Exercise 6.4

Question 1:

1. Using differentials, find the approximate value of each of the following up to 3 places of decimal

(i) $\sqrt{25.3}$ (ii) $\sqrt{49.5}$ (iii) $\sqrt{0.6}$

(iv) $(0.009)^{\frac{1}{3}}$ (v) $(0.999)^{\frac{1}{10}}$ (vi) $(15)^{\frac{1}{4}}$

(vii) $(26)^{\frac{1}{3}}$ (viii) $(255)^{\frac{1}{4}}$ (ix) $(82)^{\frac{1}{4}}$

(x) $(401)^{\frac{1}{2}}$ (xi) $(0.0037)^{\frac{1}{2}}$ (xii) $(26.57)^{\frac{1}{3}}$

(xiii) $(81.5)^{\frac{1}{4}}$ (xiv) $(3.968)^{\frac{3}{2}}$ (xv) $(32.15)^{\frac{1}{5}}$

Answer

(i) $\sqrt{25.3}$

Consider $y=\sqrt{x}$. Let x = 25 and Δx = 0.3.

Then,

$$\Delta y=\sqrt{x+\Delta x}-\sqrt{x}=\sqrt{25.3}-\sqrt{25}=\sqrt{25.3}-5$$
$$\Rightarrow \sqrt{25.3}=\Delta y+5$$

Now, dy is approximately equal to Δy and is given by,

$$dy=\left(\frac{dy}{dx}\right)\Delta x=\frac{1}{2\sqrt{x}}(0.3) \qquad \left[\text{as } y=\sqrt{x}\right]$$

$$=\frac{1}{2\sqrt{25}}(0.3)=0.03$$

Hence, the approximate value of $\sqrt{25.3}$ is 0.03 + 5 = 5.03.

(ii) $\sqrt{49.5}$

Consider $y=\sqrt{x}$. Let x = 49 and Δx = 0.5.

Then,

$$\Delta y = \sqrt{x+\Delta x} - \sqrt{x} = \sqrt{49.5} - \sqrt{49} = \sqrt{49.5} - 7$$
$$\Rightarrow \sqrt{49.5} = 7 + \Delta y$$

Now, dy is approximately equal to Δy and is given by,

$$dy = \left(\frac{dy}{dx}\right)\Delta x = \frac{1}{2\sqrt{x}}(0.5) \qquad \left[\text{as } y = \sqrt{x}\right]$$
$$= \frac{1}{2\sqrt{49}}(0.5) = \frac{1}{14}(0.5) = 0.035$$

Hence, the approximate value of $\sqrt{49.5}$ is 7 + 0.035 = 7.035.

(iii) $\sqrt{0.6}$

Consider $y = \sqrt{x}$. Let x = 1 and Δx = − 0.4.

Then,

$$\Delta y = \sqrt{x+\Delta x} - \sqrt{x} = \sqrt{0.6} - 1$$
$$\Rightarrow \sqrt{0.6} = 1 + \Delta y$$

Now, dy is approximately equal to Δy and is given by,

$$dy = \left(\frac{dy}{dx}\right)\Delta x = \frac{1}{2\sqrt{x}}(\Delta x) \qquad \left[\text{as } y = \sqrt{x}\right]$$
$$= \frac{1}{2}(-0.4) = -0.2$$

Hence, the approximate value of $\sqrt{0.6}$ is 1 + (−0.2) = 1 − 0.2 = 0.8.

(iv) $(0.009)^{\frac{1}{3}}$

Consider $y = x^{\frac{1}{3}}$. Let x = 0.008 and Δx = 0.001.

Then,

$$\Delta y = (x+\Delta x)^{\frac{1}{3}} - (x)^{\frac{1}{3}} = (0.009)^{\frac{1}{3}} - (0.008)^{\frac{1}{3}} = (0.009)^{\frac{1}{3}} - 0.2$$
$$\Rightarrow (0.009)^{\frac{1}{3}} = 0.2 + \Delta y$$

Now, dy is approximately equal to Δy and is given by,

$$dy = \left(\frac{dy}{dx}\right)\Delta x = \frac{1}{3(x)^{\frac{2}{3}}}(\Delta x) \qquad \left[\text{as } y = x^{\frac{1}{3}}\right]$$

$$= \frac{1}{3\times 0.04}(0.001) = \frac{0.001}{0.12} = 0.008$$

Hence, the approximate value of $(0.009)^{\frac{1}{3}}$ is 0.2 + 0.008 = 0.208.

