Exercise 8.2

Question 1:

Find the coefficient of x^5 in $(x + 3)^8$

Answer

It is known that $(r + 1)^{th}$ term, (T_{r+1}), in the binomial expansion of $(a + b)^n$ is given by

$$T_{r+1} = {}^nC_r a^{n-r} b^r.$$

Assuming that x^5 occurs in the $(r + 1)^{th}$ term of the expansion $(x + 3)^8$, we obtain

$$T_{r+1} = {}^8C_r (x)^{8-r} (3)^r$$

Comparing the indices of x in x^5 and in T_{r+1}, we obtain

$r = 3$

Thus, the coefficient of x^5 is ${}^8C_3 (3)^3 = \frac{8!}{3!5!} \times 3^3 = \frac{8 \cdot 7 \cdot 6 \cdot 5!}{3 \cdot 2.5!} \cdot 3^3 = 1512$

Question 2:

Find the coefficient of a^5b^7 in $(a - 2b)^{12}$

Answer

It is known that $(r + 1)^{th}$ term, (T_{r+1}), in the binomial expansion of $(a + b)^n$ is given by

$$T_{r+1} = {}^nC_r a^{n-r} b^r.$$

Assuming that a^5b^7 occurs in the $(r + 1)^{th}$ term of the expansion $(a - 2b)^{12}$, we obtain

$$T_{r+1} = {}^{12}C_r (a)^{12-r} (-2b)^r = {}^{12}C_r (-2)^r (a)^{12-r} (b)^r$$

Comparing the indices of a and b in a^5 b^7 and in T_{r+1}, we obtain

$r = 7$

Thus, the coefficient of a^5b^7 is

$${}^{12}C_7 (-2)^7 = -\frac{12!}{7!5!} \cdot 2^7 = -\frac{12 \cdot 11 \cdot 10 \cdot 9 \cdot 8.7!}{5 \cdot 4 \cdot 3 \cdot 2.7!} \cdot 2^7 = -(792)(128) = -101376$$

Question 3:

Write the general term in the expansion of $(x^2 - y)^6$

Answer

It is known that the general term T_{r+1} {which is the $(r + 1)^{th}$ term} in the binomial expansion of $(a + b)^n$ is given by $T_{r+1} = {}^nC_r a^{n-r} b^r$.

Thus, the general term in the expansion of $(x^2 – y^6)$ is

$$T_{r+1} = {}^6C_r \left(x^2\right)^{6-r} (-y)^r = (-1)^r \, {}^6C_r . x^{12-2r} . y^r$$

Question 4:

Write the general term in the expansion of $(x^2 – yx)^{12}$, $x \neq 0$

Answer

It is known that the general term T_{r+1} {which is the $(r + 1)^{th}$ term} in the binomial expansion of $(a + b)^n$ is given by $T_{r+1} = {}^nC_r a^{n-r} b^r$.

Thus, the general term in the expansion of$(x^2 – yx)^{12}$ is

$$T_{r+1} = {}^{12}C_r \left(x^2\right)^{12-r} (-yx)^r = (-1)^r \, {}^{12}C_r . x^{24-2r} . y^r . x^r = (-1)^r \, {}^{12}C_r . x^{24-r} . y^r$$

Question 5:

Find the 4^{th} term in the expansion of $(x – 2y)^{12}$.

Answer

It is known that $(r + 1)^{th}$ term, (T_{r+1}), in the binomial expansion of $(a + b)^n$ is given by

$T_{r+1} = {}^nC_r a^{n-r} b^r$.

Thus, the 4^{th} term in the expansion of $(x – 2y)^{12}$ is

$$T_4 = T_{3+1} = {}^{12}C_3 (x)^{12-3} (-2y)^3 = (-1)^3 \cdot \frac{12!}{3!9!} \cdot x^9 \cdot (2)^3 \cdot y^3 = -\frac{12 \cdot 11 \cdot 10}{3 \cdot 2} \cdot (2)^3 x^9 y^3 = -1760 x^9 y^3$$

Question 6:

Find the 13^{th} term in the expansion of $\left(9x - \frac{1}{3\sqrt{x}}\right)^{18}, \ x \neq 0$.

Answer

It is known that $(r + 1)^{th}$ term, (T_{r+1}), in the binomial expansion of $(a + b)^n$ is given by

$T_{r+1} = {}^nC_r a^{n-r} b^r$.

