Exercise 5.8

Question 1:

Verify Rolle's Theorem for the function $f(x)=x^2+2x-8$, $x\in[-4,2]$

Answer

The given function, $f(x)=x^2+2x-8$, being a polynomial function, is continuous in [−4, 2] and is differentiable in (−4, 2).

$f(-4)=(-4)^2+2\times(-4)-8=16-8-8=0$

$f(2)=(2)^2+2\times2-8=4+4-8=0$

∴ $f(-4) = f(2) = 0$

⇒ The value of $f(x)$ at −4 and 2 coincides.

Rolle's Theorem states that there is a point $c \in (-4, 2)$ such that $f'(c)=0$

$$
\begin{aligned}
& f(x)=x^2+2x-8 \\
& \Rightarrow f'(x)=2x+2 \\
& \therefore f'(c)=0 \\
& \Rightarrow 2c+2=0 \\
& \Rightarrow c=-1,\ \text{where } c=-1\in(-4,2)
\end{aligned}
$$

Hence, Rolle's Theorem is verified for the given function.

Question 2:

Examine if Rolle's Theorem is applicable to any of the following functions. Can you say some thing about the converse of Rolle's Theorem from these examples?

(i) $f(x)=[x]$ for $x\in[5,\ 9]$

(ii) $f(x)=[x]$ for $x\in[-2,\ 2]$

(iii) $f(x)=x^2-1$ for $x\in[1,\ 2]$

Answer

By Rolle's Theorem, for a function $f:[a,\ b]\to\mathbf{R}$, if

(a) *f* is continuous on [*a*, *b*]

(b) *f* is differentiable on (*a*, *b*)

(c) *f* (*a*) = *f* (*b*)

then, there exists some *c* ∈ (*a*, *b*) such that $f'(c)=0$

Therefore, Rolle's Theorem is not applicable to those functions that do not satisfy any of the three conditions of the hypothesis.

(i) $f(x)=[x]$ for $x\in[5,\ 9]$

It is evident that the given function *f* (*x*) is not continuous at every integral point.

In particular, *f*(*x*) is not continuous at *x* = 5 and *x* = 9

⇒ *f* (*x*) is not continuous in [5, 9].

$\text{Also, } f(5)=[5]=5 \text{ and } f(9)=[9]=9$

$\therefore f(5)\neq f(9)$

The differentiability of *f* in (5, 9) is checked as follows.

Let n be an integer such that $n \in (5, 9)$.

The left hand limit of f at $x = n$ is,

$$\lim_{h\to 0^-}\frac{f(n+h)-f(n)}{h}=\lim_{h\to 0^-}\frac{[n+h]-[n]}{h}=\lim_{h\to 0^-}\frac{n-1-n}{h}=\lim_{h\to 0^-}\frac{-1}{h}=\infty$$

The right hand limit of f at $x = n$ is,

$$\lim_{h\to 0^+}\frac{f(n+h)-f(n)}{h}=\lim_{h\to 0^+}\frac{[n+h]-[n]}{h}=\lim_{h\to 0^+}\frac{n-n}{h}=\lim_{h\to 0^+}0=0$$

Since the left and right hand limits of f at $x = n$ are not equal, f is not differentiable at $x = n$

$\therefore f$ is not differentiable in (5, 9).

It is observed that f does not satisfy all the conditions of the hypothesis of Rolle's Theorem.

Hence, Rolle's Theorem is not applicable for $f(x)=[x]$ for $x\in[5,\ 9]$.

(ii) $f(x)=[x]$ for $x\in[-2,\ 2]$

It is evident that the given function $f(x)$ is not continuous at every integral point.

In particular, $f(x)$ is not continuous at $x = -2$ and $x = 2$

$\Rightarrow f(x)$ is not continuous in $[-2, 2]$.

Also, $f(-2)=[-2]=-2$ and $f(2)=[2]=2$

$\therefore f(-2)\neq f(2)$

The differentiability of f in $(-2, 2)$ is checked as follows.

Let *n* be an integer such that $n \in (-2, 2)$.

The left hand limit of f at $x = n$ is,

$$\lim_{h\to 0^-} \frac{f(n+h)-f(n)}{h} = \lim_{h\to 0^-} \frac{[n+h]-[n]}{h} = \lim_{h\to 0^-} \frac{n-1-n}{h} = \lim_{h\to 0^-} \frac{-1}{h} = \infty$$

The right hand limit of f at $x = n$ is,

$$\lim_{h\to 0^+} \frac{f(n+h)-f(n)}{h} = \lim_{h\to 0^+} \frac{[n+h]-[n]}{h} = \lim_{h\to 0^+} \frac{n-n}{h} = \lim_{h\to 0^+} 0 = 0$$

Since the left and right hand limits of *f* at $x = n$ are not equal, *f* is not differentiable at $x = n$

∴*f* is not differentiable in $(-2, 2)$.