(v) $(0.999)^{\frac{1}{10}}$

Consider $y = (x)^{\frac{1}{10}}$. Let x = 1 and Δx = −0.001.

Then,

$$\Delta y = (x+\Delta x)^{\frac{1}{10}} - (x)^{\frac{1}{10}} = (0.999)^{\frac{1}{10}} - 1$$

$$\Rightarrow (0.999)^{\frac{1}{10}} = 1 + \Delta y$$

Now, dy is approximately equal to Δy and is given by,

$$dy = \left(\frac{dy}{dx}\right)\Delta x = \frac{1}{10(x)^{\frac{9}{10}}}(\Delta x) \qquad \left[\text{as } y = (x)^{\frac{1}{10}}\right]$$

$$= \frac{1}{10}(-0.001) = -0.0001$$

Hence, the approximate value of $(0.999)^{\frac{1}{10}}$ is 1 + (−0.0001) = 0.9999.

(vi) $(15)^{\frac{1}{4}}$

Consider $y = x^{\frac{1}{4}}$. Let x = 16 and Δx = −1.

Then,

$$\Delta y = (x+\Delta x)^{\frac{1}{4}} - x^{\frac{1}{4}} = (15)^{\frac{1}{4}} - (16)^{\frac{1}{4}} = (15)^{\frac{1}{4}} - 2$$

$$\Rightarrow (15)^{\frac{1}{4}} = 2 + \Delta y$$

Now, dy is approximately equal to Δy and is given by,

$$dy = \left(\frac{dy}{dx}\right)\Delta x = \frac{1}{4(x)^{\frac{3}{4}}}(\Delta x) \qquad \left[\text{as } y = x^{\frac{1}{4}}\right]$$

$$= \frac{1}{4(16)^{\frac{3}{4}}}(-1) = \frac{-1}{4\times 8} = \frac{-1}{32} = -0.03125$$

Hence, the approximate value of $(15)^{\frac{1}{4}}$ is 2 + (−0.03125) = 1.96875.

(vii) $(26)^{\frac{1}{3}}$

Consider $y = (x)^{\frac{1}{3}}$. Let x = 27 and Δx = −1.
Then,

$$\Delta y = (x+\Delta x)^{\frac{1}{3}} - (x)^{\frac{1}{3}} = (26)^{\frac{1}{3}} - (27)^{\frac{1}{3}} = (26)^{\frac{1}{3}} - 3$$

$$\Rightarrow (26)^{\frac{1}{3}} = 3 + \Delta y$$

Now, dy is approximately equal to Δy and is given by,

$$dy = \left(\frac{dy}{dx}\right)\Delta x = \frac{1}{3(x)^{\frac{2}{3}}}(\Delta x) \qquad \left[\text{as } y = (x)^{\frac{1}{3}}\right]$$

$$= \frac{1}{3(27)^{\frac{2}{3}}}(-1) = \frac{-1}{27} = -0.0\overline{370}$$

Hence, the approximate value of $(26)^{\frac{1}{3}}$ is 3 + (−0.0370) = 2.9629.

(viii) $(255)^{\frac{1}{4}}$

Consider $y = (x)^{\frac{1}{4}}$. Let x = 256 and Δx = −1.
Then,

$$\Delta y = (x+\Delta x)^{\frac{1}{4}} - (x)^{\frac{1}{4}} = (255)^{\frac{1}{4}} - (256)^{\frac{1}{4}} = (255)^{\frac{1}{4}} - 4$$

$$\Rightarrow (255)^{\frac{1}{4}} = 4 + \Delta y$$

Now, dy is approximately equal to Δy and is given by,

$$dy = \left(\frac{dy}{dx}\right)\Delta x = \frac{1}{4(x)^{\frac{3}{4}}}(\Delta x) \qquad \left[\text{as } y = x^{\frac{1}{4}}\right]$$

$$= \frac{1}{4(256)^{\frac{3}{4}}}(-1) = \frac{-1}{4\times 4^3} = -0.0039$$

Hence, the approximate value of $(255)^{\frac{1}{4}}$ is 4 + (−0.0039) = 3.9961.

