

## Exercise 6.3

**Question 1:**

Find the slope of the tangent to the curve  $y = 3x^4 - 4x$  at  $x = 4$ .

Answer

The given curve is  $y = 3x^4 - 4x$ .

Then, the slope of the tangent to the given curve at  $x = 4$  is given by,

$$\left. \frac{dy}{dx} \right]_{x=4} = 12x^3 - 4 \Big]_{x=4} = 12(4)^3 - 4 = 12(64) - 4 = 764$$

**Question 2:**

Find the slope of the tangent to the curve  $y = \frac{x-1}{x-2}$ ,  $x \neq 2$  at  $x = 10$ .

Answer

The given curve is  $y = \frac{x-1}{x-2}$ .

$$\begin{aligned} \therefore \frac{dy}{dx} &= \frac{(x-2)(1) - (x-1)(1)}{(x-2)^2} \\ &= \frac{x-2-x+1}{(x-2)^2} = \frac{-1}{(x-2)^2} \end{aligned}$$

Thus, the slope of the tangent at  $x = 10$  is given by,

$$\left. \frac{dy}{dx} \right]_{x=10} = \frac{-1}{(x-2)^2} \Big]_{x=10} = \frac{-1}{(10-2)^2} = \frac{-1}{64}$$

Hence, the slope of the tangent at  $x = 10$  is  $\frac{-1}{64}$ .

**Question 3:**

Find the slope of the tangent to curve  $y = x^3 - x + 1$  at the point whose  $x$ -coordinate is 2.

Answer

The given curve is  $y = x^3 - x + 1$ .

$$\therefore \frac{dy}{dx} = 3x^2 - 1$$

The slope of the tangent to a curve at  $(x_0, y_0)$  is  $\left. \frac{dy}{dx} \right|_{(x_0, y_0)}$ .

It is given that  $x_0 = 2$ .

Hence, the slope of the tangent at the point where the  $x$ -coordinate is 2 is given by,

$$\left. \frac{dy}{dx} \right|_{x=2} = 3x^2 - 1 \Big|_{x=2} = 3(2)^2 - 1 = 12 - 1 = 11$$

**Question 4:**

Find the slope of the tangent to the curve  $y = x^3 - 3x + 2$  at the point whose  $x$ -coordinate is 3.

Answer

The given curve is  $y = x^3 - 3x + 2$ .

$$\therefore \frac{dy}{dx} = 3x^2 - 3$$

The slope of the tangent to a curve at  $(x_0, y_0)$  is  $\left. \frac{dy}{dx} \right|_{(x_0, y_0)}$ .

Hence, the slope of the tangent at the point where the  $x$ -coordinate is 3 is given by,

$$\left. \frac{dy}{dx} \right|_{x=3} = 3x^2 - 3 \Big|_{x=3} = 3(3)^2 - 3 = 27 - 3 = 24$$

**Question 5:**

Find the slope of the normal to the curve  $x = a\cos^3\theta$ ,  $y = a\sin^3\theta$  at  $\theta = \frac{\pi}{4}$ .

Answer

It is given that  $x = a\cos^3\theta$  and  $y = a\sin^3\theta$ .

$$\therefore \frac{dx}{d\theta} = 3a \cos^2 \theta (-\sin \theta) = -3a \cos^2 \theta \sin \theta$$

$$\frac{dy}{d\theta} = 3a \sin^2 \theta (\cos \theta)$$

$$\therefore \frac{dy}{dx} = \frac{\left(\frac{dy}{d\theta}\right)}{\left(\frac{dx}{d\theta}\right)} = \frac{3a \sin^2 \theta \cos \theta}{-3a \cos^2 \theta \sin \theta} = -\frac{\sin \theta}{\cos \theta} = -\tan \theta$$

Therefore, the slope of the tangent at  $\theta = \frac{\pi}{4}$  is given by,

$$\left. \frac{dy}{dx} \right|_{\theta = \frac{\pi}{4}} = -\tan \theta \Big|_{\theta = \frac{\pi}{4}} = -\tan \frac{\pi}{4} = -1$$

Hence, the slope of the normal at  $\theta = \frac{\pi}{4}$  is given by,

$$\frac{1}{\text{slope of the tangent at } \theta = \frac{\pi}{4}} = \frac{1}{-1} = -1$$

### Question 6:

Find the slope of the normal to the curve  $x = 1 - a \sin \theta$ ,  $y = b \cos^2 \theta$  at  $\theta = \frac{\pi}{2}$ .

Answer

It is given that  $x = 1 - a \sin \theta$  and  $y = b \cos^2 \theta$ .

$$\therefore \frac{dx}{d\theta} = -a \cos \theta \text{ and } \frac{dy}{d\theta} = 2b \cos \theta (-\sin \theta) = -2b \sin \theta \cos \theta$$

$$\therefore \frac{dy}{dx} = \frac{\left(\frac{dy}{d\theta}\right)}{\left(\frac{dx}{d\theta}\right)} = \frac{-2b \sin \theta \cos \theta}{-a \cos \theta} = \frac{2b}{a} \sin \theta$$

Therefore, the slope of the tangent at  $\theta = \frac{\pi}{2}$  is given by,

$$\left. \frac{dy}{dx} \right|_{\theta=\frac{\pi}{2}} = \left. \frac{2b}{a} \sin \theta \right|_{\theta=\frac{\pi}{2}} = \frac{2b}{a} \sin \frac{\pi}{2} = \frac{2b}{a}$$

Hence, the slope of the normal at  $\theta = \frac{\pi}{2}$  is given by,

$$\frac{1}{\text{slope of the tangent at } \theta = \frac{\pi}{2}} = \frac{-1}{\left(\frac{2b}{a}\right)} = -\frac{a}{2b}$$

### Question 7:

Find points at which the tangent to the curve  $y = x^3 - 3x^2 - 9x + 7$  is parallel to the  $x$ -axis.

