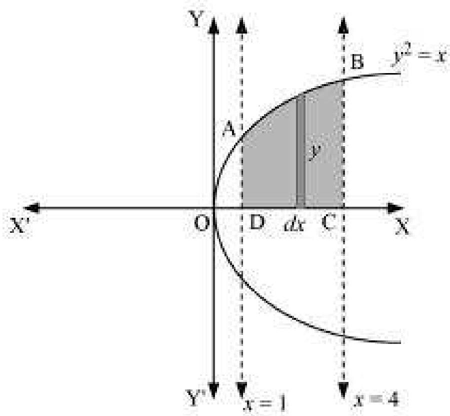
Exercise 8.1

Question 1:

Find the area of the region bounded by the curve $y^2 = x$ and the lines $x = 1$, $x = 4$ and the x-axis.

Answer



The area of the region bounded by the curve, $y^2 = x$, the lines, $x = 1$ and $x = 4$, and the x-axis is the area ABCD.

$$\text{Area of ABCD} = \int_1^4 y\,dx$$

$$= \int_1^4 \sqrt{x}\,dx$$

$$= \left[\frac{x^{\frac{3}{2}}}{\frac{3}{2}}\right]_1^4$$

$$= \frac{2}{3}\left[(4)^{\frac{3}{2}} - (1)^{\frac{3}{2}}\right]$$

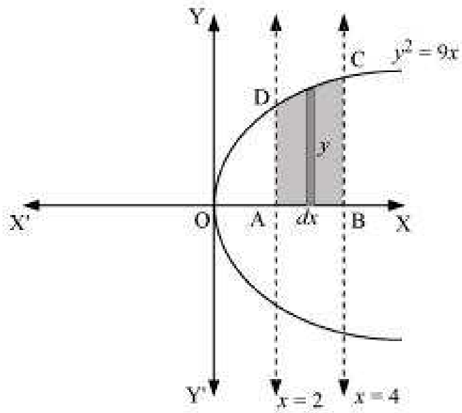
$$= \frac{2}{3}[8-1]$$

$$= \frac{14}{3} \text{ units}$$

Question 2:

Find the area of the region bounded by $y^2 = 9x$, $x = 2$, $x = 4$ and the x-axis in the first quadrant.

Answer



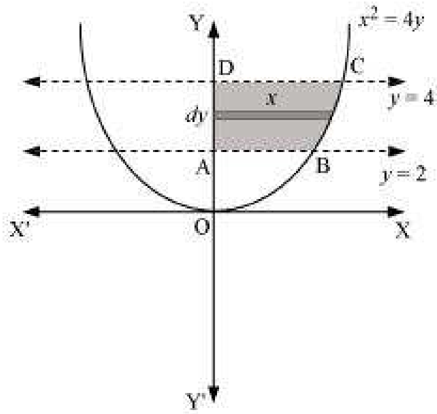
The area of the region bounded by the curve, $y^2 = 9x$, $x = 2$, and $x = 4$, and the x-axis is the area ABCD.

$$\text{Area of ABCD} = \int_2^4 y\,dx$$
$$= \int_2^4 3\sqrt{x}\,dx$$
$$= 3\left[\frac{x^{\frac{3}{2}}}{\frac{3}{2}}\right]_2^4$$
$$= 2\left[x^{\frac{3}{2}}\right]_2^4$$
$$= 2\left[(4)^{\frac{3}{2}} - (2)^{\frac{3}{2}}\right]$$
$$= 2\left[8 - 2\sqrt{2}\right]$$
$$= \left(16 - 4\sqrt{2}\right) \text{ units}$$

Question 3:

Find the area of the region bounded by $x^2 = 4y$, $y = 2$, $y = 4$ and the y-axis in the first quadrant.

Answer



The area of the region bounded by the curve, $x^2 = 4y$, $y = 2$, and $y = 4$, and the y-axis is the area ABCD.

$$\text{Area of ABCD} = \int_2^4 x\,dy$$

$$= \int_2^4 2\sqrt{y}\,dy$$

$$= 2\int_2^4 \sqrt{y}\,dy$$

$$= 2\left[\frac{y^{\frac{3}{2}}}{\frac{3}{2}}\right]_2^4$$

$$= \frac{4}{3}\left[(4)^{\frac{3}{2}} - (2)^{\frac{3}{2}}\right]$$

$$= \frac{4}{3}\left[8 - 2\sqrt{2}\right]$$

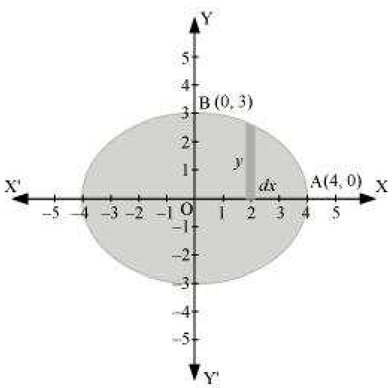
$$= \left(\frac{32 - 8\sqrt{2}}{3}\right) \text{ units}$$

Question 4:

Find the area of the region bounded by the ellipse $\frac{x^2}{16}+\frac{y^2}{9}=1$

Answer

The given equation of the ellipse, $\frac{x^2}{16}+\frac{y^2}{9}=1$, can be represented as



It can be observed that the ellipse is symmetrical about *x*-axis and *y*-axis.

∴ Area bounded by ellipse = 4 × Area of OAB

$$\begin{aligned}
\text{Area of OAB} &= \int_0^4 y\,dx \\
&= \int_0^4 3\sqrt{1-\frac{x^2}{16}}dx \\
&= \frac{3}{4}\int_0^4 \sqrt{16-x^2}\,dx \\
&= \frac{3}{4}\left[\frac{x}{2}\sqrt{16-x^2}+\frac{16}{2}\sin^{-1}\frac{x}{4}\right]_0^4 \\
&= \frac{3}{4}\left[2\sqrt{16-16}+8\sin^{-1}(1)-0-8\sin^{-1}(0)\right] \\
&= \frac{3}{4}\left[\frac{8\pi}{2}\right] \\
&= \frac{3}{4}[4\pi] \\
&= 3\pi
\end{aligned}$$

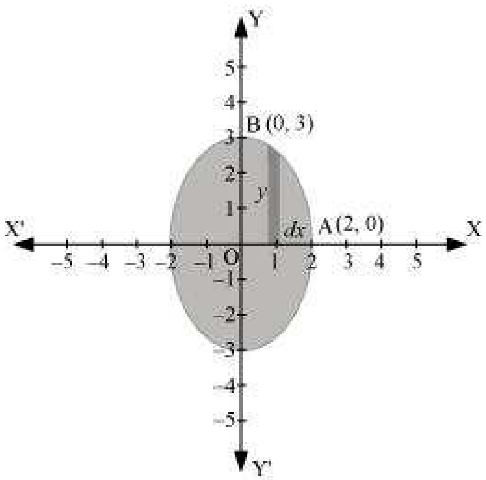
Therefore, area bounded by the ellipse = 4 × 3π = 12π units

Question 5:

Find the area of the region bounded by the ellipse $\frac{x^2}{4}+\frac{y^2}{9}=1$

Answer

The given equation of the ellipse can be represented as



$$\frac{x^2}{4}+\frac{y^2}{9}=1$$

$$\Rightarrow y=3\sqrt{1-\frac{x^2}{4}} \qquad ...(1)$$

It can be observed that the ellipse is symmetrical about *x*-axis and *y*-axis.

∴ Area bounded by ellipse = 4 × Area OAB

$$\therefore \text{Area of OAB} = \int_0^2 y\,dx$$

$$= \int_0^2 3\sqrt{1-\frac{x^2}{4}}\,dx \qquad \text{[Using (1)]}$$

$$= \frac{3}{2}\int_0^2 \sqrt{4-x^2}\,dx$$

$$= \frac{3}{2}\left[\frac{x}{2}\sqrt{4-x^2}+\frac{4}{2}\sin^{-}\frac{x}{2}\right]_0^2$$

$$= \frac{3}{2}\left[\frac{2\pi}{2}\right]$$

$$= \frac{3\pi}{2}$$

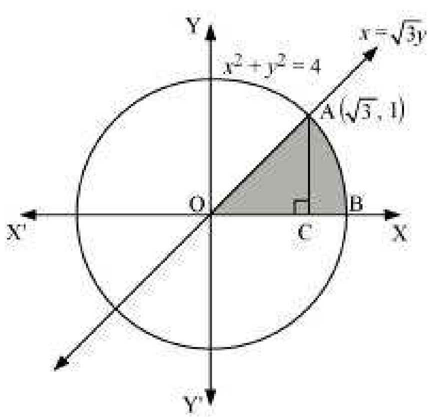
Therefore, area bounded by the ellipse = $4\times\frac{3\pi}{2} = 6\pi$ units

Question 6:

Find the area of the region in the first quadrant enclosed by *x*-axis, line $x=\sqrt{3}y$ and the circle $x^2+y^2=4$

Answer

The area of the region bounded by the circle, $x^2+y^2=4, x=\sqrt{3}y$, and the *x*-axis is the area OAB.



The point of intersection of the line and the circle in the first quadrant is $\left(\sqrt{3},1\right)$.

