a(1+\cos\theta)$

$$\text{Then, } \frac{dx}{d\theta} = a\left[\frac{d}{d\theta}(\theta) - \frac{d}{d\theta}(\sin\theta)\right] = a(1-\cos\theta)$$

$$\frac{dy}{d\theta} = a\left[\frac{d}{d\theta}(1) + \frac{d}{d\theta}(\cos\theta)\right] = a\left[0 + (-\sin\theta)\right] = -a\sin\theta$$

$$\therefore \frac{dy}{dx} = \frac{\left(\frac{dy}{d\theta}\right)}{\left(\frac{dx}{d\theta}\right)} = \frac{-a\sin\theta}{a(1-\cos\theta)} = \frac{-2\sin\frac{\theta}{2}\cos\frac{\theta}{2}}{2\sin^2\frac{\theta}{2}} = \frac{-\cos\frac{\theta}{2}}{\sin\frac{\theta}{2}} = -\cot\frac{\theta}{2}$$

Question 7:

If *x* and *y* are connected parametrically by the equation, without eliminating the parameter, find $\frac{dy}{dx}$.

$$x = \frac{\sin^3 t}{\sqrt{\cos 2t}},\ y = \frac{\cos^3 t}{\sqrt{\cos 2t}}$$

Answer

The given equations are $x = \frac{\sin^3 t}{\sqrt{\cos 2t}}$ and $y = \frac{\cos^3 t}{\sqrt{\cos 2t}}$

Then, $\dfrac{dx}{dt}=\dfrac{d}{dt}\left[\dfrac{\sin^3 t}{\sqrt{\cos 2t}}\right]$

$$=\frac{\sqrt{\cos 2t}\cdot\frac{d}{dt}\left(\sin^3 t\right)-\sin^3 t\cdot\frac{d}{dt}\sqrt{\cos 2t}}{\cos 2t}$$

$$=\frac{\sqrt{\cos 2t}\cdot 3\sin^2 t\cdot\frac{d}{dt}(\sin t)-\sin^3 t\times\frac{1}{2\sqrt{\cos 2t}}.\frac{d}{dt}(\cos 2t)}{\cos 2t}$$

$$=\frac{3\sqrt{\cos 2t}\cdot\sin^2 t\cos t-\frac{\sin^3 t}{2\sqrt{\cos 2t}}.(-2\sin 2t)}{\cos 2t}$$

$$=\frac{3\cos 2t\sin^2 t\cos t+\sin^3 t\sin 2t}{\cos 2t\sqrt{\cos 2t}}$$

$\dfrac{dy}{dt}=\dfrac{d}{dt}\left[\dfrac{\cos^3 t}{\sqrt{\cos 2t}}\right]$

$$=\frac{\sqrt{\cos 2t}.\frac{d}{dt}\left(\cos^3 t\right)-\cos^3 t\cdot\frac{d}{dt}\left(\sqrt{\cos 2t}\right)}{\cos 2t}$$

$$=\frac{\sqrt{\cos 2t}.3\cos^2 t\cdot\frac{d}{dt}(\cos t)-\cos^3 t.\frac{1}{2\sqrt{\cos 2t}}\cdot\frac{d}{dt}(\cos 2t)}{\cos 2t}$$

$$=\frac{3\sqrt{\cos 2t}.\cos^2 t(-\sin t)-\cos^3 t\cdot\frac{1}{2\sqrt{\cos 2t}}.(-2\sin 2t)}{\cos 2t}$$

$$=\frac{-3\cos 2t.\cos^2 t.\sin t+\cos^3 t\sin 2t}{\cos 2t\cdot\sqrt{\cos 2t}}$$

$$\therefore \frac{dy}{dx}=\frac{\left(\frac{dy}{dt}\right)}{\left(\frac{dx}{dt}\right)}=\frac{-3\cos 2t.\cos^2 t.\sin t+\cos^3 t\sin 2t}{3\cos 2t\sin^2 t\cos t+\sin^3 t\sin 2t}$$