Answer

The equation of the given curve is  $y = x^3 - 3x^2 - 9x + 7$ .

$$\therefore \frac{dy}{dx} = 3x^2 - 6x - 9$$

Now, the tangent is parallel to the  $x$ -axis if the slope of the tangent is zero.

$$\begin{aligned} \therefore 3x^2 - 6x - 9 = 0 &\Rightarrow x^2 - 2x - 3 = 0 \\ &\Rightarrow (x-3)(x+1) = 0 \\ &\Rightarrow x = 3 \text{ or } x = -1 \end{aligned}$$

When  $x = 3$ ,  $y = (3)^3 - 3(3)^2 - 9(3) + 7 = 27 - 27 - 27 + 7 = -20$ .

When  $x = -1$ ,  $y = (-1)^3 - 3(-1)^2 - 9(-1) + 7 = -1 - 3 + 9 + 7 = 12$ .

Hence, the points at which the tangent is parallel to the  $x$ -axis are  $(3, -20)$  and  $(-1, 12)$ .

### Question 8:

Find a point on the curve  $y = (x - 2)^2$  at which the tangent is parallel to the chord joining the points  $(2, 0)$  and  $(4, 4)$ .

Answer

If a tangent is parallel to the chord joining the points  $(2, 0)$  and  $(4, 4)$ , then the slope of the tangent = the slope of the chord.

$$\frac{4-0}{4-2} = \frac{4}{2} = 2.$$

The slope of the chord is

Now, the slope of the tangent to the given curve at a point  $(x, y)$  is given by,

$$\frac{dy}{dx} = 2(x-2)$$

Since the slope of the tangent = slope of the chord, we have:

$$2(x-2) = 2$$

$$\Rightarrow x-2 = 1 \Rightarrow x = 3$$

$$\text{When } x = 3, y = (3-2)^2 = 1.$$

Hence, the required point is  $(3, 1)$ .

**Question 9:**

Find the point on the curve  $y = x^3 - 11x + 5$  at which the tangent is  $y = x - 11$ .

Answer

The equation of the given curve is  $y = x^3 - 11x + 5$ .

The equation of the tangent to the given curve is given as  $y = x - 11$  (which is of the form  $y = mx + c$ ).

∴ Slope of the tangent = 1

Now, the slope of the tangent to the given curve at the point  $(x, y)$  is given by,

$$\frac{dy}{dx} = 3x^2 - 11$$

Then, we have:

$$3x^2 - 11 = 1$$

$$\Rightarrow 3x^2 = 12$$

$$\Rightarrow x^2 = 4$$

$$\Rightarrow x = \pm 2$$

$$\text{When } x = 2, y = (2)^3 - 11(2) + 5 = 8 - 22 + 5 = -9.$$

$$\text{When } x = -2, y = (-2)^3 - 11(-2) + 5 = -8 + 22 + 5 = 19.$$

Hence, the required points are (2, -9) and (-2, 19).

**Question 10:**

Find the equation of all lines having slope -1 that are tangents to the curve

$$y = \frac{1}{x-1}, x \neq 1$$

Answer

The equation of the given curve is  $y = \frac{1}{x-1}, x \neq 1$ .

The slope of the tangents to the given curve at any point (x, y) is given by,

$$\frac{dy}{dx} = \frac{-1}{(x-1)^2}$$

If the slope of the tangent is -1, then we have:

$$\begin{aligned}\frac{-1}{(x-1)^2} &= -1 \\ \Rightarrow (x-1)^2 &= 1 \\ \Rightarrow x-1 &= \pm 1 \\ \Rightarrow x &= 2, 0\end{aligned}$$

When  $x = 0$ ,  $y = -1$  and when  $x = 2$ ,  $y = 1$ .

Thus, there are two tangents to the given curve having slope -1. These are passing through the points (0, -1) and (2, 1).

∴ The equation of the tangent through (0, -1) is given by,

$$\begin{aligned}y - (-1) &= -1(x - 0) \\ \Rightarrow y + 1 &= -x \\ \Rightarrow y + x + 1 &= 0\end{aligned}$$

∴ The equation of the tangent through (2, 1) is given by,

$$y - 1 = -1(x - 2)$$

$$\Rightarrow y - 1 = -x + 2$$

$$\Rightarrow y + x - 3 = 0$$

Hence, the equations of the required lines are  $y + x + 1 = 0$  and  $y + x - 3 = 0$ .

**Question 11:**

Find the equation of all lines having slope 2 which are tangents to the

curve  $y = \frac{1}{x-3}, x \neq 3$ .

Answer

The equation of the given curve is  $y = \frac{1}{x-3}, x \neq 3$ .