Area OAB = Area ΔOCA + Area ACB

Area of OAC $=\frac{1}{2}\times \mathrm{OC}\times \mathrm{AC}=\frac{1}{2}\times\sqrt{3}\times 1=\frac{\sqrt{3}}{2}$ ...(1)

Area of ABC $=\int_{\sqrt{3}}^{2} y\,dx$

$$=\int_{\sqrt{3}}^{2}\sqrt{4-x^2}\,dx$$

$$=\left[\frac{x}{2}\sqrt{4-x^2}+\frac{4}{2}\sin^{-1}\frac{x}{2}\right]_{\sqrt{3}}^{2}$$

$$=\left[2\times\frac{\pi}{2}-\frac{\sqrt{3}}{2}\sqrt{4-3}-2\sin^{-1}\left(\frac{\sqrt{3}}{2}\right)\right]$$

$$=\left[\pi-\frac{\sqrt{3}\pi}{2}-2\left(\frac{}{3}\right)\right]$$

$$=\left[\pi-\frac{\sqrt{3}}{2}-\frac{2\pi}{3}\right]$$

$$=\left[\frac{\pi}{3}-\frac{\sqrt{3}}{2}\right] \qquad ...(2)$$

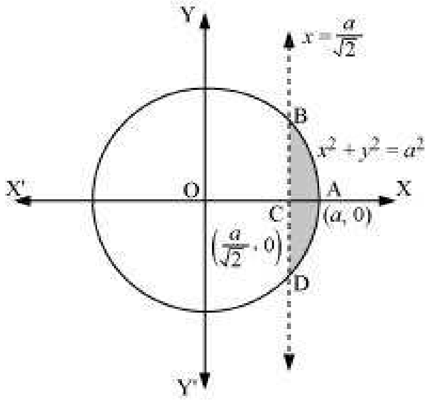
Therefore, area enclosed by *x*-axis, the line $x=\sqrt{3}y$, and the circle $x^2+y^2=4$ in the first quadrant = $\frac{\sqrt{3}\pi}{2}+\frac{}{3}-\frac{3\sqrt{\ }\pi}{2}=\frac{}{3}$ units

Question 7:

Find the area of the smaller part of the circle $x^2 + y^2 = a^2$ cut off by the line $x=\frac{a}{\sqrt{2}}$

Answer

The area of the smaller part of the circle, $x^2 + y^2 = a^2$, cut off by the line, $x = \frac{a}{\sqrt{2}}$, is the area ABCDA.



It can be observed that the area ABCD is symmetrical about x-axis.

∴ Area ABCD = 2 × Area ABC

$$Area\ of\ ABC = \int_{\frac{a}{\sqrt{2}}}^{a} y\,dx$$

$$= \int_{\frac{a}{\sqrt{2}}}^{a} \sqrt{a^2 - x^2}\,dx$$

$$= \left[\frac{x}{2}\sqrt{a^2 - x^2} + \frac{a^2}{2}\sin^{-1}\frac{x}{a}\right]_{\frac{a}{\sqrt{2}}}^{a}$$

$$= \left[\frac{a^2}{2}\left(\frac{\pi}{2}\right) - \frac{a}{2\sqrt{2}}\sqrt{a^2 - \frac{a^2}{2}} - \frac{a^2}{2}\sin^{-1}\left(\frac{1}{\sqrt{2}}\right)\right]$$

$$= \frac{a^2\pi}{4} - \frac{a}{2\sqrt{2}}\cdot\frac{a}{\sqrt{2}} - \frac{a^2}{2}\left(\frac{\pi}{4}\right)$$

$$= \frac{a^2\pi}{4} - \frac{a^2}{4} - \frac{a^2\pi}{8}$$

$$= \frac{a^2}{4}\left[\pi - 1 - \frac{\pi}{2}\right]$$

$$= \frac{a^2}{4}\left[\frac{\pi}{2} - 1\right]$$

$$\Rightarrow Area\ ABCD = 2\left[\frac{a^2}{4}\left(\frac{\pi}{2} - 1\right)\right] = \frac{a^2}{2}\left(\frac{\pi}{2} - 1\right)$$

Therefore, the area of smaller part of the circle, $x^2 + y^2 = a^2$, cut off by the line, $x = \frac{a}{\sqrt{2}}$, is $\frac{a^2}{2}\left(\frac{\pi}{2} - 1\right)$ units.

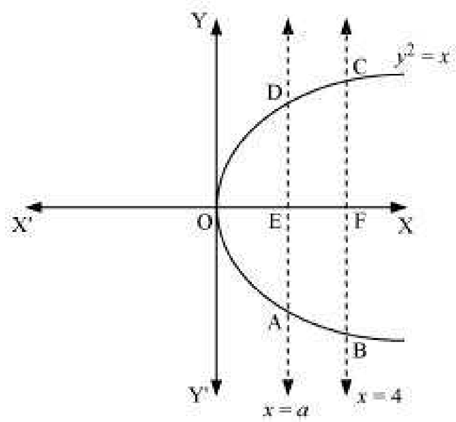
Question 8:

The area between $x = y^2$ and $x = 4$ is divided into two equal parts by the line $x = a$, find the value of a.

Answer

The line, $x = a$, divides the area bounded by the parabola and $x = 4$ into two equal parts.

∴ Area OAD = Area ABCD



It can be observed that the given area is symmetrical about *x*-axis.

⇒ Area OED = Area EFCD

$$Area\ OED = \int_0^a y\,dx$$
$$= \int_0^a \sqrt{x}\,dx$$
$$= \left[\frac{x^{\frac{3}{2}}}{\frac{3}{2}}\right]_0^a$$
$$= \frac{2}{3}(a)^{\frac{3}{2}} \qquad ...(1)$$

$$Area\ of\ EFCD = \int_0^4 \sqrt{x}dx$$
$$= \left[\frac{x^{\frac{3}{2}}}{\frac{3}{2}}\right]_0^4$$
$$= \frac{2}{3}\left[8 - a^{\frac{3}{2}}\right] \qquad ...(2)$$

From (1) and (2), we obtain

$$\frac{2}{3}(a)^{\frac{3}{2}} = \frac{2}{3}\left[8 - (a)^{\frac{3}{2}}\right]$$
$$\Rightarrow 2\cdot(a)^{\frac{3}{2}} = 8$$
$$\Rightarrow (a)^{\frac{3}{2}} = 4$$
$$\Rightarrow a = (4)^{\frac{2}{3}}$$

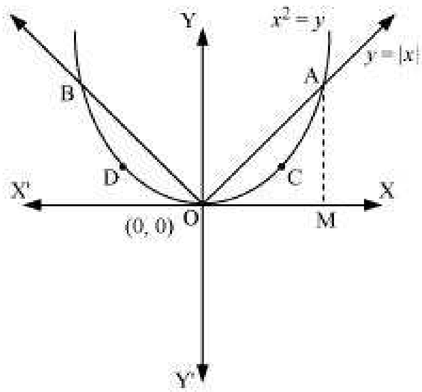
Therefore, the value of *a* is $(4)^{\frac{2}{3}}$.

Question 9:

Find the area of the region bounded by the parabola $y = x^2$ and $y = |x|$

Answer

The area bounded by the parabola, $x^2 = y$,and the line, $y = |x|$, can be represented as



The given area is symmetrical about y-axis.

∴ Area OACO = Area ODBO

The point of intersection of parabola, $x^2 = y$, and line, $y = x$, is A (1, 1).

Area of OACO = Area ΔOAB – Area OBACO

$$\therefore \text{Area of } \Delta\text{OAB} = \frac{1}{2}\times \text{OB}\times \text{AB} = \frac{1}{2}\times 1\times 1 = \frac{1}{2}$$

$$\text{Area of OBACO} = \int_0^1 y\,dx = \int_0^1 x^2\,dx = \left[\frac{x^3}{3}\right]_0^1 = \frac{1}{3}$$

⇒ Area of OACO = Area of ΔOAB – Area of OBACO

$$= \frac{1}{2} - \frac{1}{3}$$

$$= \frac{1}{6}$$

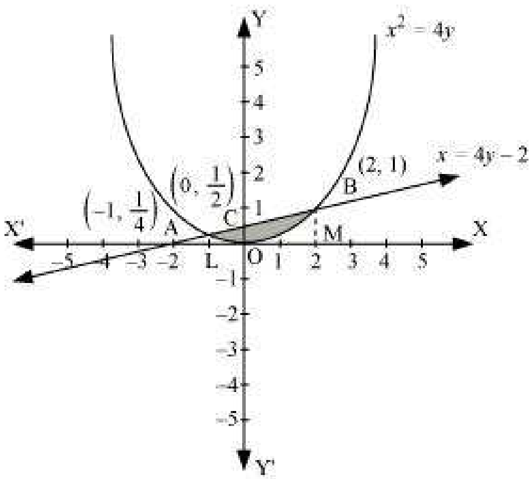
Therefore, required area = $2\left[\frac{1}{6}\right] = \frac{1}{3}$ units

Question 10:

Find the area bounded by the curve $x^2 = 4y$ and the line $x = 4y - 2$

Answer

The area bounded by the curve, $x^2 = 4y$, and line, $x = 4y - 2$, is represented by the shaded area OBAO.



Let A and B be the points of intersection of the line and parabola.

Coordinates of point A are $\left(-1, \frac{1}{4}\right)$.

Coordinates of point B are (2, 1).

We draw AL and BM perpendicular to x-axis.

It can be observed that,

Area OBAO = Area OBCO + Area OACO … (1)

Then, Area OBCO = Area OMBC – Area OMBO

$$= \int_0^2 \frac{x+2}{4}dx - \int_0^2 \frac{x^2}{4}dx$$

$$= \frac{1}{4}\left[\frac{x^2}{2}+2x\right]_0^2 - \frac{1}{4}\left[\frac{x^3}{3}\right]_0^2$$

$$= \frac{1}{4}[2+4] - \frac{1}{4}\left[\frac{8}{3}\right]$$

$$= \frac{3}{2} - \frac{2}{3}$$

$$= \frac{5}{6}$$

Similarly, Area OACO = Area OLAC – Area OLAO

$$= \int_{-1}^0 \frac{x+2}{4}dx - \int_{-1}^0 \frac{x^2}{4}dx$$

$$= \frac{1}{4}\left[\frac{x^2}{2}+2x\right]_{-1}^0 - \frac{1}{4}\left[\frac{x^3}{3}\right]_{-1}^0$$

$$= -\frac{1}{4}\left[\frac{(-1)^2}{2}+2(-1)\right] - \left[-\frac{1}{4}\left(\frac{(-1)^3}{3}\right)\right]$$

$$= -\frac{1}{4}\left[\frac{1}{2}-2\right] - \frac{1}{12}$$

$$= \frac{1}{2} - \frac{1}{8} - \frac{1}{12}$$

$$= \frac{7}{24}$$

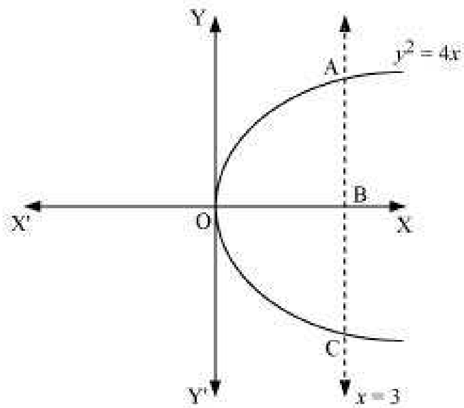
Therefore, required area = $\left(\frac{5}{6}+\frac{7}{24}\right) = \frac{9}{8}$ units

Question 11:

Find the area of the region bounded by the curve $y^2 = 4x$ and the line $x = 3$

Answer

The region bounded by the parabola, $y^2 = 4x$, and the line, $x = 3$, is the area OACO.