(ix) $(82)^{\frac{1}{4}}$

Consider $y = x^{\frac{1}{4}}$. Let x = 81 and Δx = 1.
Then,

$$\Delta y = (x+\Delta x)^{\frac{1}{4}} - (x)^{\frac{1}{4}} = (82)^{\frac{1}{4}} - (81)^{\frac{1}{4}} = (82)^{\frac{1}{4}} - 3$$

$$\Rightarrow (82)^{\frac{1}{4}} = \Delta y + 3$$

Now, dy is approximately equal to Δy and is given by,

$$dy = \left(\frac{dy}{dx}\right)\Delta x = \frac{1}{4(x)^{\frac{3}{4}}}(\Delta x) \qquad \left[\text{as } y = x^{\frac{1}{4}}\right]$$

$$= \frac{1}{4(81)^{\frac{3}{4}}}(1) = \frac{1}{4(3)^3} = \frac{1}{108} = 0.009$$

Hence, the approximate value of $(82)^{\frac{1}{4}}$ is 3 + 0.009 = 3.009.

(x) $(401)^{\frac{1}{2}}$

Consider $y = x^{\frac{1}{2}}$. Let x = 400 and Δx = 1.
Then,

$$\Delta y = \sqrt{x+\Delta x} - \sqrt{x} = \sqrt{401} - \sqrt{400} = \sqrt{401} - 20$$

$$\Rightarrow \sqrt{401} = 20 + \Delta y$$

Now, dy is approximately equal to Δy and is given by,

$$dy = \left(\frac{dy}{dx}\right)\Delta x = \frac{1}{2\sqrt{x}}(\Delta x) \qquad \left[\text{as } y = x^{\frac{1}{2}}\right]$$

$$= \frac{1}{2\times 20}(1) = \frac{1}{40} = 0.025$$

Hence, the approximate value of $\sqrt{401}$ is 20 + 0.025 = 20.025.

(xi) $(0.0037)^{\frac{1}{2}}$

Consider $y = x^{\frac{1}{2}}$. Let x = 0.0036 and Δx = 0.0001.

Then,

$$\Delta y = (x+\Delta x)^{\frac{1}{2}} - (x)^{\frac{1}{2}} = (0.0037)^{\frac{1}{2}} - (0.0036)^{\frac{1}{2}} = (0.0037)^{\frac{1}{2}} - 0.06$$

$$\Rightarrow (0.0037)^{\frac{1}{2}} = 0.06 + \Delta y$$

Now, dy is approximately equal to Δy and is given by,

$$dy = \left(\frac{dy}{dx}\right)\Delta x = \frac{1}{2\sqrt{x}}(\Delta x) \qquad \left[\text{as } y = x^{\frac{1}{2}}\right]$$

$$= \frac{1}{2\times 0.06}(0.0001)$$

$$= \frac{0.0001}{0.12} = 0.00083$$

Thus, the approximate value of $(0.0037)^{\frac{1}{2}}$ is 0.06 + 0.00083 = 0.06083.

(xii) $(26.57)^{\frac{1}{3}}$

Consider $y = x^{\frac{1}{3}}$. Let x = 27 and Δx = −0.43.

Then,

$$\Delta y = (x+\Delta x)^{\frac{1}{3}} - x^{\frac{1}{3}} = (26.57)^{\frac{1}{3}} - (27)^{\frac{1}{3}} = (26.57)^{\frac{1}{3}} - 3$$

$$\Rightarrow (26.57)^{\frac{1}{3}} = 3 + \Delta y$$

Now, dy is approximately equal to Δy and is given by,

$$dy = \left(\frac{dy}{dx}\right)\Delta x = \frac{1}{3(x)^{\frac{2}{3}}}(\Delta x) \qquad \left[\text{as } y = x^{\frac{1}{3}}\right]$$

$$= \frac{1}{3(9)}(-0.43)$$

$$= \frac{-0.43}{27} = -0.015$$

Hence, the approximate value of $(26.57)^{\frac{1}{3}}$ is 3 + (−0.015) = 2.984.

(xiii) $(81.5)^{\frac{1}{4}}$

Consider $y = x^{\frac{1}{4}}$. Let x = 81 and Δx = 0.5.
Then,

$$\Delta y = (x+\Delta x)^{\frac{1}{4}} - (x)^{\frac{1}{4}} = (81.5)^{\frac{1}{4}} - (81)^{\frac{1}{4}} = (81.5)^{\frac{1}{4}} - 3$$

$$\Rightarrow (81.5)^{\frac{1}{4}} = 3 + \Delta y$$

Now, dy is approximately equal to Δy and is given by,

$$dy = \left(\frac{dy}{dx}\right)\Delta x = \frac{1}{4(x)^{\frac{3}{4}}}(\Delta x) \qquad \left[\text{as } y = x^{\frac{1}{4}}\right]$$

$$= \frac{1}{4(3)^3}(0.5) = \frac{0.5}{108} = 0.0046$$

Hence, the approximate value of $(81.5)^{\frac{1}{4}}$ is 3 + 0.0046 = 3.0046.