The slope of the tangent to the given curve at any point (x, y) is given by,

$$\frac{dy}{dx} = \frac{-1}{(x-3)^2}$$

If the slope of the tangent is 2, then we have:

$$\frac{-1}{(x-3)^2} = 2$$

$$\Rightarrow 2(x-3)^2 = -1$$

$$\Rightarrow (x-3)^2 = \frac{-1}{2}$$

This is not possible since the L.H.S. is positive while the R.H.S. is negative.

Hence, there is no tangent to the given curve having slope 2.

**Question 12:**

Find the equations of all lines having slope 0 which are tangent to the curve

$$y = \frac{1}{x^2 - 2x + 3}.$$

Answer

The equation of the given curve is  $y = \frac{1}{x^2 - 2x + 3}$ .

The slope of the tangent to the given curve at any point  $(x, y)$  is given by,

$$\frac{dy}{dx} = \frac{-(2x-2)}{(x^2 - 2x + 3)^2} = \frac{-2(x-1)}{(x^2 - 2x + 3)^2}$$

If the slope of the tangent is 0, then we have:

$$\frac{-2(x-1)}{(x^2 - 2x + 3)^2} = 0$$

$$\Rightarrow -2(x-1) = 0$$

$$\Rightarrow x = 1$$

When  $x = 1$ ,  $y = \frac{1}{1-2+3} = \frac{1}{2}$ .

∴ The equation of the tangent through  $\left(1, \frac{1}{2}\right)$  is given by,

$$y - \frac{1}{2} = 0(x-1)$$

$$\Rightarrow y - \frac{1}{2} = 0$$

$$\Rightarrow y = \frac{1}{2}$$

Hence, the equation of the required line is  $y = \frac{1}{2}$ .

**Question 13:**

Find points on the curve  $\frac{x^2}{9} + \frac{y^2}{16} = 1$  at which the tangents are  
(i) parallel to x-axis (ii) parallel to y-axis

Answer

The equation of the given curve is  $\frac{x^2}{9} + \frac{y^2}{16} = 1$ .

On differentiating both sides with respect to  $x$ , we have:

$$\frac{2x}{9} + \frac{2y}{16} \cdot \frac{dy}{dx} = 0$$

$$\Rightarrow \frac{dy}{dx} = \frac{-16x}{9y}$$

(i) The tangent is parallel to the x-axis if the slope of the tangent is i.e.,  $0$   $\frac{-16x}{9y} = 0$ ,  
which is possible if  $x = 0$ .

Then,  $\frac{x^2}{9} + \frac{y^2}{16} = 1$  for  $x = 0$

$$\Rightarrow y^2 = 16 \Rightarrow y = \pm 4$$

Hence, the points at which the tangents are parallel to the x-axis are  
(0, 4) and (0, -4).

(ii) The tangent is parallel to the  $y$ -axis if the slope of the normal is 0, which

$$\text{gives } \frac{-1}{\left(\frac{-16x}{9y}\right)} = \frac{9y}{16x} = 0 \Rightarrow y = 0.$$

$$\text{Then, } \frac{x^2}{9} + \frac{y^2}{16} = 1 \text{ for } y = 0.$$

$$\Rightarrow x = \pm 3$$

Hence, the points at which the tangents are parallel to the  $y$ -axis are

$(3, 0)$  and  $(-3, 0)$ .

#### Question 14:

Find the equations of the tangent and normal to the given curves at the indicated points:

(i)  $y = x^4 - 6x^3 + 13x^2 - 10x + 5$  at  $(0, 5)$

(ii)  $y = x^4 - 6x^3 + 13x^2 - 10x + 5$  at  $(1, 3)$

(iii)  $y = x^3$  at  $(1, 1)$

(iv)  $y = x^2$  at  $(0, 0)$

(v)  $x = \cos t, y = \sin t$  at  $t = \frac{\pi}{4}$

Answer

(i) The equation of the curve is  $y = x^4 - 6x^3 + 13x^2 - 10x + 5$ .

On differentiating with respect to  $x$ , we get:

$$\frac{dy}{dx} = 4x^3 - 18x^2 + 26x - 10$$

$$\left. \frac{dy}{dx} \right|_{(0, 5)} = -10$$

Thus, the slope of the tangent at  $(0, 5)$  is  $-10$ . The equation of the tangent is given as:

$$y - 5 = -10(x - 0)$$

$$\Rightarrow y - 5 = -10x$$

$$\Rightarrow 10x + y = 5$$

The slope of the normal at (0, 5) is  $\frac{-1}{\text{Slope of the tangent at (0, 5)}} = \frac{1}{10}$ .

Therefore, the equation of the normal at (0, 5) is given as:

$$y - 5 = \frac{1}{10}(x - 0)$$

$$\Rightarrow 10y - 50 = x$$

$$\Rightarrow x - 10y + 50 = 0$$

(ii) The equation of the curve is  $y = x^4 - 6x^3 + 13x^2 - 10x + 5$ .