The area OACO is symmetrical about x-axis.

∴ Area of OACO = 2 (Area of OAB)

$$\text{Area OACO} = 2\left[\int_0^3 y\,dx\right]$$
$$= 2\int_0^3 2\sqrt{x}\,dx$$
$$= 4\left[\frac{x^{\frac{3}{2}}}{\frac{3}{2}}\right]_0^3$$
$$= \frac{8}{3}\left[(3)^{\frac{3}{2}}\right]$$
$$= 8\sqrt{3}$$

Therefore, the required area is $8\sqrt{3}$ units.

Question 12:

Area lying in the first quadrant and bounded by the circle $x^2 + y^2 = 4$ and the lines $x = 0$ and $x = 2$ is

A. π

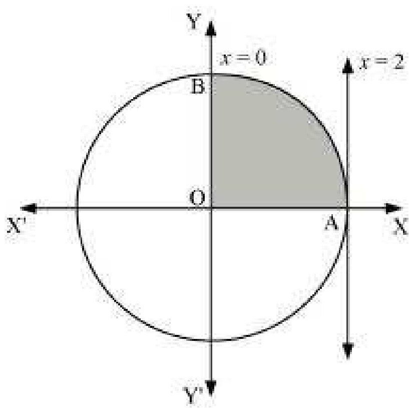
B. $\frac{\pi}{2}$

C. $\frac{\pi}{3}$

D. $\frac{\pi}{4}$

Answer

The area bounded by the circle and the lines, *x* = 0 and *x* = 2, in the first quadrant is represented as



$$\therefore \text{Area OAB} = \int_0^2 y\,dx$$

$$= \int_0^2 \sqrt{4-x^2}\,dx$$

$$= \left[\frac{x}{2}\sqrt{4-x^2} + \frac{4}{2}\sin^{-1}\frac{x}{2}\right]_0^2$$

$$= 2\left(\frac{\pi}{2}\right)$$

$$= \pi \text{ units}$$

Thus, the correct answer is A.

Question 13:

Area of the region bounded by the curve $y^2 = 4x$, y-axis and the line $y = 3$ is

A. 2

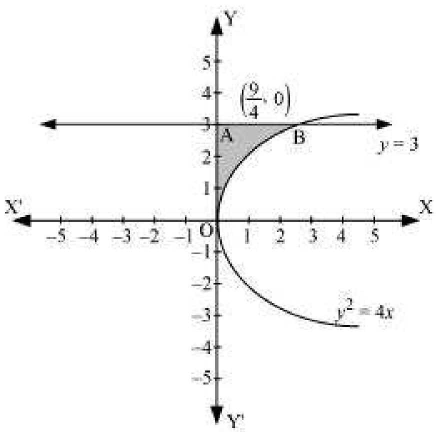
B. $\frac{9}{4}$

C. $\frac{9}{3}$

D. $\frac{9}{2}$

Answer

The area bounded by the curve, $y^2 = 4x$, y-axis, and $y = 3$ is represented as



$$\therefore \text{Area OAB} = \int_0^3 x\,dy$$

$$= \int_0^3 \frac{y^2}{4} dy$$

$$= \frac{1}{4}\left[\frac{y^3}{3}\right]_0^3$$

$$= \frac{1}{12}(27)$$

$$= \frac{9}{4} \text{ units}$$

Thus, the correct answer is B.

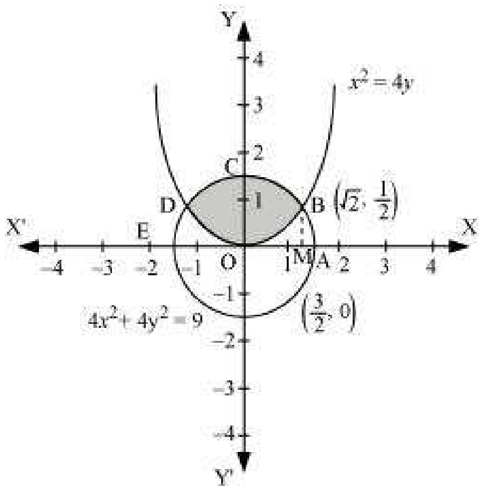
Exercise 8.2

Question 1:

Find the area of the circle $4x^2 + 4y^2 = 9$ which is interior to the parabola $x^2 = 4y$

Answer

The required area is represented by the shaded area OBCDO.



Solving the given equation of circle, $4x^2 + 4y^2 = 9$, and parabola, $x^2 = 4y$, we obtain the point of intersection as $\text{B}\left(\sqrt{2}, \frac{1}{2}\right) \text{ and D}\left(-\sqrt{2}, \frac{1}{2}\right)$.

It can be observed that the required area is symmetrical about y-axis.

∴ Area OBCDO = 2 × Area OBCO

We draw BM perpendicular to OA.

Therefore, the coordinates of M are $\left(\sqrt{2}, 0\right)$.

Therefore, Area OBCO = Area OMBCO – Area OMBO

$$= \int_0^{\sqrt{2}} \sqrt{\frac{(9-4x^2)}{4}}dx - \int_0^{\sqrt{2}} \sqrt{\frac{x^2}{4}}dx$$
$$= \frac{1}{2}\int_0^{\sqrt{2}} \sqrt{9-4x^2}\,dx - \frac{1}{4}\int_0^{\sqrt{2}} x^2 dx$$
$$= \frac{1}{4}\left[x\sqrt{9-4x^2} + \frac{9}{2}\sin^{-1}\frac{2x}{3}\right]_0^{\sqrt{2}} - \frac{1}{4}\left[\frac{x^3}{3}\right]_0^{\sqrt{2}}$$
$$= \frac{1}{4}\left[\sqrt{2}\sqrt{9-8} + \frac{9}{2}\sin^{-1}\frac{2\sqrt{2}}{3}\right] - \frac{1}{12}\left(\sqrt{2}\right)^3$$
$$= \frac{\sqrt{2}}{4} + \frac{9}{8}\sin^{-1}\frac{2\sqrt{2}}{3} - \frac{\sqrt{2}}{6}$$
$$= \frac{\sqrt{2}}{12} + \frac{9}{8}\sin^{-1}\frac{2\sqrt{2}}{3}$$
$$= \frac{1}{2}\left(\frac{\sqrt{2}}{6} + \frac{9}{4}\sin^{-1}\frac{2\sqrt{2}}{3}\right)$$

Therefore, the required area OBCDO is

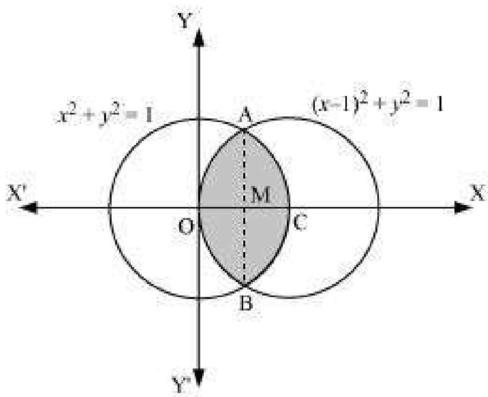
$$\left(2\times\frac{1}{2}\left[\frac{\sqrt{2}}{6} + \frac{9}{4}\sin^{-1}\frac{2\sqrt{2}}{3}\right]\right) = \left[\frac{\sqrt{2}}{6} + \frac{9}{4}\sin^{-1}\frac{2\sqrt{2}}{3}\right] \text{units}$$

Question 2:

Find the area bounded by curves $(x-1)^2 + y^2 = 1$ and $x^2 + y^2 = 1$

Answer

The area bounded by the curves, $(x-1)^2 + y^2 = 1$ and $x^2 + y^2 = 1$, is represented by the shaded area as



On solving the equations, $(x-1)^2+y^2=1$ and $x^2+y^2=1$, we obtain the point of intersection as A$\left(\frac{1}{2},\frac{\sqrt{3}}{2}\right)$and B$\left(\frac{1}{2},-\frac{\sqrt{3}}{2}\right)$.

It can be observed that the required area is symmetrical about x-axis.

∴ Area OBCAO = 2 × Area OCAO

We join AB, which intersects OC at M, such that AM is perpendicular to OC.

The coordinates of M are $\left(\frac{1}{2},0\right)$.