It is observed that *f* does not satisfy all the conditions of the hypothesis of Rolle's Theorem.

Hence, Rolle's Theorem is not applicable for $f(x) = [x]$ for $x \in [-2, 2]$.

(iii) $f(x) = x^2 - 1$ for $x \in [1, 2]$

It is evident that *f*, being a polynomial function, is continuous in [1, 2] and is differentiable in (1, 2).

$f(1) = (1)^2 - 1 = 0$

$f(2) = (2)^2 - 1 = 3$

∴$f(1) \neq f(2)$

It is observed that *f* does not satisfy a condition of the hypothesis of Rolle's Theorem.

Hence, Rolle's Theorem is not applicable for $f(x) = x^2 - 1 \text{ for } x \in [1,\ 2]$.

Question 3:

If $f : [-5,5] \to \mathbf{R}$ is a differentiable function and if $f'(x)$ does not vanish anywhere, then prove that $f(-5) \neq f(5)$.

Answer

It is given that $f : [-5,5] \to \mathbf{R}$ is a differentiable function.

Since every differentiable function is a continuous function, we obtain

(a) *f* is continuous on [−5, 5].

(b) *f* is differentiable on (−5, 5).

Therefore, by the Mean Value Theorem, there exists $c \in (-5, 5)$ such that

$$f'(c) = \frac{f(5) - f(-5)}{5 - (-5)}$$

$$\Rightarrow 10 f'(c) = f(5) - f(-5)$$

It is also given that $f'(x)$ does not vanish anywhere.

$$\therefore f'(c) \neq 0$$

$$\Rightarrow 10 f'(c) \neq 0$$

$$\Rightarrow f(5) - f(-5) \neq 0$$

$$\Rightarrow f(5) \neq f(-5)$$

Hence, proved.

Question 4:

Verify Mean Value Theorem, if $f(x) = x^2 - 4x - 3$ in the interval $[a,\ b]$, where $a = 1$ and $b = 4$.

Answer

The given function is $f(x)=x^2-4x-3$

f, being a polynomial function, is continuous in [1, 4] and is differentiable in (1, 4) whose derivative is 2*x* − 4.

$$f(1)=1^2-4\times1-3=-6,\ f(4)=4^2-4\times4-3=-3$$

$$\therefore \frac{f(b)-f(a)}{b-a}=\frac{f(4)-f(1)}{4-1}=\frac{-3-(-6)}{3}=\frac{3}{3}=1$$

Mean Value Theorem states that there is a point *c* ∈ (1, 4) such that $f'(c)=1$

$$f'(c)=1$$

$$\Rightarrow 2c-4=1$$

$$\Rightarrow c=\frac{5}{2},\ \text{where } c=\frac{5}{2}\in(1,\ 4)$$

Hence, Mean Value Theorem is verified for the given function.

Question 5:

Verify Mean Value Theorem, if $f(x)=x^3-5x^2-3x$ in the interval [*a*, *b*], where *a* = 1 and *b* = 3. Find all $c\in(1,3)$ for which $f'(c)=0$

Answer

The given function *f* is $f(x)=x^3-5x^2-3x$

f, being a polynomial function, is continuous in [1, 3] and is differentiable in (1, 3) whose derivative is $3x^2 - 10x - 3$.

$$f(1)=1^3-5\times1^2-3\times1=-7,\ f(3)=3^3-5\times3^2-3\times3=-27$$

$$\therefore \frac{f(b)-f(a)}{b-a}=\frac{f(3)-f(1)}{3-1}=\frac{-27-(-7)}{3-1}=-10$$

Mean Value Theorem states that there exist a point *c* ∈ (1, 3) such that $f'(c) = -10$

$$f'(c) = -10$$
$$\Rightarrow 3c^2 - 10c - 3 = 10$$
$$\Rightarrow 3c^2 - 10c + 7 = 0$$
$$\Rightarrow 3c^2 - 3c - 7c + 7 = 0$$
$$\Rightarrow 3c(c-1) - 7(c-1) = 0$$
$$\Rightarrow (c-1)(3c-7) = 0$$
$$\Rightarrow c = 1,\ \frac{7}{3},\ \text{where } c = \frac{7}{3} \in (1,\ 3)$$

Hence, Mean Value Theorem is verified for the given function and $c = \frac{7}{3} \in (1,\ 3)$ is the only point for which $f'(c) = 0$

Question 6:

Examine the applicability of Mean Value Theorem for all three functions given in the above exercise 2.