On differentiating with respect to  $x$ , we get:

$$\frac{dy}{dx} = 4x^3 - 18x^2 + 26x - 10$$

$$\left. \frac{dy}{dx} \right|_{(1, 3)} = 4 - 18 + 26 - 10 = 2$$

Thus, the slope of the tangent at (1, 3) is 2. The equation of the tangent is given as:

$$y - 3 = 2(x - 1)$$

$$\Rightarrow y - 3 = 2x - 2$$

$$\Rightarrow y = 2x + 1$$

The slope of the normal at (1, 3) is  $\frac{-1}{\text{Slope of the tangent at (1, 3)}} = \frac{-1}{2}$ .

Therefore, the equation of the normal at (1, 3) is given as:

$$y-3 = -\frac{1}{2}(x-1)$$

$$\Rightarrow 2y-6 = -x+1$$

$$\Rightarrow x+2y-7=0$$

(iii) The equation of the curve is  $y = x^3$ .

On differentiating with respect to  $x$ , we get:

$$\frac{dy}{dx} = 3x^2$$

$$\left. \frac{dy}{dx} \right|_{(1,1)} = 3(1)^2 = 3$$

Thus, the slope of the tangent at  $(1, 1)$  is 3 and the equation of the tangent is given as:

$$y-1 = 3(x-1)$$

$$\Rightarrow y = 3x-2$$

The slope of the normal at  $(1, 1)$  is  $\frac{-1}{\text{Slope of the tangent at } (1, 1)} = \frac{-1}{3}$ .

Therefore, the equation of the normal at  $(1, 1)$  is given as:

$$y-1 = \frac{-1}{3}(x-1)$$

$$\Rightarrow 3y-3 = -x+1$$

$$\Rightarrow x+3y-4=0$$

(iv) The equation of the curve is  $y = x^2$ .

On differentiating with respect to  $x$ , we get:

$$\frac{dy}{dx} = 2x$$

$$\left. \frac{dy}{dx} \right|_{(0,0)} = 0$$

Thus, the slope of the tangent at  $(0, 0)$  is 0 and the equation of the tangent is given as:

$$y-0 = 0(x-0)$$

$$\Rightarrow y = 0$$

The slope of the normal at  $(0, 0)$  is  $\frac{-1}{\text{Slope of the tangent at } (0, 0)} = \frac{-1}{0}$ , which is not defined.

Therefore, the equation of the normal at  $(x_0, y_0) = (0, 0)$  is given by

$$x = x_0 = 0.$$

(v) The equation of the curve is  $x = \cos t$ ,  $y = \sin t$ .

$$x = \cos t \text{ and } y = \sin t$$

$$\therefore \frac{dx}{dt} = -\sin t, \quad \frac{dy}{dt} = \cos t$$

$$\therefore \frac{dy}{dx} = \frac{\left(\frac{dy}{dt}\right)}{\left(\frac{dx}{dt}\right)} = \frac{\cos t}{-\sin t} = -\cot t$$

$$\left. \frac{dy}{dx} \right|_{t=\frac{\pi}{4}} = -\cot t = -1$$

□ The slope of the tangent at  $t = \frac{\pi}{4}$  is  $-1$ .

When  $t = \frac{\pi}{4}$ ,  $x = \frac{1}{\sqrt{2}}$  and  $y = \frac{1}{\sqrt{2}}$ .

Thus, the equation of the tangent to the given curve at  $t = \frac{\pi}{4}$  i.e., at  $\left[\left(\frac{1}{\sqrt{2}}, \frac{1}{\sqrt{2}}\right)\right]$  is

$$y - \frac{1}{\sqrt{2}} = -1 \left( x - \frac{1}{\sqrt{2}} \right).$$

$$\Rightarrow x + y - \frac{1}{\sqrt{2}} - \frac{1}{\sqrt{2}} = 0$$

$$\Rightarrow x + y - \sqrt{2} = 0$$

The slope of the normal at  $t = \frac{\pi}{4}$  is  $\frac{-1}{\text{Slope of the tangent at } t = \frac{\pi}{4}} = 1$ .

Therefore, the equation of the normal to the given curve at  $t = \frac{\pi}{4}$  i.e., at  $\left[\left(\frac{1}{\sqrt{2}}, \frac{1}{\sqrt{2}}\right)\right]$  is

$$y - \frac{1}{\sqrt{2}} = 1 \left( x - \frac{1}{\sqrt{2}} \right).$$

$$\Rightarrow x = y$$

**Question 15:**

Find the equation of the tangent line to the curve  $y = x^2 - 2x + 7$  which is

- (a) parallel to the line  $2x - y + 9 = 0$   
 (b) perpendicular to the line  $5y - 15x = 13$ .

Answer

The equation of the given curve is  $y = x^2 - 2x + 7$ .

On differentiating with respect to  $x$ , we get:

$$\frac{dy}{dx} = 2x - 2$$

(a) The equation of the line is  $2x - y + 9 = 0$ .

$$2x - y + 9 = 0 \Rightarrow y = 2x + 9$$

This is of the form  $y = mx + c$ .

$$\square \text{ Slope of the line} = 2$$

If a tangent is parallel to the line  $2x - y + 9 = 0$ , then the slope of the tangent is equal to the slope of the line.