$\Rightarrow Area\ OCAO = Area\ OMAO + Area\ MCAM$

$$= \left[\int_0^{\frac{1}{2}} \sqrt{1-(x-1)^2}\,dx + \int_{\frac{1}{2}}^{1} \sqrt{1-x^2}\,dx\right]$$

$$= \left[\frac{x-1}{2}\sqrt{1-(x-1)^2} + \frac{1}{2}\sin^{-1}(x-1)\right]_0^{\frac{1}{2}} + \left[\frac{x}{2}\sqrt{1-x^2} + \frac{1}{2}\sin^{-1} x\right]_{\frac{1}{2}}^{1}$$

$$= \left[-\frac{1}{4}\sqrt{1-\left(-\frac{1}{2}\right)^2} + \frac{1}{2}\sin^{-1}\left(\frac{1}{2}-1\right) - \frac{1}{2}\sin^{-1}(-1)\right] +$$

$$\left[\frac{1}{2}\sin^{-1}(1) - \frac{1}{4}\sqrt{1-\left(\frac{1}{2}\right)^2} - \frac{1}{2}\sin^{-1}\left(\frac{1}{2}\right)\right]$$

$$= \left[-\frac{\sqrt{3}}{8} + \frac{1}{2}\left(-\frac{\pi}{6}\right) - \frac{1}{2}\left(-\frac{\pi}{2}\right)\right] + \left[\frac{1}{2}\left(\frac{\pi}{2}\right) - \frac{\sqrt{3}}{8} - \frac{1}{2}\left(\frac{\pi}{6}\right)\right]$$

$$= \left[-\frac{\sqrt{3}}{4} - \frac{\pi}{12} + \frac{\pi}{4} + \frac{\pi}{4} - \frac{\pi}{12}\right]$$

$$= \left[-\frac{\sqrt{3}}{4} - \frac{\pi}{6} + \frac{\pi}{2}\right]$$

$$= \left[\frac{2\pi}{6} - \frac{\sqrt{3}}{4}\right]$$

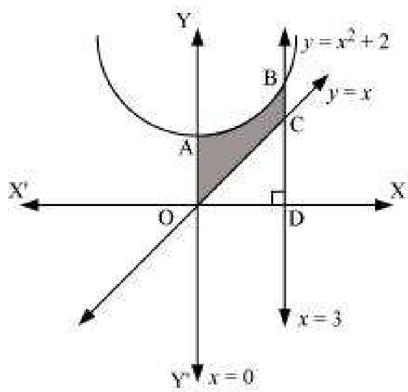
Therefore, required area OBCAO = $2\times\left(\frac{2\pi}{6} - \frac{\sqrt{3}}{4}\right) = \left(\frac{2\pi}{3} - \frac{\sqrt{3}}{2}\right)$ units

Question 3:

Find the area of the region bounded by the curves $y = x^2 + 2$, $y = x$, $x = 0$ and $x = 3$

Answer

The area bounded by the curves, $y = x^2 + 2$, $y = x$, $x = 0$, and $x = 3$, is represented by the shaded area OCBAO as



Then, Area OCBAO = Area ODBAO – Area ODCO

$$= \int_0^3 \left(x^2 + 2\right) dx - \int_0^3 x\, dx$$

$$= \left[\frac{x^3}{3} + 2x\right]_0^3 - \left[\frac{x^2}{2}\right]_0^3$$

$$= \left[9 + 6\right] - \left[\frac{9}{2}\right]$$

$$= 15 - \frac{9}{2}$$

$$= \frac{21}{2} \text{ units}$$

Question 4:

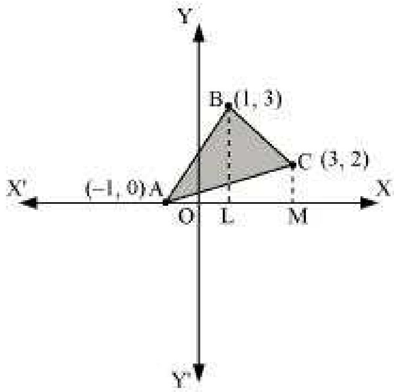
Using integration finds the area of the region bounded by the triangle whose vertices are (–1, 0), (1, 3) and (3, 2).

Answer

BL and CM are drawn perpendicular to *x*-axis.

It can be observed in the following figure that,

Area (ΔACB) = Area (ALBA) + Area (BLMCB) – Area (AMCA) … (1)



Equation of line segment AB is

$$y-0=\frac{3-0}{1+1}(x+1)$$

$$y=\frac{3}{2}(x+1)$$

$$\therefore \text{Area}(\text{ALBA})=\int_{-1}^{1}\frac{3}{2}(x+1)\,dx=\frac{3}{2}\left[\frac{x^2}{2}+x\right]_{-1}^{1}=\frac{3}{2}\left[\frac{1}{2}+1-\frac{1}{2}+1\right]=3 \text{ units}$$

Equation of line segment BC is

$$y-3=\frac{2-3}{3-1}(x-1)$$

$$y=\frac{1}{2}(-x+7)$$

$$\therefore \text{Area}(\text{BLMCB})=\int_{1}^{3}\frac{1}{2}(-x+7)\,dx=\frac{1}{2}\left[-\frac{x^2}{2}+7x\right]_{1}^{3}=\frac{1}{2}\left[-\frac{9}{2}+21+\frac{1}{2}-7\right]=5 \text{ units}$$

Equation of line segment AC is

$$y-0=\frac{2-0}{3+1}(x+1)$$

$$y=\frac{1}{2}(x+1)$$

$$\therefore \text{Area}(\text{AMCA})=\frac{1}{2}\int_{-1}^{3}(x+1)\,dx=\frac{1}{2}\left[\frac{x^2}{2}+x\right]_{-1}^{3}=\frac{1}{2}\left[\frac{9}{2}+3-\frac{1}{2}+1\right]=4 \text{ units}$$

Therefore, from equation (1), we obtain

Area (ΔABC) = (3 + 5 – 4) = 4 units

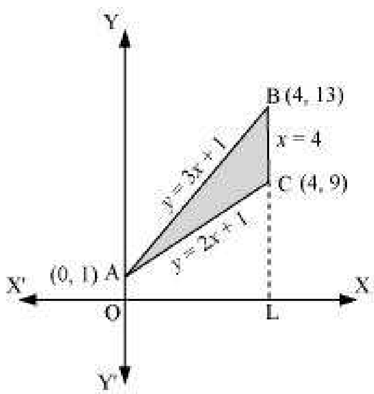
Question 5:

Using integration find the area of the triangular region whose sides have the equations $y = 2x + 1$, $y = 3x + 1$ and $x = 4$.

Answer

The equations of sides of the triangle are $y = 2x + 1$, $y = 3x + 1$, and $x = 4$.

On solving these equations, we obtain the vertices of triangle as A(0, 1), B(4, 13), and C (4, 9).



It can be observed that,

Area (ΔACB) = Area (OLBAO) –Area (OLCAO)

$$= \int_0^4 (3x+1)\,dx - \int_0^4 (2x+1)\,dx$$

$$= \left[\frac{3x^2}{2}+x\right]_0^4 - \left[\frac{2x^2}{2}+x\right]_0^4$$

$$= (24+4)-(16+4)$$

$$= 28-20$$

$$= 8 \text{ units}$$

Question 6:

Smaller area enclosed by the circle $x^2 + y^2 = 4$ and the line $x + y = 2$ is

A. 2 (π – 2)

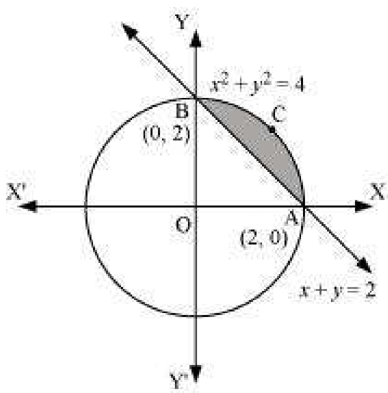
B. π – 2

C. 2π – 1

D. 2 (π + 2)

Answer

The smaller area enclosed by the circle, $x^2 + y^2 = 4$, and the line, $x + y = 2$, is represented by the shaded area ACBA as



It can be observed that,

Area ACBA = Area OACBO – Area (ΔOAB)

$$= \int_0^2 \sqrt{4-x^2}\,dx - \int_0^2 (2-x)\,dx$$

$$= \left[\frac{x}{2}\sqrt{4-x^2} + \frac{4}{2}\sin^{-1}\frac{x}{2}\right]_0^2 - \left[2x - \frac{x^2}{2}\right]_0^2$$

$$= \left[2\cdot\frac{\pi}{2}\right] - [4-2]$$

$$= (\pi - 2) \text{ units}$$

Thus, the correct answer is B.

Question 7:

Area lying between the curve $y^2 = 4x$ and $y = 2x$ is

A. $\frac{2}{3}$

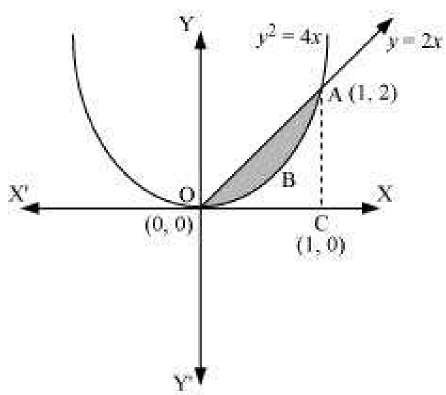
B. $\frac{1}{3}$

C. $\frac{1}{4}$

D. $\frac{3}{4}$

Answer

The area lying between the curve, $y^2 = 4x$ and $y = 2x$, is represented by the shaded area OBAO as



The points of intersection of these curves are O (0, 0) and A (1, 2).

We draw AC perpendicular to x-axis such that the coordinates of C are (1, 0).

∴ Area OBAO = Area (ΔOCA) – Area (OCABO)

$$= \int_0^1 2x\,dx - \int_0^1 2\sqrt{x}\,dx$$

$$= 2\left[\frac{x^2}{2}\right]_0^1 - 2\left[\frac{x^{\frac{3}{2}}}{\frac{3}{2}}\right]_0^1$$

$$= \left|1 - \frac{4}{3}\right|$$

$$= \left|-\frac{1}{3}\right|$$

$$= \frac{1}{3} \text{ units}$$

Thus, the correct answer is B.