Answer

Mean Value Theorem states that for a function $f : [a,\ \text{b}] \to \mathbf{R}$, if

(a) *f* is continuous on [*a*, *b*]

(b) *f* is differentiable on (*a*, *b*)

then, there exists some *c* ∈ (*a*, *b*) such that $f'(c) = \frac{f(b) - f(a)}{b - a}$

Therefore, Mean Value Theorem is not applicable to those functions that do not satisfy any of the two conditions of the hypothesis.

(i) $f(x)=[x] \text{ for } x \in [5,\ 9]$

It is evident that the given function *f* (*x*) is not continuous at every integral point.

In particular, *f*(*x*) is not continuous at *x* = 5 and *x* = 9

⇒ *f* (*x*) is not continuous in [5, 9].

The differentiability of *f* in (5, 9) is checked as follows.

Let *n* be an integer such that *n* ∈ (5, 9).

The left hand limit of f at $x=n$ is,

$$\lim_{h\to 0^-}\frac{f(n+h)-f(n)}{h}=\lim_{h\to 0^-}\frac{[n+h]-[n]}{h}=\lim_{h\to 0^-}\frac{n-1-n}{h}=\lim_{h\to 0^-}\frac{-1}{h}=\infty$$

The right hand limit of f at $x=n$ is,

$$\lim_{h\to 0^+}\frac{f(n+h)-f(n)}{h}=\lim_{h\to 0^+}\frac{[n+h]-[n]}{h}=\lim_{h\to 0^+}\frac{n-n}{h}=\lim_{h\to 0^+}0=0$$

Since the left and right hand limits of *f* at *x* = *n* are not equal, *f* is not differentiable at *x* = *n*

∴*f* is not differentiable in (5, 9).

It is observed that *f* does not satisfy all the conditions of the hypothesis of Mean Value Theorem.

Hence, Mean Value Theorem is not applicable for $f(x)=[x] \text{ for } x \in [5,\ 9]$.

(ii) $f(x)=[x] \text{ for } x \in [-2,\ 2]$

It is evident that the given function *f* (*x*) is not continuous at every integral point.

In particular, $f(x)$ is not continuous at $x = -2$ and $x = 2$

⇒ $f(x)$ is not continuous in [−2, 2].

The differentiability of f in (−2, 2) is checked as follows.

Let n be an integer such that $n \in (-2, 2)$.

The left hand limit of f at $x = n$ is,

$$\lim_{h\to 0^-} \frac{f(n+h)-f(n)}{h} = \lim_{h\to 0^-} \frac{[n+h]-[n]}{h} = \lim_{h\to 0^-} \frac{n-1-n}{h} = \lim_{h\to 0^-} \frac{-1}{h} = \infty$$

The right hand limit of f at $x = n$ is,

$$\lim_{h\to 0^+} \frac{f(n+h)-f(n)}{h} = \lim_{h\to 0^+} \frac{[n+h]-[n]}{h} = \lim_{h\to 0^+} \frac{n-n}{h} = \lim_{h\to 0^+} 0 = 0$$

Since the left and right hand limits of f at $x = n$ are not equal, f is not differentiable at $x = n$

∴f is not differentiable in (−2, 2).

It is observed that f does not satisfy all the conditions of the hypothesis of Mean Value Theorem.

Hence, Mean Value Theorem is not applicable for $f(x) = [x] \text{ for } x \in [-2,\ 2]$.

(iii) $f(x) = x^2 - 1 \text{ for } x \in [1,\ 2]$

It is evident that f, being a polynomial function, is continuous in [1, 2] and is differentiable in (1, 2).

It is observed that f satisfies all the conditions of the hypothesis of Mean Value Theorem.

Hence, Mean Value Theorem is applicable for $f(x)=x^2-1$ for $x \in [1, 2]$.

It can be proved as follows.

$f(1)=1^2-1=0,\ f(2)=2^2-1=3$

$\therefore \frac{f(b)-f(a)}{b-a}=\frac{f(2)-f(1)}{2-1}=\frac{3-0}{1}=3$

$f'(x)=2x$

$\therefore f'(c)=3$

$\Rightarrow 2c=3$

$\Rightarrow c=\frac{3}{2}=1.5,$ where $1.5 \in [1, 2]$