Therefore, we have:

$$2 = 2x - 2$$

$$\Rightarrow 2x = 4$$

$$\Rightarrow x = 2$$

Now,  $x = 2$

$$\Rightarrow y = 4 - 4 + 7 = 7$$

Thus, the equation of the tangent passing through  $(2, 7)$  is given by,

$$y - 7 = 2(x - 2)$$

$$\Rightarrow y - 2x - 3 = 0$$

Hence, the equation of the tangent line to the given curve (which is parallel to line  $2x - y + 9 = 0$ ) is  $y - 2x - 3 = 0$ .

(b) The equation of the line is  $5y - 15x = 13$ .

$$5y - 15x = 13 \quad \square \quad y = 3x + \frac{13}{5}$$

This is of the form  $y = mx + c$ .

□ Slope of the line = 3

If a tangent is perpendicular to the line  $5y - 15x = 13$ , then the slope of the tangent is

$$\frac{-1}{\text{slope of the line}} = \frac{-1}{3}$$

$$\Rightarrow 2x - 2 = \frac{-1}{3}$$

$$\Rightarrow 2x = \frac{-1}{3} + 2$$

$$\Rightarrow 2x = \frac{5}{3}$$

$$\Rightarrow x = \frac{5}{6}$$

$$\text{Now, } x = \frac{5}{6}$$

$$\Rightarrow y = \frac{25}{36} - \frac{10}{6} + 7 = \frac{25 - 60 + 252}{36} = \frac{217}{36}$$

Thus, the equation of the tangent passing through  $\left(\frac{5}{6}, \frac{217}{36}\right)$  is given by,

$$y - \frac{217}{36} = -\frac{1}{3}\left(x - \frac{5}{6}\right)$$

$$\Rightarrow \frac{36y - 217}{36} = \frac{-1}{18}(6x - 5)$$

$$\Rightarrow 36y - 217 = -2(6x - 5)$$

$$\Rightarrow 36y - 217 = -12x + 10$$

$$\Rightarrow 36y + 12x - 227 = 0$$

Hence, the equation of the tangent line to the given curve (which is perpendicular to line  $5y - 15x = 13$ ) is  $36y + 12x - 227 = 0$ .

**Question 16:**

Show that the tangents to the curve  $y = 7x^3 + 11$  at the points where  $x = 2$  and  $x = -2$  are parallel.

Answer

The equation of the given curve is  $y = 7x^3 + 11$ .

$$\therefore \frac{dy}{dx} = 21x^2$$

The slope of the tangent to a curve at  $(x_0, y_0)$  is  $\left. \frac{dy}{dx} \right|_{(x_0, y_0)}$ .

Therefore, the slope of the tangent at the point where  $x = 2$  is given by,

$$\left. \frac{dy}{dx} \right|_{x=2} = 21(2)^2 = 84$$

It is observed that the slopes of the tangents at the points where  $x = 2$  and  $x = -2$  are equal.

Hence, the two tangents are parallel.

**Question 17:**

Find the points on the curve  $y = x^3$  at which the slope of the tangent is equal to the  $y$ -coordinate of the point.

Answer

The equation of the given curve is  $y = x^3$ .

$$\therefore \frac{dy}{dx} = 3x^2$$

The slope of the tangent at the point  $(x, y)$  is given by,

$$\left. \frac{dy}{dx} \right|_{(x, y)} = 3x^2$$

When the slope of the tangent is equal to the  $y$ -coordinate of the point, then  $y = 3x^2$ .

Also, we have  $y = x^3$ .

$$\square 3x^2 = x^3$$

$$\square x^2 (x - 3) = 0$$

$$\square x = 0, x = 3$$

When  $x = 0$ , then  $y = 0$  and when  $x = 3$ , then  $y = 3(3)^2 = 27$ .

Hence, the required points are  $(0, 0)$  and  $(3, 27)$ .

### Question 18:

For the curve  $y = 4x^3 - 2x^5$ , find all the points at which the tangents passes through the origin.

Answer

The equation of the given curve is  $y = 4x^3 - 2x^5$ .

$$\therefore \frac{dy}{dx} = 12x^2 - 10x^4$$

Therefore, the slope of the tangent at a point  $(x, y)$  is  $12x^2 - 10x^4$ .

The equation of the tangent at  $(x, y)$  is given by,

$$Y - y = (12x^2 - 10x^4)(X - x) \quad \dots(1)$$

When the tangent passes through the origin  $(0, 0)$ , then  $X = Y = 0$ .

Therefore, equation (1) reduces to:

$$-y = (12x^2 - 10x^4)(-x)$$

$$y = 12x^3 - 10x^5$$

Also, we have  $y = 4x^3 - 2x^5$ .

$$\therefore 12x^3 - 10x^5 = 4x^3 - 2x^5$$

$$\Rightarrow 8x^3 - 8x^5 = 0$$

$$\Rightarrow x^3 - x^5 = 0$$

$$\Rightarrow x^3(x^2 - 1) = 0$$

$$\Rightarrow x = 0, \pm 1$$

When  $x = 0$ ,  $y = 4(0)^3 - 2(0)^5 = 0$ .

When  $x = 1$ ,  $y = 4(1)^3 - 2(1)^5 = 2$ .

When  $x = -1$ ,  $y = 4(-1)^3 - 2(-1)^5 = -2$ .

Hence, the required points are (0, 0), (1, 2), and (–1, –2).

**Question 19:**

Find the points on the curve  $x^2 + y^2 - 2x - 3 = 0$  at which the tangents are parallel to the x-axis.