Miscellaneous Solutions

Question 1:

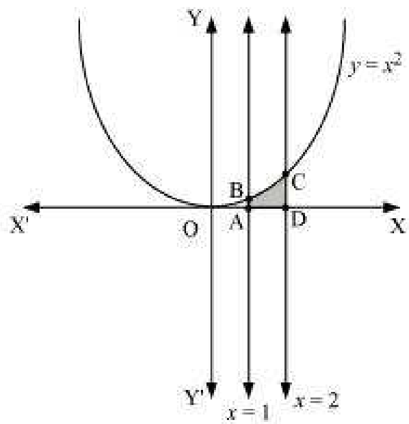
Find the area under the given curves and given lines:

(i) $y = x^2$, $x = 1$, $x = 2$ and x-axis

(ii) $y = x^4$, $x = 1$, $x = 5$ and x –axis

Answer

i. The required area is represented by the shaded area ADCBA as



$$\text{Area ADCBA} = \int_1^2 y\,dx$$

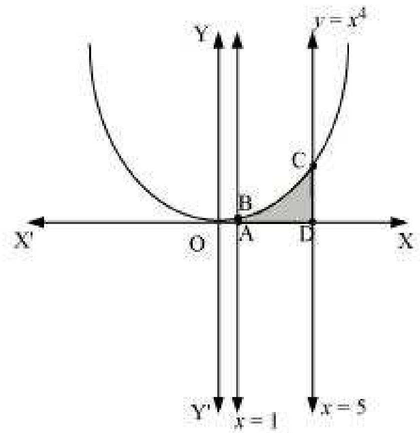
$$= \int_1^2 x^2\,dx$$

$$= \left[\frac{x^3}{3}\right]_1^2$$

$$= \frac{8}{3} - \frac{1}{3}$$

$$= \frac{7}{3} \text{ units}$$

ii. The required area is represented by the shaded area ADCBA as



$$\text{Area ADCBA} = \int_1^5 x^4 dx$$

$$= \left[\frac{x^5}{5}\right]_1^5$$

$$= \frac{(5)^5}{5} - \frac{1}{5}$$

$$= (5)^4 - \frac{1}{5}$$

$$= 625 - \frac{1}{5}$$

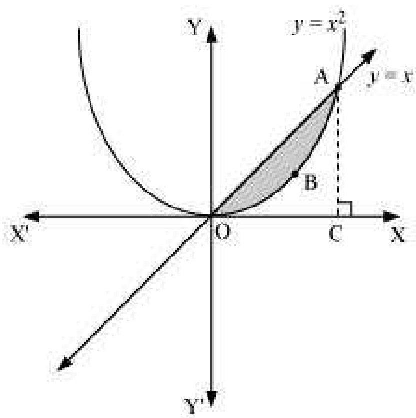
$$= 624.8 \text{ units}$$

Question 2:

Find the area between the curves $y = x$ and $y = x^2$

Answer

The required area is represented by the shaded area OBAO as



The points of intersection of the curves, $y = x$ and $y = x^2$, is A (1, 1).

We draw AC perpendicular to x-axis.

∴ Area (OBAO) = Area (ΔOCA) – Area (OCABO) … (1)

$$= \int_0^1 x\,dx - \int_0^1 x^2 dx$$

$$= \left[\frac{x^2}{2}\right]_0^1 - \left[\frac{x^3}{3}\right]_0^1$$

$$= \frac{1}{2} - \frac{1}{3}$$

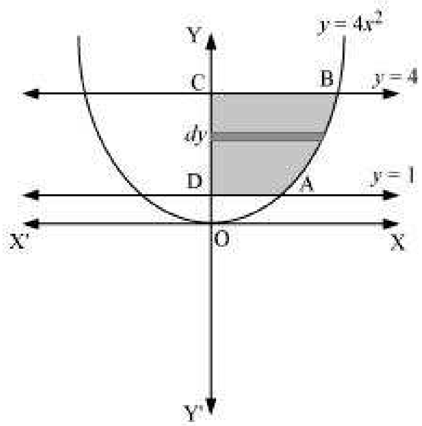
$$= \frac{1}{6} \text{ units}$$

Question 3:

Find the area of the region lying in the first quadrant and bounded by $y = 4x^2$, $x = 0$, $y = 1$ and $y = 4$

Answer

The area in the first quadrant bounded by $y = 4x^2$, $x = 0$, $y = 1$, and $y = 4$ is represented by the shaded area ABCDA as



$$\therefore \text{Area ABCD} = \int_1^4 x\,dx$$

$$= \int_1^4 \frac{\sqrt{y}}{2} dx$$

$$= \frac{1}{2}\left[\frac{y^{\frac{3}{2}}}{\frac{3}{2}}\right]_1^4$$

$$= \frac{1}{3}\left[(4)^{\frac{3}{2}} - 1\right]$$

$$= \frac{1}{3}[8-1]$$

$$= \frac{7}{3} \text{ units}$$

Question 4:

Sketch the graph of $y = |x+3|$ and evaluate $\int_{-6}^{0} |x+3| dx$

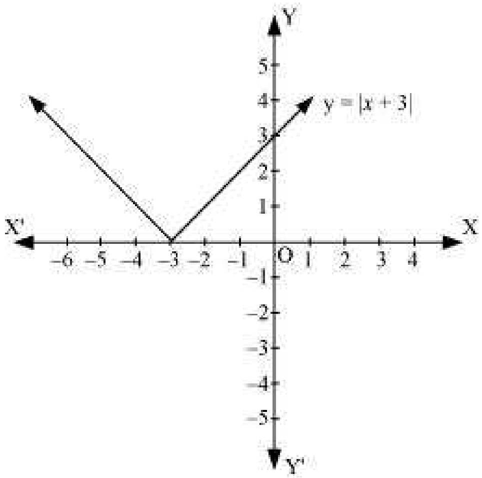
Answer

The given equation is $y=|x+3|$

The corresponding values of *x* and *y* are given in the following table.

x	− 6	− 5	− 4	− 3	− 2	− 1	0
y	3	2	1	0	1	2	3

On plotting these points, we obtain the graph of $y=|x+3|$ as follows.



It is known that, $(x+3)\le 0$ for $-6\le x\le -3$ and $(x+3)\ge 0$ for $-3\le x\le 0$

$$\therefore \int_{-6}^{0}|(x+3)|dx=-\int_{-6}^{-3}(x+3)dx+\int_{-3}^{0}(x+3)dx$$

$$=-\left[\frac{x^2}{2}+3x\right]_{-6}^{-3}+\left[\frac{x^2}{2}+3x\right]_{-3}^{0}$$

$$=-\left[\left(\frac{(-3)^2}{2}+3(-3)\right)-\left(\frac{(-6)^2}{2}+3(-6)\right)\right]+\left[0-\left(\frac{(-3)^2}{2}+3(-3)\right)\right]$$

$$=-\left[-\frac{9}{2}\right]-\left[-\frac{9}{2}\right]$$

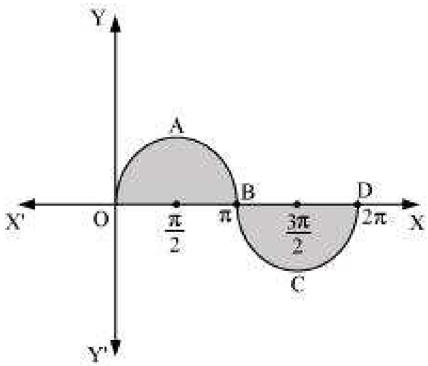
$$=9$$

Question 5:

Find the area bounded by the curve y = sin x between x = 0 and x = 2π

Answer

The graph of y = sin x can be drawn as



∴ Required area = Area OABO + Area BCDB

$$= \int_0^{\pi} \sin x\, dx + \left| \int_{\pi}^{2\pi} \sin x\, dx \right|$$

$$= \left[-\cos x\right]_0^{\pi} + \left| \left[-\cos x\right]_{\pi}^{2\pi} \right|$$

$$= \left[-\cos \pi + \cos 0\right] + \left| -\cos 2\pi + \cos \pi \right|$$

$$= 1 + 1 + \left| (-1-1) \right|$$

$$= 2 + \left| -2 \right|$$

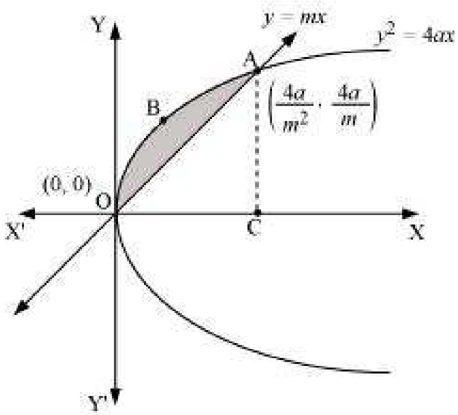
$$= 2 + 2 = 4 \text{ units}$$

Question 6:

Find the area enclosed between the parabola $y^2 = 4ax$ and the line $y = mx$

Answer

The area enclosed between the parabola, $y^2 = 4ax$, and the line, $y = mx$, is represented by the shaded area OABO as



The points of intersection of both the curves are (0, 0) and $\left(\frac{4a}{m^2}, \frac{4a}{m}\right)$.

We draw AC perpendicular to *x*-axis.

