Exercise 7.4

Question 1:

$$\frac{3x^2}{x^6 + 1}$$

Answer

Let $x^3 = t$

$\therefore 3x^2\,dx = dt$

$$\Rightarrow \int \frac{3x^2}{x^6 + 1} dx = \int \frac{dt}{t^2 + 1}$$
$$= \tan^1 t + C$$
$$= \tan^{-1}(x^3) + C$$

Question 2:

$$\frac{1}{\sqrt{1 + 4x^2}}$$

Answer

Let $2x = t$

$\therefore 2dx = dt$

$$\Rightarrow \int \frac{1}{\sqrt{1+4x^2}} dx = \frac{1}{2} \int \frac{dt}{\sqrt{1+t^2}}$$

$$= \frac{1}{2} \left[\log \left| t + \sqrt{t^2+1} \right| \right] + C \qquad \left[\int \frac{1}{\sqrt{x^2+a^2}} dt = \log \left| x + \sqrt{x^2+a^2} \right| \right]$$

$$= \frac{1}{2} \log \left| 2x + \sqrt{4x^2+1} \right| + C$$

Question 3:

$$\frac{1}{\sqrt{(2-x)^2 + 1}}$$

Answer

Let $2 - x = t$

$\Rightarrow -dx = dt$

$$\Rightarrow \int \frac{1}{\sqrt{(2-x)^2 + 1}} dx = -\int \frac{1}{\sqrt{t^2+1}} dt$$

$$= -\log \left| t + \sqrt{t^2+1} \right| + C \qquad \left[\int \frac{1}{\sqrt{x^2+a^2}} dt = \log \left| x + \sqrt{x^2+a^2} \right| \right]$$

$$= -\log \left| 2-x + \sqrt{(2-x)^2 + 1} \right| + C$$

$$= \log \left| \frac{1}{(2-x) + \sqrt{x^2 - 4x + 5}} \right| + C$$

Question 4:

$$\frac{1}{\sqrt{9 - 25x^2}}$$

Answer

Let $5x = t$

$\therefore 5dx = dt$

$$\Rightarrow \int \frac{1}{\sqrt{9 - 25x^2}} dx = \frac{1}{5} \int \frac{1}{9 - t^2} dt$$

$$= \frac{1}{5} \int \frac{1}{\sqrt{3^2 - t^2}} dt$$

$$= \frac{1}{5} \sin^{-1} \left(\frac{t}{3} \right) + C$$

$$= \frac{1}{5} \sin^{-1} \left(\frac{5x}{3} \right) + C$$

Question 5:

$$\frac{3x}{1 + 2x^4}$$

Answer

Let $\sqrt{2}x^2 = t$

$\therefore 2\sqrt{2}x \, dx = dt$

$$\Rightarrow \int \frac{3x}{1 + 2x^4} dx = \frac{3}{2\sqrt{2}} \int \frac{dt}{1 + t^2}$$

$$= \frac{3}{2\sqrt{2}} \left[\tan^{-1} t \right] + C$$

$$= \frac{3}{2\sqrt{2}} \tan^{-1} \left(\sqrt{2}x^2 \right) + C$$

Question 6:

$$\frac{x^2}{1 - x^6}$$

Answer

Let $x^3 = t$

$\therefore$ *3x² dx = dt*

$$\Rightarrow \int \frac{x^2}{1-x^6} dx = \frac{1}{3} \int \frac{dt}{1-t^2}$$

$$= \frac{1}{3} \left[\frac{1}{2} \log \left| \frac{1+t}{1-t} \right| \right] + C$$

$$= \frac{1}{6} \log \left| \frac{1+x^3}{1-x^3} \right| + C$$

Question 7:

$$\frac{x-1}{\sqrt{x^2-1}}$$

Answer

$$\int \frac{x-1}{\sqrt{x^2-1}} \, dx = \int \frac{x}{\sqrt{x^2-1}} \, dx - \int \frac{1}{\sqrt{x^2-1}} \, dx \qquad \ldots(1)$$