Answer

The equation of the given curve is  $x^2 + y^2 - 2x - 3 = 0$ .

On differentiating with respect to  $x$ , we have:

$$2x + 2y \frac{dy}{dx} - 2 = 0$$

$$\Rightarrow y \frac{dy}{dx} = 1 - x$$

$$\Rightarrow \frac{dy}{dx} = \frac{1-x}{y}$$

Now, the tangents are parallel to the x-axis if the slope of the tangent is 0.

$$\therefore \frac{1-x}{y} = 0 \Rightarrow 1-x = 0 \Rightarrow x = 1$$

But,  $x^2 + y^2 - 2x - 3 = 0$  for  $x = 1$ .

$$\Rightarrow y^2 = 4 \quad \square \quad y = \pm 2$$

Hence, the points at which the tangents are parallel to the x-axis are (1, 2) and (1, –2).

**Question 20:**

Find the equation of the normal at the point  $(am^2, am^3)$  for the curve  $ay^2 = x^3$ .

Answer

The equation of the given curve is  $ay^2 = x^3$ .

On differentiating with respect to  $x$ , we have:

$$2ay \frac{dy}{dx} = 3x^2$$

$$\Rightarrow \frac{dy}{dx} = \frac{3x^2}{2ay}$$

The slope of a tangent to the curve at  $(x_0, y_0)$  is  $\left. \frac{dy}{dx} \right|_{(x_0, y_0)}$ .

⇒ The slope of the tangent to the given curve at  $(am^2, am^3)$  is

$$\left. \frac{dy}{dx} \right|_{(am^2, am^3)} = \frac{3(am^2)^2}{2a(am^3)} = \frac{3a^2m^4}{2a^2m^3} = \frac{3m}{2}.$$

□ Slope of normal at  $(am^2, am^3)$

$$= \frac{-1}{\text{slope of the tangent at } (am^2, am^3)} = \frac{-2}{3m}$$

Hence, the equation of the normal at  $(am^2, am^3)$  is given by,

$$\begin{aligned} y - am^3 &= \frac{-2}{3m}(x - am^2) \\ \Rightarrow 3my - 3am^4 &= -2x + 2am^2 \\ \Rightarrow 2x + 3my - am^2(2 + 3m^2) &= 0 \end{aligned}$$

#### Question 21:

Find the equation of the normals to the curve  $y = x^3 + 2x + 6$  which are parallel to the line  $x + 14y + 4 = 0$ .

Answer

The equation of the given curve is  $y = x^3 + 2x + 6$ .

The slope of the tangent to the given curve at any point  $(x, y)$  is given by,

$$\frac{dy}{dx} = 3x^2 + 2$$

□ Slope of the normal to the given curve at any point  $(x, y)$

$$= \frac{-1}{\text{Slope of the tangent at the point } (x, y)}$$

$$= \frac{-1}{3x^2 + 2}$$

The equation of the given line is  $x + 14y + 4 = 0$ .

$$x + 14y + 4 = 0 \quad \square \quad y = -\frac{1}{14}x - \frac{4}{14} \text{ (which is of the form } y = mx + c)$$

$$\square \text{ Slope of the given line} = \frac{-1}{14}$$

If the normal is parallel to the line, then we must have the slope of the normal being equal to the slope of the line.

$$\therefore \frac{-1}{3x^2 + 2} = \frac{-1}{14}$$

$$\Rightarrow 3x^2 + 2 = 14$$

$$\Rightarrow 3x^2 = 12$$

$$\Rightarrow x^2 = 4$$

$$\Rightarrow x = \pm 2$$

When  $x = 2$ ,  $y = 8 + 4 + 6 = 18$ .

When  $x = -2$ ,  $y = -8 - 4 + 6 = -6$ .

Therefore, there are two normals to the given curve with slope  $\frac{-1}{14}$  and passing through the points  $(2, 18)$  and  $(-2, -6)$ .

Thus, the equation of the normal through  $(2, 18)$  is given by,

$$y - 18 = \frac{-1}{14}(x - 2)$$

$$\Rightarrow 14y - 252 = -x + 2$$

$$\Rightarrow x + 14y - 254 = 0$$

And, the equation of the normal through  $(-2, -6)$  is given by,

$$y - (-6) = \frac{-1}{14}[x - (-2)]$$

$$\Rightarrow y + 6 = \frac{-1}{14}(x + 2)$$

$$\Rightarrow 14y + 84 = -x - 2$$

$$\Rightarrow x + 14y + 86 = 0$$

Hence, the equations of the normals to the given curve (which are parallel to the given line) are  $x + 14y - 254 = 0$  and  $x + 14y + 86 = 0$ .

**Question 22:**

Find the equations of the tangent and normal to the parabola  $y^2 = 4ax$  at the point  $(at^2, 2at)$ .

Answer

The equation of the given parabola is  $y^2 = 4ax$ .

On differentiating  $y^2 = 4ax$  with respect to  $x$ , we have:

$$2y \frac{dy}{dx} = 4a$$

$$\Rightarrow \frac{dy}{dx} = \frac{2a}{y}$$

□ The slope of the tangent at  $(at^2, 2at)$  is  $\left. \frac{dy}{dx} \right|_{(at^2, 2at)} = \frac{2a}{2at} = \frac{1}{t}$ .