Thus, 13th term in the expansion of $\left(9x-\frac{1}{3\sqrt{x}}\right)^{18}$ is

$$T_{13}=T_{12+1}={}^{18}C_{12}(9x)^{18-12}\left(-\frac{1}{3\sqrt{x}}\right)^{12}$$

$$=(-1)^{12}\frac{18!}{12!6!}(9)^6(x)^6\left(\frac{1}{3}\right)^{12}\left(\frac{1}{\sqrt{x}}\right)^{12}$$

$$=\frac{18\cdot17\cdot16\cdot15\cdot14\cdot13.12!}{12!.6\cdot5\cdot4\cdot3\cdot2}\cdot x^6.\left(\frac{1}{x^6}\right).3^{12}\left(\frac{1}{3^{12}}\right) \qquad \left[9^6=\left(3^2\right)^6=3^{12}\right]$$

$$=18564$$

Question 7:

Find the middle terms in the expansions of $\left(3-\frac{x^3}{6}\right)^7$

Answer

It is known that in the expansion of $(a+b)^n$, if n is odd, then there are two middle terms, namely, $\left(\frac{n+1}{2}\right)^{th}$ term and $\left(\frac{n+1}{2}+1\right)^{th}$ term.

Therefore, the middle terms in the expansion of $\left(3-\frac{x^3}{6}\right)^7$ are $\left(\frac{7+1}{2}\right)^{th}=4^{th}$ term and $\left(\frac{7+1}{2}+1\right)^{th}=5^{th}$ term

$$T_4=T_{3+1}={}^7C_3(3)^{7-3}\left(-\frac{x^3}{6}\right)^3=(-1)^3\frac{7!}{3!4!}.3^4\cdot\frac{x^9}{6^3}$$

$$=-\frac{7\cdot6\cdot5.4!}{3\cdot2.4!}\cdot3^4\cdot\frac{1}{2^3\cdot3^3}\cdot x^9=-\frac{105}{8}x^9$$

$$T_5=T_{4+1}={}^7C_4(3)^{7-4}\left(-\frac{x^3}{6}\right)^4=(-1)^4\frac{7!}{4!3!}(3)^3\cdot\frac{x^{12}}{6^4}$$

$$=\frac{7\cdot6\cdot5.4!}{4!.3\cdot2}\cdot\frac{3^3}{2^4\cdot3^4}\cdot x^{12}=\frac{35}{48}x^{12}$$

Thus, the middle terms in the expansion of $\left(3-\frac{x^3}{6}\right)^7$ are $-\frac{105}{8}x^9$ and $\frac{35}{48}x^{12}$.

Question 8:

Find the middle terms in the expansions of $\left(\frac{x}{3}+9y\right)^{10}$

Answer

It is known that in the expansion $(a + b)^n$, if n is even, then the middle term is $\left(\frac{n}{2}+1\right)^{th}$ term.

Therefore, the middle term in the expansion of $\left(\frac{x}{3}+9y\right)^{10}$ is $\left(\frac{10}{2}+1\right)^{th} = 6^{th}$ term

$$T_6 = T_{5+1} = {}^{10}C_5\left(\frac{x}{3}\right)^{10-5}(9y)^5 = \frac{10!}{5!5!}\cdot\frac{x^5}{3^5}\cdot 9^5\cdot y^5$$
$$= \frac{10.9\cdot 8\cdot 7\cdot 6.5!}{5\cdot 4\cdot 3\cdot 2.5!}\cdot\frac{1}{3^5}\cdot 3^{10}\cdot x^5y^5 \qquad \left[9^5 = \left(3^2\right)^5 = 3^{10}\right]$$
$$= 252\times 3^5\cdot x^5\cdot y^5 = 61236x^5y^5$$

Thus, the middle term in the expansion of $\left(\frac{x}{3}+9y\right)^{10}$ is 61236 x^5y^5.

Question 9:

In the expansion of $(1 + a)^{m + n}$, prove that coefficients of a^m and a^n are equal.

Answer

It is known that $(r + 1)^{th}$ term, (T_{r+1}), in the binomial expansion of $(a + b)^n$ is given by

$T_{r+1} = {}^nC_r a^{n-r} b^r$.