For $\int \frac{x}{\sqrt{x^2-1}} dx$, let $x^2 - 1 = t \Rightarrow 2x \, dx = dt$

$$\therefore \int \frac{x}{\sqrt{x^2-1}} dx = \frac{1}{2} \int \frac{dt}{\sqrt{t}}$$

$$= \frac{1}{2} \int t^{-\frac{1}{2}} dt$$

$$= \frac{1}{2} \left[2t^{\frac{1}{2}} \right]$$

$$= \sqrt{t}$$

$$= \sqrt{x^2-1}$$

From (1), we obtain

$$\int \frac{x-1}{\sqrt{x^2-1}}dx = \int \frac{x}{\sqrt{x^2-1}}dx - \int \frac{1}{\sqrt{x^2-1}}dx$$

$$\left[\int \frac{1}{\sqrt{x^2-a^2}}dt = \log\left|x+\sqrt{x^2-a^2}\right|\right]$$

$$= \sqrt{x^2-1} - \log\left|x+\sqrt{x^2-1}\right| + C$$

Question 8:

$$\frac{x^2}{\sqrt{x^6+a^6}}$$

Answer

Let $x^3 = t$

$\Rightarrow 3x^2\ dx = dt$

$$\therefore \int \frac{x^2}{\sqrt{x^6+a^6}}dx = \frac{1}{3}\int \frac{dt}{\sqrt{t^2+\left(a^3\right)^2}}$$

$$= \frac{1}{3}\log\left|t+\sqrt{t^2+a^6}\right| + C$$

$$= \frac{1}{3}\log\left|x^3+\sqrt{x^6+a^6}\right| + C$$

Question 9:

$$\frac{\sec^2 x}{\sqrt{\tan^2 x+4}}$$

Answer

Let $\tan x = t$

$\therefore \sec^2 x\ dx = dt$

$$\Rightarrow \int \frac{\sec^2 x}{\sqrt{\tan^2 x + 4}} dx = \int \frac{dt}{\sqrt{t^2 + 2^2}}$$

$$= \log\left|t + \sqrt{t^2 + 4}\right| + C$$

$$= \log\left|\tan x + \sqrt{\tan^2 x + 4}\right| + C$$

Question 10:

$$\frac{1}{\sqrt{x^2 + 2x + 2}}$$

Answer

$$\int \frac{1}{\sqrt{x^2 + 2x + 2}} dx = \int \frac{1}{\sqrt{(x+1)^2 + (1)^2}} dx$$

Let $x + 1 = t$

$\therefore \; dx = dt$

$$\Rightarrow \int \frac{1}{\sqrt{x^2 + 2x + 2}} dx = \int \frac{1}{\sqrt{t^2 + 1}} dt$$

$$= \log\left|t + \sqrt{t^2 + 1}\right| + C$$

$$= \log\left|(x+1) + \sqrt{(x+1)^2 + 1}\right| + C$$

$$= \log\left|(x+1) + \sqrt{x^2 + 2x + 2}\right| + C$$

Question 11:

$$\frac{1}{\sqrt{9x^2 + 6x + 5}}$$

Answer

$$\int \frac{1}{9x^2 + 6x + 5} dx = \int \frac{1}{(3x+1)^2 + (2)^2} dx$$

$\text{Let} (3x+1) = t$

$\therefore \ 3dx = dt$

$$\Rightarrow \int \frac{1}{(3x+1)^2 + (2)^2} dx = \frac{1}{3} \int \frac{1}{t^2 + 2^2} dt$$

$$= \frac{1}{3} \left[\frac{1}{2} \tan^{-1} \left(\frac{t}{2} \right) \right] + C$$

$$= \frac{1}{6} \tan^{-1} \left(\frac{3x+1}{2} \right) + C$$

Question 12:

$$\frac{1}{\sqrt{7 - 6x - x^2}}$$

Answer

$7 - 6x - x^2$ can be written as $7 - (x^2 + 6x + 9 - 9)$.

Therefore,

$7 - (x^2 + 6x + 9 - 9)$

$= 16 - (x^2 + 6x + 9)$

$= 16 - (x+3)^2$

$= (4)^2 - (x+3)^2$

$$\therefore \int \frac{1}{\sqrt{7 - 6x - x^2}} dx = \int \frac{1}{\sqrt{(4)^2 - (x+3)^2}} dx$$

$\text{Let } x + 3 = t$

$\Rightarrow dx = dt$

$$\Rightarrow \int \frac{1}{\sqrt{(4)^2 - (x+3)^2}} dx = \int \frac{1}{\sqrt{(4)^2 - (t)^2}} dt$$

$$= \sin^{-1} \left(\frac{t}{4} \right) + C$$

$$= \sin^{-1} \left(\frac{x+3}{4} \right) + C$$

Question 13:

$$\frac{1}{\sqrt{(x-1)(x-2)}}$$

Answer

$(x-1)(x-2)$ can be written as $x^2 - 3x + 2$.