$$=\frac{-3\cos 2t.\cos^2 t.\sin t+\cos^3 t\left(2\sin t\cos t\right)}{3\cos 2t\sin^2 t\cos t+\sin^3 t\left(2\sin t\cos t\right)}$$

$$=\frac{\sin t\cos t\left[-3\cos 2t.\cos t+2\cos^3 t\right]}{\sin t\cos t\left[3\cos 2t\sin t+2\sin^3 t\right]}$$

$$=\frac{\left[-3\left(2\cos^2 t-1\right)\cos t+2\cos^3 t\right]}{\left[3\left(1-2\sin^2 t\right)\sin t+2\sin^3 t\right]} \qquad \begin{bmatrix}\cos 2t=\left(2\cos^2 t-1\right),\\ \cos 2t=\left(1-2\sin^2 t\right)\end{bmatrix}$$

$$=\frac{-4\cos^3 t+3\cos t}{3\sin t-4\sin^3 t}$$

$$=\frac{-\cos 3t}{\sin 3t} \qquad \begin{bmatrix}\cos 3t=4\cos^3 t-3\cos t,\\ \sin 3t=3\sin t-4\sin^3 t\end{bmatrix}$$

$$=-\cot 3t$$

Question 8:

If x and y are connected parametrically by the equation, without eliminating the parameter, find $\frac{dy}{dx}$.

$x=a\left(\cos t+\log\tan\frac{t}{2}\right),\ y=a\sin t$

Answer

The given equations are $x=a\left(\cos t+\log\tan\frac{t}{2}\right)$ and $y=a\sin t$

Then, $\frac{dx}{dt} = a \cdot \left[\frac{d}{dt}(\cos t) + \frac{d}{dt}\left(\log \tan \frac{t}{2}\right)\right]$

$$= a\left[-\sin t + \frac{1}{\tan \frac{t}{2}} \cdot \frac{d}{dt}\left(\tan \frac{t}{2}\right)\right]$$

$$= a\left[-\sin t + \cot \frac{t}{2} \cdot \sec^2 \frac{t}{2} \cdot \frac{d}{dt}\left(\frac{t}{2}\right)\right]$$

$$= a\left[-\sin t + \frac{\cos \frac{t}{2}}{\sin \frac{t}{2}} \times \frac{1}{\cos^2 \frac{t}{2}} \times \frac{1}{2}\right]$$

$$= a\left[-\sin t + \frac{1}{2\sin \frac{t}{2} \cos \frac{t}{2}}\right]$$

$$= a\left(-\sin t + \frac{1}{\sin t}\right)$$

$$= a\left(\frac{-\sin^2 t + 1}{\sin t}\right)$$

$$= a\frac{\cos^2 t}{\sin t}$$

$$\frac{dy}{dt} = a\frac{d}{dt}(\sin t) = a\cos t$$

$$\therefore \frac{dy}{dx} = \frac{\left(\frac{dy}{dt}\right)}{\left(\frac{dx}{dt}\right)} = \frac{a\cos t}{\left(a\frac{\cos^2 t}{\sin t}\right)} = \frac{\sin t}{\cos t} = \tan t$$

Question 9:

If x and y are connected parametrically by the equation, without eliminating the parameter, find $\frac{dy}{dx}$.

$x = a\sec\theta,\ y = b\tan\theta$

Answer

The given equations are $x = a\sec\theta$ and $y = b\tan\theta$

Then, $\frac{dx}{d\theta} = a \cdot \frac{d}{d\theta}(\sec\theta) = a\sec\theta\tan\theta$

$\frac{dy}{d\theta} = b \cdot \frac{d}{d\theta}(\tan\theta) = b\sec^2\theta$

$\therefore \frac{dy}{dx} = \frac{\left(\frac{dy}{d\theta}\right)}{\left(\frac{dx}{d\theta}\right)} = \frac{b\sec^2\theta}{a\sec\theta\tan\theta} = \frac{b}{a}\sec\theta\cot\theta = \frac{b\cos\theta}{a\cos\theta\sin\theta} = \frac{b}{a} \times \frac{1}{\sin\theta} = \frac{b}{a}\operatorname{cosec}\theta$

Question 10:

If x and y are connected parametrically by the equation, without eliminating the parameter, find $\frac{dy}{dx}$.