(xiv) $(3.968)^{\frac{3}{2}}$

Consider $y = x^{\frac{3}{2}}$. Let x = 4 and Δx = − 0.032.
Then,

$$\Delta y = (x+\Delta x)^{\frac{3}{2}} - x^{\frac{3}{2}} = (3.968)^{\frac{3}{2}} - (4)^{\frac{3}{2}} = (3.968)^{\frac{3}{2}} - 8$$

$$\Rightarrow (3.968)^{\frac{3}{2}} = 8 + \Delta y$$

Now, dy is approximately equal to Δy and is given by,

$$dy=\left(\frac{dy}{dx}\right)\Delta x=\frac{3}{2}(x)^{\frac{1}{2}}(\Delta x) \qquad \left[\text{as } y=x^{\frac{3}{2}}\right]$$

$$=\frac{3}{2}(2)(-0.032)$$

$$=-0.096$$

Hence, the approximate value of $(3.968)^{\frac{3}{2}}$ is 8 + (−0.096) = 7.904.

(xv) $(32.15)^{\frac{1}{5}}$

Consider $y=x^{\frac{1}{5}}$. Let x = 32 and Δx = 0.15.

Then,

$$\Delta y=(x+\Delta x)^{\frac{1}{5}}-x^{\frac{1}{5}}=(32.15)^{\frac{1}{5}}-(32)^{\frac{1}{5}}=(32.15)^{\frac{1}{5}}-2$$

$$\Rightarrow (32.15)^{\frac{1}{5}}=2+\Delta y$$

Now, dy is approximately equal to Δy and is given by,

$$dy=\left(\frac{dy}{dx}\right)\Delta x=\frac{1}{5(x)^{\frac{4}{5}}}\cdot(\Delta x) \qquad \left[\text{as } y=x^{\frac{1}{5}}\right]$$

$$=\frac{1}{5\times(2)^4}(0.15)$$

$$=\frac{0.15}{80}=0.00187$$

Hence, the approximate value of $(32.15)^{\frac{1}{5}}$ is 2 + 0.00187 = 2.00187.

Question 2:

Find the approximate value of f (2.01), where $f(x) = 4x^2 + 5x + 2$

Answer

Let x = 2 and Δx = 0.01. Then, we have:

$f(2.01) = f(x + \Delta x) = 4(x + \Delta x)^2 + 5(x + \Delta x) + 2$

Now, $\Delta y = f(x + \Delta x) - f(x)$

☐ $f(x + \Delta x) = f(x) + \Delta y$

$$\approx f(x)+f'(x)\cdot\Delta x \qquad (\text{as } dx=\Delta x)$$

$$\Rightarrow f(2.01)\approx\left(4x^2+5x+2\right)+\left(8x+5\right)\Delta x$$
$$=\left[4(2)^2+5(2)+2\right]+\left[8(2)+5\right](0.01) \qquad \left[\text{as } x=2,\ \Delta x=0.01\right]$$
$$=(16+10+2)+(16+5)(0.01)$$
$$=28+(21)(0.01)$$
$$=28+0.21$$
$$=28.21$$

Hence, the approximate value of f (2.01) is 28.21.

Question 3:

Find the approximate value of f (5.001), where $f(x) = x^3 - 7x^2 + 15$.

Answer

Let $x = 5$ and $\Delta x = 0.001$. Then, we have:

$$f(5.001)=f(x+\Delta x)=(x+\Delta x)^3-7(x+\Delta x)^2+15$$
$$\text{Now}, \Delta y=f(x+\Delta x)-f(x)$$
$$\therefore f(x+\Delta x)=f(x)+\Delta y$$
$$\approx f(x)+f'(x)\cdot\Delta x \qquad (\text{as } dx=\Delta x)$$
$$\Rightarrow f(5.001)\approx\left(x^3-7x^2+15\right)+\left(3x^2-14x\right)\Delta x$$
$$=\left[(5)^3-7(5)^2+15\right]+\left[3(5)^2-14(5)\right](0.001) \qquad \left[x=5, \Delta x=0.001\right]$$
$$=(125-175+15)+(75-70)(0.001)$$
$$=-35+(5)(0.001)$$
$$=-35+0.005$$
$$=-34.995$$

Hence, the approximate value of f (5.001) is −34.995.

Question 4:

Find the approximate change in the volume V of a cube of side x metres caused by increasing side by 1%.