Therefore,

$x^2 - 3x + 2$

$= x^2 - 3x + \dfrac{9}{4} - \dfrac{9}{4} + 2$

$= \left(x - \dfrac{3}{2}\right)^2 - \dfrac{1}{4}$

$= \left(x - \dfrac{3}{2}\right)^2 - \left(\dfrac{1}{2}\right)^2$

$\therefore \int \dfrac{1}{\sqrt{(x-1)(x-2)}}\, dx = \int \dfrac{1}{\sqrt{\left(x - \dfrac{3}{2}\right)^2 - \left(\dfrac{1}{2}\right)^2}}\, dx$

Let $x - \dfrac{3}{2} = t$

$\therefore dx = dt$

$\Rightarrow \int \dfrac{1}{\sqrt{\left(x - \dfrac{3}{2}\right)^2 - \left(\dfrac{1}{2}\right)^2}}\, dx = \int \dfrac{1}{\sqrt{t^2 - \left(\dfrac{1}{2}\right)^2}}\, dt$

$= \log\left| t + \sqrt{t^2 - \left(\dfrac{1}{2}\right)^2} \right| + C$

$= \log\left| \left(x - \dfrac{3}{2}\right) + \sqrt{x^2 - 3x + 2} \right| + C$

Question 14:

$$\frac{1}{\sqrt{8 + 3x - x^2}}$$

Answer

$8 + 3x - x^2$ can be written as $8 - \left(x^2 - 3x + \dfrac{9}{4} - \dfrac{9}{4} \right)$.

Therefore,

$8 - \left(x^2 - 3x + \dfrac{9}{4} - \dfrac{9}{4} \right)$

$= \dfrac{41}{4} - \left(x - \dfrac{3}{2} \right)^2$

$\Rightarrow \displaystyle\int \dfrac{1}{\sqrt{8 + 3x - x^2}}\, dx = \displaystyle\int \dfrac{1}{\sqrt{\dfrac{41}{4} - \left(x - \dfrac{3}{2} \right)^2}}\, dx$

Let $x - \dfrac{3}{2} = t$

$\therefore dx = dt$

$\Rightarrow \displaystyle\int \dfrac{1}{\sqrt{\dfrac{41}{4} - \left(x - \dfrac{3}{2} \right)^2}}\, dx = \displaystyle\int \dfrac{1}{\sqrt{\left(\dfrac{\sqrt{41}}{2} \right)^2 - t^2}}\, dt$

$= \sin^{-1}\left(\dfrac{t}{\dfrac{\sqrt{41}}{2}} \right) + C$

$= \sin^{-1}\left(\dfrac{x - \dfrac{3}{2}}{\dfrac{\sqrt{41}}{2}} \right) + C$

$= \sin^{-1}\left(\dfrac{2x - 3}{\sqrt{41}} \right) + C$

Question 15:

$\dfrac{1}{\sqrt{(x - a)(x - b)}}$

Answer

$(x-a)(x-b)$ can be written as $x^2 - (a+b)x + ab$.

Therefore,

$x^2 - (a+b)x + ab$

$= x^2 - (a+b)x + \dfrac{(a+b)^2}{4} - \dfrac{(a+b)^2}{4} + ab$

$= \left[x - \left(\dfrac{a+b}{2} \right) \right]^2 - \dfrac{(a-b)^2}{4}$

$\Rightarrow \displaystyle\int \dfrac{1}{\sqrt{(x-a)(x-b)}} dx = \int \dfrac{1}{\sqrt{\left\{ x - \left(\dfrac{a+b}{2} \right) \right\}^2 - \left(\dfrac{a-b}{2} \right)^2}} dx$

Let $x - \left(\dfrac{a+b}{2} \right) = t$

$\therefore dx = dt$

$\Rightarrow \displaystyle\int \dfrac{1}{\sqrt{\left\{ x - \left(\dfrac{a+b}{2} \right) \right\}^2 - \left(\dfrac{a-b}{2} \right)^2}} dx = \int \dfrac{1}{\sqrt{t^2 - \left(\dfrac{a-b}{2} \right)^2}} dt$

$= \log \left| t + \sqrt{t^2 - \left(\dfrac{a-b}{2} \right)^2} \right| + C$

$= \log \left| \left\{ x - \left(\dfrac{a+b}{2} \right) \right\} + \sqrt{(x-a)(x-b)} \right| + C$

Question 16:

$\dfrac{4x+1}{\sqrt{2x^2 + x - 3}}$

Answer

Let $4x + 1 = A \dfrac{d}{dx}\left(2x^2 + x - 3 \right) + B$

$\Rightarrow 4x + 1 = A(4x + 1) + B$

$\Rightarrow 4x + 1 = 4Ax + A + B$

Equating the coefficients of *x* and constant term on both sides, we obtain

$4A = 4 \Rightarrow A = 1$

$A + B = 1 \Rightarrow B = 0$

Let $2x^2 + x - 3 = t$

$\therefore (4x + 1)\ dx = dt$

$$\Rightarrow \int \frac{4x+1}{\sqrt{2x^2+x-3}}dx = \int \frac{1}{\sqrt{t}}dt$$
$$= 2\sqrt{t} + C$$
$$= 2\sqrt{2x^2+x-3} + C$$