∴ Area OABO = Area OCABO – Area (ΔOCA)

$$= \int_0^{\frac{4a}{m^2}} 2\sqrt{ax}\, dx - \int_0^{\frac{4a}{m^2}} mx\, dx$$

$$= 2\sqrt{a}\left[\frac{x^{\frac{3}{2}}}{\frac{3}{2}}\right]_0^{\frac{4a}{m^2}} - m\left[\frac{x^2}{2}\right]_0^{\frac{4a}{m^2}}$$

$$= \frac{4}{3}\sqrt{a}\left(\frac{4a}{m^2}\right)^{\frac{3}{2}} - \frac{m}{2}\left[\left(\frac{4a}{m^2}\right)^2\right]$$

$$= \frac{32a^2}{3m^3} - \frac{m}{2}\left(\frac{16a^2}{m^4}\right)$$

$$= \frac{32a^2}{3m^3} - \frac{8a^2}{m^3}$$

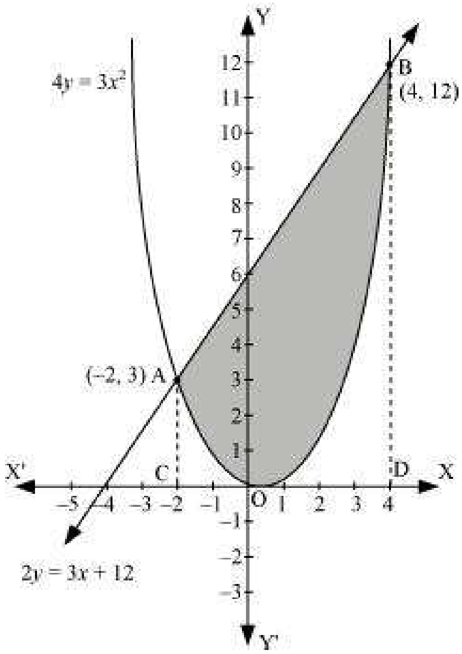
$$= \frac{8a^2}{3m^3} \text{ units}$$

Question 7:

Find the area enclosed by the parabola $4y = 3x^2$ and the line $2y = 3x + 12$

Answer

The area enclosed between the parabola, $4y = 3x^2$, and the line, $2y = 3x + 12$, is represented by the shaded area OBAO as



The points of intersection of the given curves are A (–2, 3) and (4, 12).

We draw AC and BD perpendicular to x-axis.

∴ Area OBAO = Area CDBA – (Area ODBO + Area OACO)

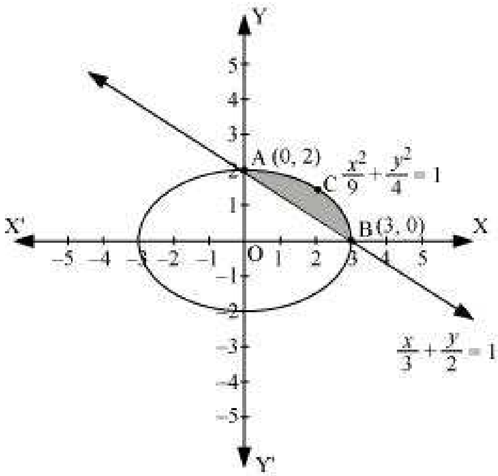
$$= \int_{-2}^{4} \frac{1}{2}(3x+12)\,dx - \int_{-2}^{4} \frac{3x^2}{4}\,dx$$
$$= \frac{1}{2}\left[\frac{3x^2}{2}+12x\right]_{-2}^{4} - \frac{3}{4}\left[\frac{x^3}{3}\right]_{-2}^{4}$$
$$= \frac{1}{2}[24+48-6+24] - \frac{1}{4}[64+8]$$
$$= \frac{1}{2}[90] - \frac{1}{4}[72]$$
$$= 45-18$$
$$= 27 \text{ units}$$

Question 8:

Find the area of the smaller region bounded by the ellipse $\frac{x^2}{9}+\frac{y^2}{4}=1$ and the line $\frac{x}{3}+\frac{y}{2}=1$

Answer

The area of the smaller region bounded by the ellipse, $\frac{x^2}{9}+\frac{y^2}{4}=1$, and the line, $\frac{x}{3}+\frac{y}{2}=1$, is represented by the shaded region BCAB as



∴ Area BCAB = Area (OBCAO) – Area (OBAO)

$$= \int_0^3 2\sqrt{1-\frac{x^2}{9}}\,dx - \int_0^3 2\left(1-\frac{x}{3}\right)dx$$

$$= \frac{2}{3}\left[\int_0^3 \sqrt{9-x^2}\,dx\right] - \frac{2}{3}\int_0^3 (3-x)\,dx$$

$$= \frac{2}{3}\left[\frac{x}{2}\sqrt{9-x^2}+\frac{9}{2}\sin^{-1}\frac{x}{3}\right]_0^3 - \frac{2}{3}\left[3x-\frac{x^2}{2}\right]_0^3$$

$$= \frac{2}{3}\left[\frac{9}{2}\left(\frac{\pi}{2}\right)\right] - \frac{2}{3}\left[9-\frac{9}{2}\right]$$

$$= \frac{2}{3}\left[\frac{9\pi}{4}-\frac{9}{2}\right]$$

$$= \frac{2}{3}\times\frac{9}{4}(\pi-2)$$

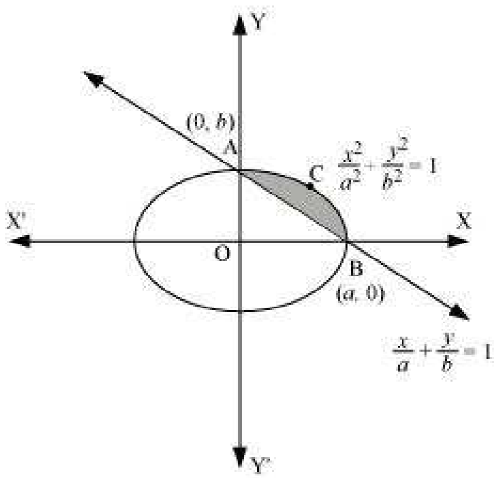
$$= \frac{3}{2}(\pi-2)\ \textit{units}$$

Question 9:

Find the area of the smaller region bounded by the ellipse $\frac{x^2}{a^2}+\frac{y^2}{b^2}=1$ and the line $\frac{x}{a}+\frac{y}{b}=1$

Answer

The area of the smaller region bounded by the ellipse, $\frac{x^2}{a^2}+\frac{y^2}{b^2}=1$, and the line, $\frac{x}{a}+\frac{y}{b}=1$, is represented by the shaded region BCAB as



∴ Area BCAB = Area (OBCAO) – Area (OBAO)

$$= \int_0^a b\sqrt{1-\frac{x^2}{a^2}}dx - \int_0^a b\left(1-\frac{x}{a}\right)dx$$

$$= \frac{b}{a}\int_0^a \sqrt{a^2-x^2}dx - \frac{b}{a}\int_0^a (a-x)dx$$

$$= \frac{b}{a}\left[\left\{\frac{x}{2}\sqrt{a^2-x^2}+\frac{a^2}{2}\sin^{-1}\frac{x}{a}\right\}_0^a - \left\{ax-\frac{x^2}{2}\right\}_0^a\right]$$

$$= \frac{b}{a}\left[\left\{\frac{a^2}{2}\left(\frac{\pi}{2}\right)\right\} - \left\{a^2-\frac{a^2}{2}\right\}\right]$$

$$= \frac{b}{a}\left[\frac{a^2\pi}{4}-\frac{a^2}{2}\right]$$

$$= \frac{ba^2}{2a}\left[\frac{\pi}{2}-1\right]$$

$$= \frac{ab}{2}\left[\frac{\pi}{2}-1\right]$$

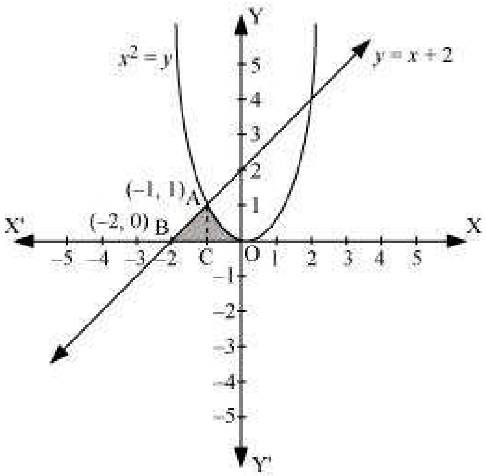
$$= \frac{ab}{4}(\pi-2)$$

Question 10:

Find the area of the region enclosed by the parabola $x^2 = y$, the line $y = x + 2$ and x-axis

Answer

The area of the region enclosed by the parabola, $x^2 = y$, the line, $y = x + 2$, and x-axis is represented by the shaded region OABCO as



The point of intersection of the parabola, $x^2 = y$, and the line, $y = x + 2$, is A (−1, 1).

∴ Area OABCO = Area (BCA) + Area COAC

$$= \int_{-2}^{-1}(x+2)dx + \int_{-1}^{0} x^2 dx$$

$$= \left[\frac{x^2}{2} + 2x\right]_{-2}^{-1} + \left[\frac{x^3}{3}\right]_{-1}^{0}$$

$$= \left[\frac{(-1)^2}{2} + 2(-1) - \frac{(-2)^2}{2} - 2(-2)\right] + \left[-\frac{(-1)^3}{3}\right]$$

$$= \left[\frac{1}{2} - 2 - 2 + 4 + \frac{1}{3}\right]$$

$$= \frac{5}{6} \text{ units}$$

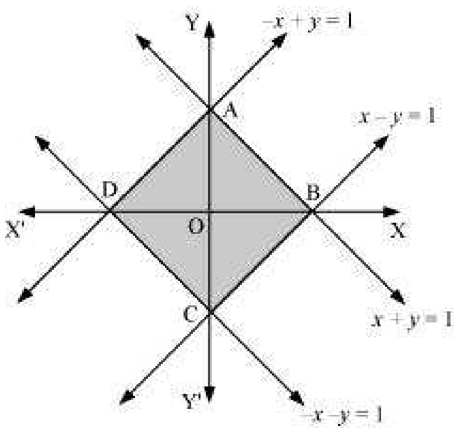
Question 11:

Using the method of integration find the area bounded by the curve $|x|+|y|=1$

[**Hint:** the required region is bounded by lines *x* + *y* = 1, *x* – *y* = 1, – *x* + *y* = 1 and – *x* – *y* = 11]

Answer

The area bounded by the curve, $|x|+|y|=1$, is represented by the shaded region ADCB as



The curve intersects the axes at points A (0, 1), B (1, 0), C (0, –1), and D (–1, 0).