Answer

The volume of a cube (*V*) of side *x* is given by $V = x^3$.

$$\therefore dV = \left(\frac{dV}{dx}\right)\Delta x$$
$$= \left(3x^2\right)\Delta x$$
$$= \left(3x^2\right)\left(0.01x\right) \qquad [\text{as } 1\% \text{ of } x \text{ is } 0.01x]$$
$$= 0.03x^3$$

Hence, the approximate change in the volume of the cube is $0.03x^3$ m^3.

Question 5:

Find the approximate change in the surface area of a cube of side *x* metres caused by decreasing the side by 1%

Answer

The surface area of a cube (*S*) of side *x* is given by $S = 6x^2$.

$$\therefore \frac{dS}{dx} = \left(\frac{dS}{dx}\right)\Delta x$$
$$= \left(12x\right)\Delta x$$
$$= \left(12x\right)\left(0.01x\right) \qquad [\text{as } 1\% \text{ of } x \text{ is } 0.01x]$$
$$= 0.12x^2$$

Hence, the approximate change in the surface area of the cube is $0.12x^2$ m^2.

Question 6:

If the radius of a sphere is measured as 7 m with an error of 0.02m, then find the approximate error in calculating its volume.

Answer

Let *r* be the radius of the sphere and Δr be the error in measuring the radius.

Then,

$r = 7$ m and $\Delta r = 0.02$ m

Now, the volume *V* of the sphere is given by,

$$V = \frac{4}{3}\pi r^3$$

$$\therefore \frac{dV}{dr} = 4\pi r^2$$

$$\therefore dV = \left(\frac{dV}{dr}\right)\Delta r$$

$$= \left(4\pi r^2\right)\Delta r$$

$$= 4\pi\left(7\right)^2\left(0.02\right)\ \text{m}^3 = 3.92\pi\ \text{m}^3$$

Hence, the approximate error in calculating the volume is 3.92 π m^3.

Question 7:

If the radius of a sphere is measured as 9 m with an error of 0.03 m, then find the approximate error in calculating in surface area.

Answer

Let *r* be the radius of the sphere and Δ*r* be the error in measuring the radius.

Then,

$r = 9$ m and $\Delta r = 0.03$ m

Now, the surface area of the sphere (*S*) is given by,

$S = 4\pi r^2$

$$\therefore \frac{dS}{dr} = 8\pi r$$

$$\therefore dS = \left(\frac{dS}{dr}\right)\Delta r$$

$$= \left(8\pi r\right)\Delta r$$

$$= 8\pi\left(9\right)\left(0.03\right)\ \text{m}^2$$

$$= 2.16\pi\ \text{m}^2$$

Hence, the approximate error in calculating the surface area is 2.16π m^2.

Question 8:

If $f(x) = 3x^2 + 15x + 5$, then the approximate value of $f(3.02)$ is

A. 47.66 **B.** 57.66 **C.** 67.66 **D.** 77.66

Answer

Let x = 3 and Δx = 0.02. Then, we have:

$$f(3.02) = f(x+\Delta x) = 3(x+\Delta x)^2 + 15(x+\Delta x) + 5$$

$$\text{Now, } \Delta y = f(x+\Delta x) - f(x)$$

$$\Rightarrow f(x+\Delta x) = f(x) + \Delta y$$

$$\approx f(x) + f'(x)\Delta x \qquad (\text{As } dx = \Delta x)$$

$$\Rightarrow f(3.02) \approx (3x^2 + 15x + 5) + (6x + 15)\Delta x$$

$$= \left[3(3)^2 + 15(3) + 5\right] + \left[6(3) + 15\right](0.02) \qquad [\text{As } x = 3,\ \Delta x = 0.02]$$

$$= (27 + 45 + 5) + (18 + 15)(0.02)$$

$$= 77 + (33)(0.02)$$

$$= 77 + 0.66$$

$$= 77.66$$

Hence, the approximate value of $f(3.02)$ is 77.66.

The correct answer is D.

Question 9:

The approximate change in the volume of a cube of side x metres caused by increasing the side by 3% is

A. 0.06 x^3 m^3 **B.** 0.6 x^3 m^3 **C.** 0.09 x^3 m^3 **D.** 0.9 x^3 m^3

Answer

The volume of a cube (V) of side x is given by $V = x^3$.

$$\therefore dV = \left(\frac{dV}{dx}\right)\Delta x$$

$$= (3x^2)\Delta x$$

$$= (3x^2)(0.03x) \qquad [\text{As 3\% of } x \text{ is } 0.03x]$$

$$= 0.09x^3 \text{ m}^3$$

Hence, the approximate change in the volume of the cube is $0.09x^3$ m^3.

The correct answer is C.