Question 17:

$$\frac{x+2}{\sqrt{x^2-1}}$$

Answer

$$\text{Let } x+2 = A\frac{d}{dx}\left(x^2-1\right)+B \qquad \text{...(1)}$$
$$\Rightarrow x+2 = A(2x)+B$$

Equating the coefficients of x and constant term on both sides, we obtain

$$2A = 1 \Rightarrow A = \frac{1}{2}$$
$$B = 2$$

From (1), we obtain

$(x+2) = \dfrac{1}{2}(2x) + 2$

Then, $\displaystyle\int \dfrac{x+2}{\sqrt{x^2-1}}dx = \int \dfrac{\dfrac{1}{2}(2x)+2}{\sqrt{x^2-1}}dx$

$\qquad\qquad = \dfrac{1}{2}\displaystyle\int \dfrac{2x}{\sqrt{x^2-1}}dx + \int \dfrac{2}{\sqrt{x^2-1}}dx$ ...(2)

In $\dfrac{1}{2}\displaystyle\int \dfrac{2x}{\sqrt{x^2-1}}dx$, let $x^2-1 = t \Rightarrow 2xdx = dt$

$\dfrac{1}{2}\displaystyle\int \dfrac{2x}{\sqrt{x^2-1}}dx = \dfrac{1}{2}\int \dfrac{dt}{\sqrt{t}}$

$\qquad\qquad = \dfrac{1}{2}\left[2\sqrt{t}\right]$

$\qquad\qquad = \sqrt{t}$

$\qquad\qquad = \sqrt{x^2-1}$

Then, $\displaystyle\int \dfrac{2}{\sqrt{x^2-1}}dx = 2\int \dfrac{1}{\sqrt{x^2-1}}dx = 2\log\left|x+\sqrt{x^2-1}\right|$

From equation (2), we obtain

$\displaystyle\int \dfrac{x+2}{\sqrt{x^2-1}}dx = \sqrt{x^2-1} + 2\log\left|x+\sqrt{x^2-1}\right| + C$

Question 18:

$\dfrac{5x-2}{1+2x+3x^2}$

Answer

Let $5x-2 = A\dfrac{d}{dx}\left(1+2x+3x^2\right) + B$

$\Rightarrow 5x-2 = A(2+6x) + B$

Equating the coefficient of *x* and constant term on both sides, we obtain

$$5 = 6A \Rightarrow A = \frac{5}{6}$$

$$2A + B = -2 \Rightarrow B = -\frac{11}{3}$$

$$\therefore 5x - 2 = \frac{5}{6}(2 + 6x) + \left(-\frac{11}{3}\right)$$

$$\Rightarrow \int \frac{5x - 2}{1 + 2x + 3x^2} dx = \int \frac{\frac{5}{6}(2 + 6x) - \frac{11}{3}}{1 + 2x + 3x^2} dx$$

$$= \frac{5}{6} \int \frac{2 + 6x}{1 + 2x + 3x^2} dx - \frac{11}{3} \int \frac{1}{1 + 2x + 3x^2} dx$$

Let $I_1 = \int \frac{2 + 6x}{1 + 2x + 3x^2} dx$ and $I_2 = \int \frac{1}{1 + 2x + 3x^2} dx$

$$\therefore \int \frac{5x - 2}{1 + 2x + 3x^2} dx = \frac{5}{6} I_1 - \frac{11}{3} I_2 \qquad \ldots(1)$$

$$I_1 = \int \frac{2 + 6x}{1 + 2x + 3x^2} dx$$

Let $1 + 2x + 3x^2 = t$

$$\Rightarrow (2 + 6x) dx = dt$$

$$\therefore I_1 = \int \frac{dt}{t}$$

$$I_1 = \log|t|$$

$$I_1 = \log\left|1 + 2x + 3x^2\right| \qquad \ldots(2)$$

$$I_2 = \int \frac{1}{1 + 2x + 3x^2} dx$$

$1+2x+3x^2$ can be written as $1+3\left(x^2+\dfrac{2}{3}x\right)$.