It can be observed that the given curve is symmetrical about *x*-axis and *y*-axis.

∴ Area ADCB = 4 × Area OBAO

$$= 4\int_0^1 (1-x)\,dx$$

$$= 4\left(x-\frac{x^2}{2}\right)_0^1$$

$$= 4\left[1-\frac{1}{2}\right]$$

$$= 4\left(\frac{1}{2}\right)$$

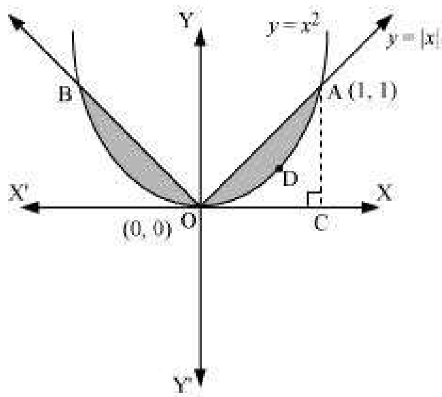
$$= 2 \text{ units}$$

Question 12:

Find the area bounded by curves $\{(x, y): y \geq x^2 \text{ and } y = |x|\}$

Answer

The area bounded by the curves, $\{(x, y): y \geq x^2 \text{ and } y = |x|\}$, is represented by the shaded region as



It can be observed that the required area is symmetrical about *y*-axis.

$$\text{Required area} = 2\left[\text{Area}(\text{OCAO}) - \text{Area}(\text{OCADO})\right]$$

$$= 2\left[\int_0^1 x\,dx - \int_0^1 x^2\,dx\right]$$

$$= 2\left[\left[\frac{x^2}{2}\right]_0^1 - \left[\frac{x^3}{3}\right]_0^1\right]$$

$$= 2\left[\frac{1}{2} - \frac{1}{3}\right]$$

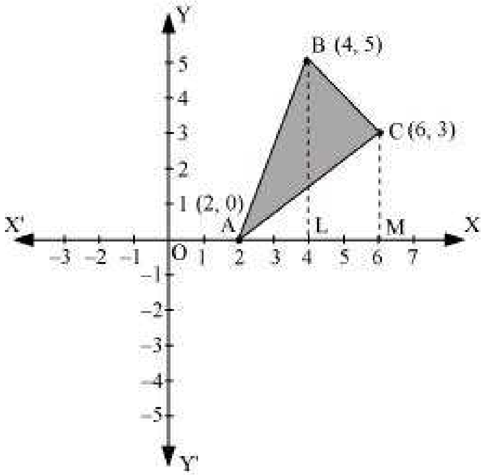
$$= 2\left[\frac{1}{6}\right] = \frac{1}{3} \text{ units}$$

Question 13:

Using the method of integration find the area of the triangle ABC, coordinates of whose vertices are A (2, 0), B (4, 5) and C (6, 3)

Answer

The vertices of ΔABC are A (2, 0), B (4, 5), and C (6, 3).



Equation of line segment AB is

$$y-0=\frac{5-0}{4-2}(x-2)$$

$$2y=5x-10$$

$$y=\frac{5}{2}(x-2) \qquad \ldots(1)$$

Equation of line segment BC is

$$y-5=\frac{3-5}{6-4}(x-4)$$

$$2y-10=-2x+8$$

$$2y=-2x+18$$

$$y=-x+9 \qquad \ldots(2)$$

Equation of line segment CA is

$$y-3=\frac{0-3}{2-6}(x-6)$$

$$-4y+12=-3x+18$$

$$4y=3x-6$$

$$y=\frac{3}{4}(x-2) \qquad \ldots(3)$$

Area (ΔABC) = Area (ABLA) + Area (BLMCB) – Area (ACMA)

$$= \int_2^4 \frac{5}{2}(x-2)\,dx + \int_4^6 (-x+9)\,dx - \int_2^6 \frac{3}{4}(x-2)\,dx$$

$$= \frac{5}{2}\left[\frac{x^2}{2} - 2x\right]_2^4 + \left[\frac{-x^2}{2} + 9x\right]_4^6 - \frac{3}{4}\left[\frac{x^2}{2} - 2x\right]_2^6$$

$$= \frac{5}{2}[8-8-2+4] + [-18+54+8-36] - \frac{3}{4}[18-12-2+4]$$

$$= 5 + 8 - \frac{3}{4}(8)$$

$$= 13 - 6$$

$$= 7 \text{ units}$$

Question 14:

Using the method of integration find the area of the region bounded by lines:

$2x + y = 4$, $3x - 2y = 6$ and $x - 3y + 5 = 0$

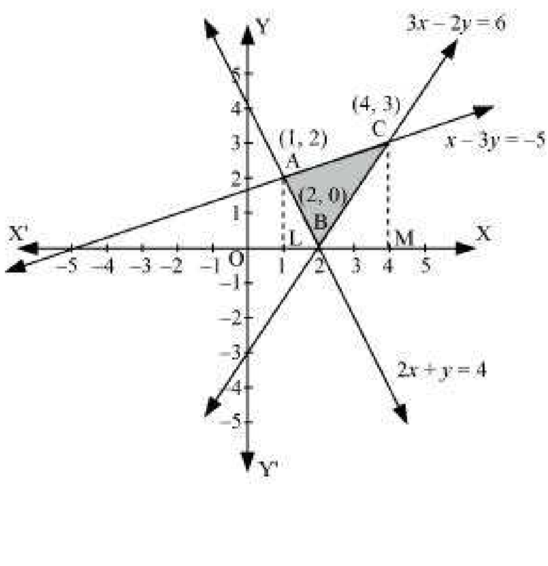
Answer

The given equations of lines are

$2x + y = 4$ … (1)

$3x - 2y = 6$ … (2)

And, $x - 3y + 5 = 0$ … (3)



The area of the region bounded by the lines is the area of ΔABC. AL and CM are the perpendiculars on *x*-axis.

Area (ΔABC) = Area (ALMCA) – Area (ALB) – Area (CMB)

$$= \int_1^4 \left(\frac{x+5}{3}\right) dx - \int_1^2 (4-2x) dx - \int_2^4 \left(\frac{3x-6}{2}\right) dx$$

$$= \frac{1}{3}\left[\frac{x^2}{2}+5x\right]_1^4 - \left[4x-x^2\right]_1^2 - \frac{1}{2}\left[\frac{3x^2}{2}-6x\right]_2^4$$

$$= \frac{1}{3}\left[8+20-\frac{1}{2}-5\right] - [8-4-4+1] - \frac{1}{2}[24-24-6+12]$$

$$= \left(\frac{1}{3}\times\frac{45}{2}\right) - (1) - \frac{1}{2}(6)$$

$$= \frac{15}{2} - 1 - 3$$

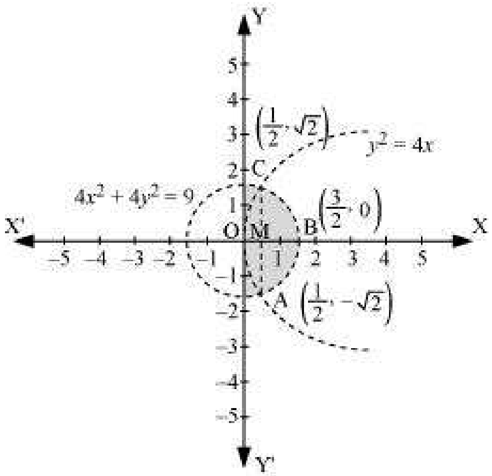
$$= \frac{15}{2} - 4 = \frac{15-8}{2} = \frac{7}{2} \text{ units}$$

Question 15:

Find the area of the region $\{(x, y) : y^2 \le 4x, 4x^2 + 4y^2 \le 9\}$

Answer

The area bounded by the curves, $\{(x, y) : y^2 \le 4x, 4x^2 + 4y^2 \le 9\}$, is represented as



The points of intersection of both the curves are $\left(\frac{1}{2}, \sqrt{2}\right)$ and $\left(\frac{1}{2}, -\sqrt{2}\right)$.

The required area is given by OABCO.

It can be observed that area OABCO is symmetrical about x-axis.

∴ Area OABCO = 2 × Area OBC

Area OBCO = Area OMC + Area MBC

$$= \int_0^{\frac{1}{2}} 2\sqrt{x}\, dx + \int_{\frac{1}{2}}^{\frac{3}{2}} \frac{1}{2}\sqrt{9-4x^2}\, dx$$

$$= \int_0^{\frac{1}{2}} 2\sqrt{x}\, dx + \int_{\frac{1}{2}}^{\frac{3}{2}} \frac{1}{2}\sqrt{(3)^2-(2x)^2}\, dx$$

Question 16:

Area bounded by the curve $y = x^3$, the x-axis and the ordinates $x = -2$ and $x = 1$ is

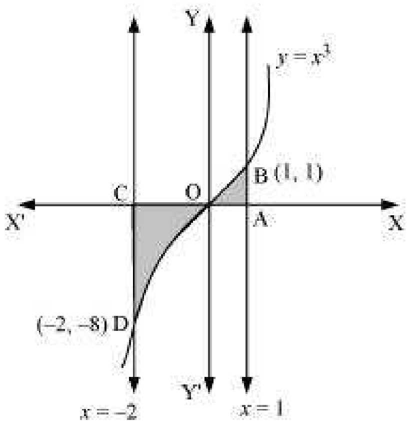
A. – 9

B. $-\frac{15}{4}$

C. $\frac{15}{4}$

D. $\frac{17}{4}$

Answer



$$\text{Required area} = \int_{-2}^{1} y\,dx$$

$$= \int_{-2}^{1} x^3\,dx$$

$$= \left[\frac{x^4}{4}\right]_{-2}^{1}$$

$$= \left[\frac{1}{4} - \frac{(-2)^4}{4}\right]$$

$$= \left(\frac{1}{4} - 4\right) = -\frac{15}{4} \text{ units}$$

Thus, the correct answer is B.