$x = a(\cos\theta + \theta\sin\theta),\ y = a(\sin\theta - \theta\cos\theta)$

Answer

The given equations are $x = a(\cos\theta + \theta\sin\theta)$ and $y = a(\sin\theta - \theta\cos\theta)$

Then, $\frac{dx}{d\theta} = a\left[\frac{d}{d\theta}\cos\theta + \frac{d}{d\theta}(\theta\sin\theta)\right] = a\left[-\sin\theta + \theta\frac{d}{d\theta}(\sin\theta) + \sin\theta\frac{d}{d\theta}(\theta)\right]$

$= a[-\sin\theta + \theta\cos\theta + \sin\theta] = a\theta\cos\theta$

$\frac{dy}{d\theta} = a\left[\frac{d}{d\theta}(\sin\theta) - \frac{d}{d\theta}(\theta\cos\theta)\right] = a\left[\cos\theta - \left\{\theta\frac{d}{d\theta}(\cos\theta) + \cos\theta \cdot \frac{d}{d\theta}(\theta)\right\}\right]$

$= a[\cos\theta + \theta\sin\theta - \cos\theta]$

$= a\theta\sin\theta$

$\therefore \frac{dy}{dx} = \frac{\left(\frac{dy}{d\theta}\right)}{\left(\frac{dx}{d\theta}\right)} = \frac{a\theta\sin\theta}{a\theta\cos\theta} = \tan\theta$

Question 11:

If $x=\sqrt{a^{\sin^{-1}t}},\ y=\sqrt{a^{\cos^{-1}t}}$, show that $\frac{dy}{dx}=-\frac{y}{x}$

Answer

The given equations are $x=\sqrt{a^{\sin^{-1}t}}$ and $y=\sqrt{a^{\cos^{-1}t}}$

$x=\sqrt{a^{\sin^{-1}t}}$ and $y=\sqrt{a^{\cos^{-1}t}}$

$\Rightarrow x=\left(a^{\sin^{-1}t}\right)^{\frac{1}{2}}$ and $y=\left(a^{\cos^{-1}t}\right)^{\frac{1}{2}}$

$\Rightarrow x=a^{\frac{1}{2}\sin^{-1}t}$ and $y=a^{\frac{1}{2}\cos^{-1}t}$

Consider $x=a^{\frac{1}{2}\sin^{-1}t}$

Taking logarithm on both the sides, we obtain

$$\log x=\frac{1}{2}\sin^{-1}t\log a$$

$$\therefore \frac{1}{x}\cdot\frac{dx}{dt}=\frac{1}{2}\log a\cdot\frac{d}{dt}\left(\sin^{-1}t\right)$$

$$\Rightarrow \frac{dx}{dt}=\frac{x}{2}\log a\cdot\frac{1}{\sqrt{1-t^2}}$$

$$\Rightarrow \frac{dx}{dt}=\frac{x\log a}{2\sqrt{1-t^2}}$$

Then, consider $y=a^{\frac{1}{2}\cos^{-1}t}$

Taking logarithm on both the sides, we obtain

$$\log y=\frac{1}{2}\cos^{-1}t\log a$$

$$\therefore \frac{1}{y}\cdot\frac{dy}{dx}=\frac{1}{2}\log a\cdot\frac{d}{dt}\left(\cos^{-1}t\right)$$

$$\Rightarrow \frac{dy}{dt}=\frac{y\log a}{2}\cdot\left(\frac{-1}{\sqrt{1-t^2}}\right)$$

$$\Rightarrow \frac{dy}{dt}=\frac{-y\log a}{2\sqrt{1-t^2}}$$

$$\therefore \frac{dy}{dx} = \frac{\left(\frac{dy}{dt}\right)}{\left(\frac{dx}{dt}\right)} = \frac{\left(\frac{-y\log a}{2\sqrt{1-t^2}}\right)}{\left(\frac{x\log a}{2\sqrt{1-t^2}}\right)} = -\frac{y}{x}.$$

Hence, proved.