Therefore,

$1+3\left(x^2+\dfrac{2}{3}x\right)$

$=1+3\left(x^2+\dfrac{2}{3}x+\dfrac{1}{9}-\dfrac{1}{9}\right)$

$=1+3\left(x+\dfrac{1}{3}\right)^2-\dfrac{1}{3}$

$=\dfrac{2}{3}+3\left(x+\dfrac{1}{3}\right)^2$

$=3\left[\left(x+\dfrac{1}{3}\right)^2+\dfrac{2}{9}\right]$

$=3\left[\left(x+\dfrac{1}{3}\right)^2+\left(\dfrac{\sqrt{2}}{3}\right)^2\right]$

$I_2=\dfrac{1}{3}\displaystyle\int\dfrac{1}{\left[\left(x+\dfrac{1}{3}\right)^2+\left(\dfrac{\sqrt{2}}{3}\right)^2\right]}dx$

$=\dfrac{1}{3}\left[\dfrac{1}{\dfrac{\sqrt{2}}{3}}\tan^{-1}\left(\dfrac{x+\dfrac{1}{3}}{\dfrac{\sqrt{2}}{3}}\right)\right]$

$=\dfrac{1}{3}\left[\dfrac{3}{\sqrt{2}}\tan^{-1}\left(\dfrac{3x+1}{\sqrt{2}}\right)\right]$

$=\dfrac{1}{\sqrt{2}}\tan^{-1}\left(\dfrac{3x+1}{\sqrt{2}}\right)$ ...(3)

Substituting equations (2) and (3) in equation (1), we obtain

$$\int \frac{5x-2}{1+2x+3x^2}dx = \frac{5}{6}\left[\log\left|1+2x+3x^2\right|\right] - \frac{11}{3}\left[\frac{1}{\sqrt{2}}\tan^{-1}\left(\frac{3x+1}{\sqrt{2}}\right)\right]+C$$

$$= \frac{5}{6}\log\left|1+2x+3x^2\right| - \frac{11}{3\sqrt{2}}\tan^{-1}\left(\frac{3x+1}{\sqrt{2}}\right)+C$$

Question 19:

$$\frac{6x+7}{\sqrt{(x-5)(x-4)}}$$

Answer

$$\frac{6x+7}{\sqrt{(x-5)(x-4)}} = \frac{6x+7}{\sqrt{x^2-9x+20}}$$

$$\text{Let } 6x+7 = A\frac{d}{dx}\left(x^2-9x+20\right)+B$$

$$\Rightarrow 6x+7 = A(2x-9)+B$$

Equating the coefficients of *x* and constant term, we obtain

$2A = 6 \Rightarrow A = 3$

$-9A + B = 7 \Rightarrow B = 34$

∴ $6x + 7 = 3 (2x − 9) + 34$

$$\int \frac{6x+7}{\sqrt{x^2-9x+20}} = \int \frac{3(2x-9)+34}{\sqrt{x^2-9x+20}}dx$$

$$= 3\int \frac{2x-9}{\sqrt{x^2-9x+20}}dx + 34\int \frac{1}{\sqrt{x^2-9x+20}}dx$$

Let $I_1 = \int \frac{2x-9}{\sqrt{x^2-9x+20}}dx$ and $I_2 = \int \frac{1}{\sqrt{x^2-9x+20}}dx$

$$\therefore \int \frac{6x+7}{\sqrt{x^2-9x+20}} = 3I_1 + 34I_2 \qquad\qquad ...(1)$$

Then,

$$I_1 = \int \frac{2x-9}{\sqrt{x^2-9x+20}}dx$$

Let $x^2 - 9x + 20 = t$

$$\Rightarrow (2x-9)dx = dt$$

$$\Rightarrow I_1 = \frac{dt}{\sqrt{t}}$$

$$I_1 = 2\sqrt{t}$$

$$I_1 = 2\sqrt{x^2-9x+20} \qquad\qquad ...(2)$$

and $I_2 = \int \frac{1}{\sqrt{x^2-9x+20}}dx$

$x^2 - 9x + 20$ can be written as $x^2 - 9x + 20 + \dfrac{81}{4} - \dfrac{81}{4}$.