Question 17:

The area bounded by the curve $y = x|x|$, x-axis and the ordinates $x = -1$ and $x = 1$ is given by

[Hint: $y = x^2$ if $x > 0$ and $y = -x^2$ if $x < 0$]

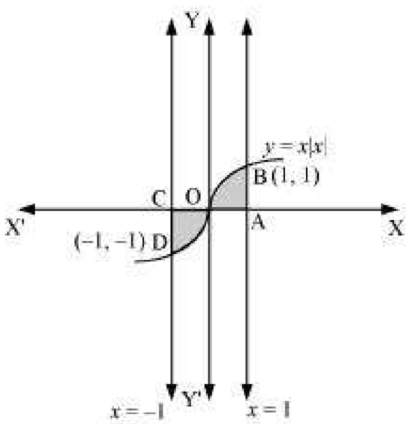
A. 0

B. $\frac{1}{3}$

C. $\frac{2}{3}$

D. $\frac{4}{3}$

Answer



Required area $= \int_{-1}^{1} y\,dx$

$= \int_{-1}^{1} x|x|dx$

$= \int_{-1}^{0} x^2 dx + \int_{0}^{1} x^2 dx$

$= \left[\frac{x^3}{3}\right]_{-1}^{0} + \left[\frac{x^3}{3}\right]_{0}^{1}$

$= -\left(-\frac{1}{3}\right) + \frac{1}{3}$

$= \frac{2}{3}$ units

Thus, the correct answer is C.

Question 18:

The area of the circle $x^2 + y^2 = 16$ exterior to the parabola $y^2 = 6x$ is

A. $\frac{4}{3}\left(4\pi - \sqrt{3}\right)$

B. $\frac{4}{3}\left(4\pi + \sqrt{3}\right)$

C. $\frac{4}{3}\left(8\pi - \sqrt{3}\right)$

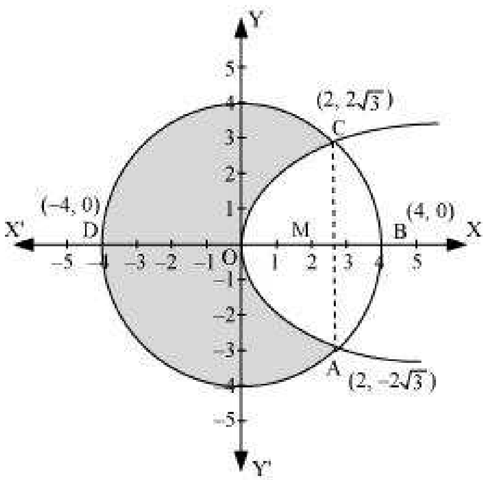
D. $\frac{4}{3}\left(4\pi + \sqrt{3}\right)$

Answer

The given equations are

$x^2 + y^2 = 16$ … (1)

$y^2 = 6x$ … (2)



Area bounded by the circle and parabola

$$= 2\left[\text{Area}(\text{OADO}) + \text{Area}(\text{ADBA})\right]$$

$$= 2\left[\int_0^2 \sqrt{16x}\,dx + \int_2^4 \sqrt{16 - x^2}\,dx\right]$$

$$= 2\left[\sqrt{6}\left\{\frac{x^{\frac{3}{2}}}{\frac{3}{2}}\right\}_0^2\right] + 2\left[\frac{x}{2}\sqrt{16 - x^2} + \frac{16}{2}\sin^{-1}\frac{x}{4}\right]_2^4$$

$$= 2\sqrt{6} \times \frac{2}{3}\left[x^{\frac{3}{2}}\right]_0^2 + 2\left[8 \cdot \frac{\pi}{2} - \sqrt{16 - 4} - 8\sin^{-1}\left(\frac{1}{2}\right)\right]$$

$$= \frac{4\sqrt{6}}{3}\left(2\sqrt{2}\right) + 2\left[4\pi - \sqrt{12} - 8\frac{\pi}{6}\right]$$

$$= \frac{16\sqrt{3}}{3} + 8\pi - 4\sqrt{3} - \frac{8}{3}\pi$$

$$= \frac{4}{3}\left[4\sqrt{3} + 6\pi - 3\sqrt{3} - 2\pi\right]$$

$$= \frac{4}{3}\left[\sqrt{3} + 4\pi\right]$$

$$= \frac{4}{3}\left[4\pi + \sqrt{3}\right] \text{ units}$$

Area of circle = π $(r)^2$

= π $(4)^2$

= 16π units

$$\therefore \text{Required area} = 16\pi - \frac{4}{3}\left[4\pi + \sqrt{3}\right]$$

$$= \frac{4}{3}\left[4 \times 3\pi - 4\pi - \sqrt{3}\right]$$

$$= \frac{4}{3}\left(8\pi - \sqrt{3}\right) \text{ units}$$

Thus, the correct answer is C.

Question 19:

The area bounded by the *y*-axis, *y* = cos *x* and *y* = sin *x* when $0 \le x \le \frac{\pi}{2}$

A. $2\left(\sqrt{2}-1\right)$

B. $\sqrt{2}-1$

C. $\sqrt{2}+1$

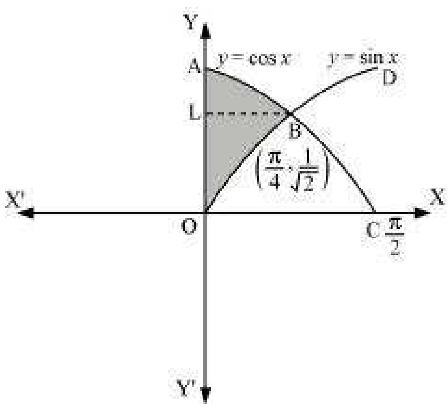
D. $\sqrt{2}$

Answer

The given equations are

y = cos *x* … (1)

And, *y* = sin *x* … (2)



Required area = Area (ABLA) + area (OBLO)

$$= \int_{\frac{1}{\sqrt{2}}}^{1} x dy + \int_{0}^{\frac{1}{\sqrt{2}}} x dy$$

$$= \int_{\frac{1}{\sqrt{2}}}^{1} \cos^{-1} y dy + \int_{0}^{\frac{1}{\sqrt{2}}} \sin^{-1} x dy$$

Integrating by parts, we obtain

$$=\left[y\cos^{-1}y-\sqrt{1-y^2}\right]_{\frac{1}{\sqrt{2}}}^{1}+\left[x\sin^{-1}x+\sqrt{1-x^2}\right]_0^{\frac{1}{\sqrt{2}}}$$

$$=\left[\cos^{-1}(1)-\frac{1}{\sqrt{2}}\cos^{-1}\left(\frac{1}{\sqrt{2}}\right)+\sqrt{1-\frac{1}{2}}\right]+\left[\frac{1}{\sqrt{2}}\sin^{-1}\left(\frac{1}{\sqrt{2}}\right)+\sqrt{1-\frac{1}{2}}-1\right]$$

$$=\frac{-\pi}{4\sqrt{2}}+\frac{1}{\sqrt{2}}+\frac{\pi}{4\sqrt{2}}+\frac{1}{\sqrt{2}}-1$$

$$=\frac{2}{\sqrt{2}}-1$$

$$=\sqrt{2}-1 \text{ units}$$

Thus, the correct answer is B.

Put $2x=t\Rightarrow dx=\dfrac{dt}{2}$

When $x=\dfrac{3}{2}$, $t=3$ and when $x=\dfrac{1}{2}$, $t=1$

$$=\int_0^{\frac{1}{2}}2\sqrt{x}\,dx+\frac{1}{4}\int_1^3\sqrt{(3)^2-(t)^2}\,dt$$

$$=2\left[\frac{x^{\frac{3}{2}}}{\frac{3}{2}}\right]_0^{\frac{1}{2}}+\frac{1}{4}\left[\frac{t}{2}\sqrt{9-t^2}+\frac{9}{2}\sin^{-1}\left(\frac{t}{3}\right)\right]_1^3$$

$$=2\left[\frac{2}{3}\left(\frac{1}{2}\right)^{\frac{3}{2}}\right]+\frac{1}{4}\left[\left\{\frac{3}{2}\sqrt{9-(3)^2}+\frac{9}{2}\sin^{-1}\left(\frac{3}{3}\right)\right\}-\left\{\frac{1}{2}\sqrt{9-(1)^2}+\frac{9}{2}\sin^{-1}\left(\frac{1}{3}\right)\right\}\right]$$

$$=\frac{2}{3\sqrt{2}}+\frac{1}{4}\left[\left\{0+\frac{9}{2}\sin^{-1}(1)\right\}-\left\{\frac{1}{2}\sqrt{8}+\frac{9}{2}\sin^{-1}\left(\frac{1}{3}\right)\right\}\right]$$

$$=\frac{\sqrt{2}}{3}+\frac{1}{4}\left[\frac{9\pi}{4}-\sqrt{2}-\frac{9}{2}\sin^{-1}\left(\frac{1}{3}\right)\right]$$

$$=\frac{\sqrt{2}}{3}+\frac{9\pi}{16}-\frac{\sqrt{2}}{4}-\frac{9}{8}\sin^{-1}\left(\frac{1}{3}\right)$$

$$=\frac{9\pi}{16}-\frac{9}{8}\sin^{-1}\left(\frac{1}{3}\right)+\frac{\sqrt{2}}{12}$$

Therefore, the required area is $\left[2\times\left(\frac{9\pi}{16}-\frac{9}{8}\sin^{-1}\left(\frac{1}{3}\right)+\frac{\sqrt{2}}{12}\right)\right]=\frac{9\pi}{8}-\frac{9}{4}\sin^{-1}\left(\frac{1}{3}\right)+\frac{1}{3\sqrt{2}}$ units