Then, the equation of the tangent at  $(at^2, 2at)$  is given by,

$$y - 2at = \frac{1}{t}(x - at^2)$$

$$\Rightarrow ty - 2at^2 = x - at^2$$

$$\Rightarrow ty = x + at^2$$

Now, the slope of the normal at  $(at^2, 2at)$  is given by,

$$\frac{-1}{\text{Slope of the tangent at } (at^2, 2at)} = -t$$

Thus, the equation of the normal at  $(at^2, 2at)$  is given as:

$$y - 2at = -t(x - at^2)$$

$$\Rightarrow y - 2at = -tx + at^3$$

$$\Rightarrow y = -tx + 2at + at^3$$

**Question 23:**

Prove that the curves  $x = y^2$  and  $xy = k$  cut at right angles if  $8k^2 = 1$ . [**Hint:** Two curves intersect at right angle if the tangents to the curves at the point of intersection are perpendicular to each other.]

Answer

The equations of the given curves are given as  $x = y^2$  and  $xy = k$ .

Putting  $x = y^2$  in  $xy = k$ , we get:

$$y^3 = k \Rightarrow y = k^{\frac{1}{3}}$$

$$\therefore x = k^{\frac{2}{3}}$$

Thus, the point of intersection of the given curves is  $\left(k^{\frac{2}{3}}, k^{\frac{1}{3}}\right)$ .

Differentiating  $x = y^2$  with respect to  $x$ , we have:

$$1 = 2y \frac{dy}{dx} \Rightarrow \frac{dy}{dx} = \frac{1}{2y}$$

Therefore, the slope of the tangent to the curve  $x = y^2$  at  $\left(k^{\frac{2}{3}}, k^{\frac{1}{3}}\right)$  is  $\left. \frac{dy}{dx} \right|_{\left(k^{\frac{2}{3}}, k^{\frac{1}{3}}\right)} = \frac{1}{2k^{\frac{1}{3}}}$ .

On differentiating  $xy = k$  with respect to  $x$ , we have:

$$x \frac{dy}{dx} + y = 0 \Rightarrow \frac{dy}{dx} = \frac{-y}{x}$$

□ Slope of the tangent to the curve  $xy = k$  at  $\left(k^{\frac{2}{3}}, k^{\frac{1}{3}}\right)$  is given by,

$$\left. \frac{dy}{dx} \right|_{\left(k^{\frac{2}{3}}, k^{\frac{1}{3}}\right)} = \left. \frac{-y}{x} \right|_{\left(k^{\frac{2}{3}}, k^{\frac{1}{3}}\right)} = -\frac{k^{\frac{1}{3}}}{k^{\frac{2}{3}}} = \frac{-1}{k^{\frac{1}{3}}}$$

We know that two curves intersect at right angles if the tangents to the curves at the

point of intersection i.e., at  $\left(k^{\frac{2}{3}}, k^{\frac{1}{3}}\right)$  are perpendicular to each other.

This implies that we should have the product of the tangents as  $-1$ .

Thus, the given two curves cut at right angles if the product of the slopes of their

respective tangents at  $\left(k^{\frac{2}{3}}, k^{\frac{1}{3}}\right)$  is  $-1$ .

$$\text{i.e., } \left( \frac{1}{2k^{\frac{1}{3}}} \right) \left( \frac{-1}{k^{\frac{1}{3}}} \right) = -1$$

$$\Rightarrow 2k^{\frac{2}{3}} = 1$$

$$\Rightarrow \left( 2k^{\frac{2}{3}} \right)^3 = (1)^3$$

$$\Rightarrow 8k^2 = 1$$

Hence, the given two curves cut at right angles if  $8k^2 = 1$ .

**Question 24:**

Find the equations of the tangent and normal to the hyperbola  $\frac{x^2}{a^2} - \frac{y^2}{b^2} = 1$  at the point  $(x_0, y_0)$ .

Answer

Differentiating  $\frac{x^2}{a^2} - \frac{y^2}{b^2} = 1$  with respect to  $x$ , we have:

$$\frac{2x}{a^2} - \frac{2y}{b^2} \frac{dy}{dx} = 0$$

$$\Rightarrow \frac{2y}{b^2} \frac{dy}{dx} = \frac{2x}{a^2}$$

$$\Rightarrow \frac{dy}{dx} = \frac{b^2 x}{a^2 y}$$

Therefore, the slope of the tangent at  $(x_0, y_0)$  is  $\left. \frac{dy}{dx} \right|_{(x_0, y_0)} = \frac{b^2 x_0}{a^2 y_0}$ .