Therefore,

$x^2 - 9x + 20 + \dfrac{81}{4} - \dfrac{81}{4}$

$= \left(x - \dfrac{9}{2} \right)^2 - \dfrac{1}{4}$

$= \left(x - \dfrac{9}{2} \right)^2 - \left(\dfrac{1}{2} \right)^2$

$\Rightarrow I_2 = \int \dfrac{1}{\sqrt{\left(x - \dfrac{9}{2} \right)^2 - \left(\dfrac{1}{2} \right)^2}} dx$

$I_2 = \log \left| \left(x - \dfrac{9}{2} \right) + \sqrt{x^2 - 9x + 20} \right|$ ...(3)

Substituting equations (2) and (3) in (1), we obtain

$\int \dfrac{6x + 7}{\sqrt{x^2 - 9x + 20}} dx = 3 \left[2\sqrt{x^2 - 9x + 20} \right] + 34 \log \left[\left(x - \dfrac{9}{2} \right) + \sqrt{x^2 - 9x + 20} \right] + C$

$= 6\sqrt{x^2 - 9x + 20} + 34 \log \left[\left(x - \dfrac{9}{2} \right) + \sqrt{x^2 - 9x + 20} \right] + C$

Question 20:

$\dfrac{x + 2}{\sqrt{4x - x^2}}$

Answer

Let $x + 2 = A \dfrac{d}{dx} \left(4x - x^2 \right) + B$

$\Rightarrow x + 2 = A \left(4 - 2x \right) + B$

Equating the coefficients of x and constant term on both sides, we obtain

$-2A = 1 \Rightarrow A = -\dfrac{1}{2}$

$4A + B = 2 \Rightarrow B = 4$

$\Rightarrow (x+2) = -\dfrac{1}{2}(4-2x)+4$

$\therefore \int \dfrac{x+2}{\sqrt{4x-x^2}} dx = \int \dfrac{-\dfrac{1}{2}(4-2x)+4}{\sqrt{4x-x^2}} dx$

$\qquad\qquad = -\dfrac{1}{2}\int \dfrac{4-2x}{\sqrt{4x-x^2}} dx + 4\int \dfrac{1}{\sqrt{4x-x^2}} dx$

Let $I_1 = \int \dfrac{4-2x}{\sqrt{4x-x^2}} dx$ and $I_2 \int \dfrac{1}{\sqrt{4x-x^2}} dx$

$\therefore \int \dfrac{x+2}{\sqrt{4x-x^2}} dx = -\dfrac{1}{2}I_1 + 4I_2$ \qquad\qquad ...(1)

Then, $I_1 = \int \dfrac{4-2x}{\sqrt{4x-x^2}} dx$

Let $4x - x^2 = t$

$\Rightarrow (4-2x) dx = dt$

$\Rightarrow I_1 = \int \dfrac{dt}{\sqrt{t}} = 2\sqrt{t} = 2\sqrt{4x-x^2}$ \qquad\qquad ...(2)

$I_2 = \int \dfrac{1}{\sqrt{4x-x^2}} dx$

$\Rightarrow 4x - x^2 = -\left(-4x+x^2\right)$

$\qquad\qquad = \left(-4x + x^2 + 4 - 4\right)$

$\qquad\qquad = 4 - (x-2)^2$

$\qquad\qquad = (2)^2 - (x-2)^2$

$\therefore I_2 = \int \dfrac{1}{\sqrt{(2)^2 - (x-2)^2}} dx = \sin^{-1}\left(\dfrac{x-2}{2}\right)$ \qquad\qquad ...(3)

Using equations (2) and (3) in (1), we obtain

$$\int \frac{x+2}{\sqrt{4x-x^2}} dx = -\frac{1}{2}\left(2\sqrt{4x-x^2}\right)+4\sin^{-1}\left(\frac{x-2}{2}\right)+C$$

$$= -\sqrt{4x-x^2}+4\sin^{-1}\left(\frac{x-2}{2}\right)+C$$

Question 21:

$$\frac{x+2}{\sqrt{x^2+2x+3}}$$

Answer

$$\int \frac{(x+2)}{\sqrt{x^2+2x+3}} dx = \frac{1}{2}\int \frac{2(x+2)}{\sqrt{x^2+2x+3}} dx$$

$$= \frac{1}{2}\int \frac{2x+4}{\sqrt{x^2+2x+3}} dx$$

$$= \frac{1}{2}\int \frac{2x+2}{\sqrt{x^2+2x+3}} dx + \frac{1}{2}\int \frac{2}{\sqrt{x^2+2x+3}} dx$$

$$= \frac{1}{2}\int \frac{2x+2}{\sqrt{x^2+2x+3}} dx + \int \frac{1}{\sqrt{x^2+2x+3}} dx$$

Let $I_1 = \int \frac{2x+2}{\sqrt{x^2+2x+3}} dx$ and $I_2 = \int \frac{1}{\sqrt{x^2+2x+3}} dx$

$$\therefore \int \frac{x+2}{\sqrt{x^2+2x+3}} dx = \frac{1}{2}I_1 + I_2 \qquad \text{...(1)}$$

Then, $I_1 = \int \frac{2x+2}{\sqrt{x^2+2x+3}} dx$

Let $x^2 + 2x + 3 = t$

$\Rightarrow (2x + 2) dx = dt$

$$I_1 = \int \frac{dt}{\sqrt{t}} = 2\sqrt{t} = 2\sqrt{x^2 + 2x + 3} \qquad \qquad ...(2)$$

$$I_2 = \int \frac{1}{\sqrt{x^2 + 2x + 3}} dx$$

$$\Rightarrow x^2 + 2x + 3 = x^2 + 2x + 1 + 2 = (x+1)^2 + \left(\sqrt{2}\right)^2$$

$$\therefore I_2 = \int \frac{1}{\sqrt{(x+1)^2 + \left(\sqrt{2}\right)^2}} dx = \log \left|(x+1) + \sqrt{x^2 + 2x + 3}\right| \qquad ...(3)$$

Using equations (2) and (3) in (1), we obtain

$$\int \frac{x+2}{\sqrt{x^2 + 2x + 3}} dx = \frac{1}{2}\left[2\sqrt{x^2 + 2x + 3}\right] + \log \left|(x+1) + \sqrt{x^2 + 2x + 3}\right| + C$$

$$= \sqrt{x^2 + 2x + 3} + \log \left|(x+1) + \sqrt{x^2 + 2x + 3}\right| + C$$

Question 22:

$$\frac{x+3}{x^2 - 2x - 5}$$

Answer

Let $(x+3) = A\dfrac{d}{dx}\left(x^2 - 2x - 5\right) + B$

$(x+3) = A(2x-2) + B$

Equating the coefficients of *x* and constant term on both sides, we obtain

$$2A = 1 \Rightarrow A = \frac{1}{2}$$

$$-2A + B = 3 \Rightarrow B = 4$$

$$\therefore (x+3) = \frac{1}{2}(2x-2) + 4$$

$$\Rightarrow \int \frac{x+3}{x^2 - 2x - 5} dx = \int \frac{\frac{1}{2}(2x-2) + 4}{x^2 - 2x - 5} dx$$

$$= \frac{1}{2}\int \frac{2x-2}{x^2 - 2x - 5} dx + 4 \int \frac{1}{x^2 - 2x - 5} dx$$

Let $I_1 = \int \dfrac{2x-2}{x^2-2x-5} dx$ and $I_2 = \int \dfrac{1}{x^2-2x-5} dx$

$\therefore \int \dfrac{x+3}{\left(x^2-2x-5\right)} dx = \dfrac{1}{2} I_1 + 4I_2$ ...(1)

Then, $I_1 = \int \dfrac{2x-2}{x^2-2x-5} dx$

Let $x^2-2x-5 = t$

$\Rightarrow (2x-2) dx = dt$

$\Rightarrow I_1 = \int \dfrac{dt}{t} = \log|t| = \log\left|x^2-2x-5\right|$ ...(2)

$I_2 = \int \dfrac{1}{x^2-2x-5} dx$

$\quad = \int \dfrac{1}{\left(x^2-2x+1\right)-6} dx$

$\quad = \int \dfrac{1}{(x-1)^2+\left(\sqrt{6}\right)^2} dx$

$\quad = \dfrac{1}{2\sqrt{6}} \log\left(\dfrac{x-1-\sqrt{6}}{x-1+\sqrt{6}}\right)$ ...(3)

Substituting (2) and (3) in (1), we obtain

$\int \dfrac{x+3}{x^2-2x-5} dx = \dfrac{1}{2} \log\left|x^2-2x-5\right| + \dfrac{4}{2\sqrt{6}} \log\left|\dfrac{x-1-\sqrt{6}}{x-1+\sqrt{6}}\right| + C$

$\qquad\qquad = \dfrac{1}{2} \log\left|x^2-2x-5\right| + \dfrac{2}{\sqrt{6}} \log\left|\dfrac{x-1-\sqrt{6}}{x-1+\sqrt{6}}\right| + C$

Question 23:

$\dfrac{5x+3}{\sqrt{x^2+4x+10}}$

Answer

Let $5x+3 = A\dfrac{d}{dx}\left(x^2+4x+10\right)+B$

$\Rightarrow 5x+3 = A(2x+4)+B$

Equating the coefficients of x and constant term, we obtain

$2A = 5 \Rightarrow A = \dfrac{5}{2}$

$4A+B = 3 \Rightarrow B = -7$

$\therefore 5x+3 = \dfrac{5}{2}(2x+4)-7$

$\Rightarrow \displaystyle\int \dfrac{5x+3}{\sqrt{x^2+4x+10}}dx = \displaystyle\int \dfrac{\dfrac{5}{2}(2x+4)-7}{\sqrt{x^2+4x+10}}dx$