Then, the equation of the tangent at  $(x_0, y_0)$  is given by,

$$\begin{aligned}
 y - y_0 &= \frac{b^2 x_0}{a^2 y_0} (x - x_0) \\
 \Rightarrow a^2 y y_0 - a^2 y_0^2 &= b^2 x x_0 - b^2 x_0^2 \\
 \Rightarrow b^2 x x_0 - a^2 y y_0 - b^2 x_0^2 + a^2 y_0^2 &= 0 \\
 \Rightarrow \frac{x x_0}{a^2} - \frac{y y_0}{b^2} - \left( \frac{x_0^2}{a^2} - \frac{y_0^2}{b^2} \right) &= 0 && \left[ \text{On dividing both sides by } a^2 b^2 \right] \\
 \Rightarrow \frac{x x_0}{a^2} - \frac{y y_0}{b^2} - 1 &= 0 && \left[ (x_0, y_0) \text{ lies on the hyperbola } \frac{x^2}{a^2} - \frac{y^2}{b^2} = 1 \right] \\
 \Rightarrow \frac{x x_0}{a^2} - \frac{y y_0}{b^2} &= 1
 \end{aligned}$$

Now, the slope of the normal at  $(x_0, y_0)$  is given by,

$$\text{Slope of the tangent at } (x_0, y_0) = \frac{-1}{\frac{-a^2 y_0}{b^2 x_0}}$$

Hence, the equation of the normal at  $(x_0, y_0)$  is given by,

$$\begin{aligned}
 y - y_0 &= \frac{-a^2 y_0}{b^2 x_0} (x - x_0) \\
 \Rightarrow \frac{y - y_0}{a^2 y_0} &= \frac{-(x - x_0)}{b^2 x_0} \\
 \Rightarrow \frac{y - y_0}{a^2 y_0} + \frac{(x - x_0)}{b^2 x_0} &= 0
 \end{aligned}$$

#### Question 25:

Find the equation of the tangent to the curve  $y = \sqrt{3x-2}$  which is parallel to the line  $4x - 2y + 5 = 0$ .

Answer

The equation of the given curve is  $y = \sqrt{3x-2}$ .

The slope of the tangent to the given curve at any point  $(x, y)$  is given by,

$$\frac{dy}{dx} = \frac{3}{2\sqrt{3x-2}}$$

The equation of the given line is  $4x - 2y + 5 = 0$ .

$$4x - 2y + 5 = 0 \Rightarrow y = 2x + \frac{5}{2} \text{ (which is of the form } y = mx + c \text{)}$$

□ Slope of the line = 2

Now, the tangent to the given curve is parallel to the line  $4x - 2y - 5 = 0$  if the slope of the tangent is equal to the slope of the line.

$$\frac{3}{2\sqrt{3x-2}} = 2$$

$$\Rightarrow \sqrt{3x-2} = \frac{3}{4}$$

$$\Rightarrow 3x-2 = \frac{9}{16}$$

$$\Rightarrow 3x = \frac{9}{16} + 2 = \frac{41}{16}$$

$$\Rightarrow x = \frac{41}{48}$$

$$\text{When } x = \frac{41}{48}, y = \sqrt{3\left(\frac{41}{48}\right) - 2} = \sqrt{\frac{41}{16} - 2} = \sqrt{\frac{41-32}{16}} = \sqrt{\frac{9}{16}} = \frac{3}{4}.$$

□ Equation of the tangent passing through the point  $\left(\frac{41}{48}, \frac{3}{4}\right)$  is given by,

$$y - \frac{3}{4} = 2\left(x - \frac{41}{48}\right)$$

$$\Rightarrow \frac{4y-3}{4} = 2\left(\frac{48x-41}{48}\right)$$

$$\Rightarrow 4y-3 = \frac{48x-41}{6}$$

$$\Rightarrow 24y-18 = 48x-41$$

$$\Rightarrow 48x-24y = 23$$

Hence, the equation of the required tangent is  $48x - 24y = 23$ .

#### Question 26:

The slope of the normal to the curve  $y = 2x^2 + 3 \sin x$  at  $x = 0$  is

(A) 3 (B)  $\frac{1}{3}$  (C) -3 (D)  $-\frac{1}{3}$

Answer

The equation of the given curve is  $y = 2x^2 + 3\sin x$ .

Slope of the tangent to the given curve at  $x = 0$  is given by,

$$\left. \frac{dy}{dx} \right|_{x=0} = 4x + 3\cos x \Big|_{x=0} = 0 + 3\cos 0 = 3$$

Hence, the slope of the normal to the given curve at  $x = 0$  is

$$\frac{-1}{\text{Slope of the tangent at } x = 0} = \frac{-1}{3}.$$

The correct answer is D.

**Question 27:**

The line  $y = x + 1$  is a tangent to the curve  $y^2 = 4x$  at the point

(A) (1, 2) (B) (2, 1) (C) (1, -2) (D) (-1, 2)

Answer

The equation of the given curve is  $y^2 = 4x$ .

Differentiating with respect to  $x$ , we have:

$$2y \frac{dy}{dx} = 4 \Rightarrow \frac{dy}{dx} = \frac{2}{y}$$

Therefore, the slope of the tangent to the given curve at any point  $(x, y)$  is given by,

$$\frac{dy}{dx} = \frac{2}{y}$$

The given line is  $y = x + 1$  (which is of the form  $y = mx + c$ )

□ Slope of the line = 1

The line  $y = x + 1$  is a tangent to the given curve if the slope of the line is equal to the slope of the tangent. Also, the line must intersect the curve.

Thus, we must have:

$$\frac{2}{y} = 1$$

$$\Rightarrow y = 2$$

$$\text{Now, } y = x + 1 \Rightarrow x = y - 1 \Rightarrow x = 2 - 1 = 1$$

Hence, the line  $y = x + 1$  is a tangent to the given curve at the point  $(1, 2)$ .

The correct answer is A.