$\qquad\qquad\qquad = \dfrac{5}{2}\displaystyle\int \dfrac{2x+4}{\sqrt{x^2+4x+10}}dx - 7\displaystyle\int \dfrac{1}{\sqrt{x^2+4x+10}}dx$

Let $I_1 = \displaystyle\int \dfrac{2x+4}{\sqrt{x^2+4x+10}}dx$ and $I_2 = \displaystyle\int \dfrac{1}{\sqrt{x^2+4x+10}}dx$

$\therefore \displaystyle\int \dfrac{5x+3}{\sqrt{x^2+4x+10}}dx = \dfrac{5}{2}I_1 - 7I_2$ ...(1)

Then, $I_1 = \displaystyle\int \dfrac{2x+4}{\sqrt{x^2+4x+10}}dx$

Let $x^2+4x+10 = t$

$\therefore (2x+4)dx = dt$

$\Rightarrow I_1 = \displaystyle\int \dfrac{dt}{\sqrt{t}} = 2\sqrt{t} = 2\sqrt{x^2+4x+10}$ ...(2)

$I_2 = \displaystyle\int \dfrac{1}{\sqrt{x^2+4x+10}}dx$

$\quad = \displaystyle\int \dfrac{1}{\sqrt{(x^2+4x+4)+6}}dx$

$\quad = \displaystyle\int \dfrac{1}{(x+2)^2+\left(\sqrt{6}\right)^2}dx$

$\quad = \log\left|(x+2)\sqrt{x^2+4x+10}\right|$ ...(3)

Using equations (2) and (3) in (1), we obtain

$$\int \frac{5x+3}{\sqrt{x^2+4x+10}}dx = \frac{5}{2}\left[2\sqrt{x^2+4x+10}\right] - 7\log\left|(x+2)+\sqrt{x^2+4x+10}\right| + C$$

$$= 5\sqrt{x^2+4x+10} - 7\log\left|(x+2)+\sqrt{x^2+4x+10}\right| + C$$

Question 24:

$\int \dfrac{dx}{x^2+2x+2}$ equals

A. $x \tan^{-1}(x+1) + C$

B. $\tan^{-1}(x+1) + C$

C. $(x+1)\tan^{-1}x + C$

D. $\tan^{-1}x + C$

Answer

$$\int \frac{dx}{x^2+2x+2} = \int \frac{dx}{(x^2+2x+1)+1}$$

$$= \int \frac{1}{(x+1)^2+(1)^2}dx$$

$$= \left[\tan^{-1}(x+1)\right] + C$$

Hence, the correct Answer is B.

Question 25:

$\int \dfrac{dx}{\sqrt{9x-4x^2}}$ equals

A. $\dfrac{1}{9}\sin^{-1}\left(\dfrac{9x-8}{8}\right) + C$

B. $\dfrac{1}{2}\sin^{-1}\left(\dfrac{8x-9}{9}\right) + C$

C. $\dfrac{1}{3}\sin^{-1}\left(\dfrac{9x-8}{8}\right) + C$

D. $\dfrac{1}{2}\sin^{-1}\left(\dfrac{9x-8}{9}\right)+C$

Answer

$$\int \dfrac{dx}{\sqrt{9x-4x^2}}$$

$$= \int \dfrac{1}{\sqrt{-4\left(x^2-\dfrac{9}{4}x\right)}}dx$$

$$= \int \dfrac{1}{\sqrt{-4\left(x^2-\dfrac{9}{4}x+\dfrac{81}{64}-\dfrac{81}{64}\right)}}dx$$

$$= \int \dfrac{1}{\sqrt{-4\left[\left(x-\dfrac{9}{8}\right)^2-\left(\dfrac{9}{8}\right)^2\right]}}dx$$

$$= \dfrac{1}{2}\int \dfrac{1}{\sqrt{\left(\dfrac{9}{8}\right)^2-\left(x-\dfrac{9}{8}\right)^2}}dx$$

$$= \dfrac{1}{2}\left[\sin^{-1}\left(\dfrac{x-\dfrac{9}{8}}{\dfrac{9}{8}}\right)\right]+C \qquad\qquad \left(\int \dfrac{dy}{\sqrt{a^2-y^2}}=\sin^{-1}\dfrac{y}{a}+C\right)$$

$$= \dfrac{1}{2}\sin^{-1}\left(\dfrac{8x-9}{9}\right)+C$$

Hence, the correct Answer is B.