Assuming that a^m occurs in the $(r + 1)^{th}$ term of the expansion $(1 + a)^{m + n}$, we obtain

$$T_{r+1} = {}^{m+n}C_r(1)^{m+n-r}(a)^r = {}^{m+n}C_r a^r$$

Comparing the indices of a in a^m and in $T_{r + 1}$, we obtain

$r = m$

Therefore, the coefficient of a^m is

$$^{m+n}C_m = \frac{(m+n)!}{m!(m+n-m)!} = \frac{(m+n)!}{m!n!} \qquad ...(1)$$

Assuming that a^n occurs in the $(k + 1)^{th}$ term of the expansion $(1 + a)^{m+n}$, we obtain

$$T_{k+1} = {}^{m+n}C_k (1)^{m+n-k} (a)^k = {}^{m+n}C_k (a)^k$$

Comparing the indices of a in a^n and in T_{k+1}, we obtain

$k = n$

Therefore, the coefficient of a^n is

$$^{m+n}C_n = \frac{(m+n)!}{n!(m+n-n)!} = \frac{(m+n)!}{n!m!} \qquad ...(2)$$

Thus, from (1) and (2), it can be observed that the coefficients of a^m and a^n in the expansion of $(1 + a)^{m+n}$ are equal.

Question 10:

The coefficients of the $(r - 1)^{th}$, r^{th} and $(r + 1)^{th}$ terms in the expansion of $(x + 1)^n$ are in the ratio 1:3:5. Find n and r.

Answer

It is known that $(k + 1)^{th}$ term, (T_{k+1}), in the binomial expansion of $(a + b)^n$ is given by

$$T_{k+1} = {}^nC_k a^{n-k} b^k.$$

Therefore, $(r - 1)^{th}$ term in the expansion of $(x + 1)^n$ is

$$T_{r-1} = {}^nC_{r-2} (x)^{n-(r-2)} (1)^{(r-2)} = {}^nC_{r-2} x^{n-r+2}$$

r^{th} term in the expansion of $(x + 1)^n$ is $T_r = {}^nC_{r-1} (x)^{n-(r-1)} (1)^{(r-1)} = {}^nC_{r-1} x^{n-r+1}$

$(r + 1)^{th}$ term in the expansion of $(x + 1)^n$ is $T_{r+1} = {}^nC_r (x)^{n-r} (1)^r = {}^nC_r x^{n-r}$

Therefore, the coefficients of the $(r - 1)^{th}$, r^{th}, and $(r + 1)^{th}$ terms in the expansion of $(x + 1)^n$ are ${}^nC_{r-2}$, ${}^nC_{r-1}$, and ${}^nC_r$ respectively. Since these coefficients are in the ratio 1:3:5, we obtain

$$\frac{{}^nC_{r-2}}{{}^nC_{r-1}} = \frac{1}{3} \text{ and } \frac{{}^nC_{r-1}}{{}^nC_r} = \frac{3}{5}$$

$$\frac{{}^nC_{r-2}}{{}^nC_{r-1}} = \frac{n!}{(r-2)!(n-r+2)!} \times \frac{(r-1)!(n-r+1)!}{n!} = \frac{(r-1)(r-2)!(n-r+1)!}{(r-2)!(n-r+2)(n-r+1)!}$$

$$= \frac{r-1}{n-r+2}$$

$$\therefore \frac{r-1}{n-r+2} = \frac{1}{3}$$

$$\Rightarrow 3r-3 = n-r+2$$

$$\Rightarrow n-4r+5 = 0 \qquad ...(1)$$

$$\frac{{}^nC_{r-1}}{{}^nC_r} = \frac{n!}{(r-1)!(n-r+1)} \times \frac{r!(n-r)!}{n!} = \frac{r(r-1)!(n-r)!}{(r-1)!(n-r+1)(n-r)!}$$

$$= \frac{r}{n-r+1}$$

$$\therefore \frac{r}{n-r+1} = \frac{3}{5}$$

$$\Rightarrow 5r = 3n-3r+3$$

$$\Rightarrow 3n-8r+3 = 0 \qquad ...(2)$$

Multiplying (1) by 3 and subtracting it from (2), we obtain

$4r - 12 = 0$

$\Rightarrow r = 3$

Putting the value of r in (1), we obtain

$n - 12 + 5 = 0$

$\Rightarrow n = 7$

Thus, $n = 7$ and $r = 3$

Question 11:

Prove that the coefficient of x^n in the expansion of $(1 + x)^{2n}$ is twice the coefficient of x^n in the expansion of $(1 + x)^{2n-1}$.

Answer

It is known that $(r + 1)^{\text{th}}$ term, (T_{r+1}), in the binomial expansion of $(a + b)^n$ is given by

$$T_{r+1} = {}^nC_r a^{n-r} b^r.$$

Assuming that x^n occurs in the $(r + 1)^{th}$ term of the expansion of $(1 + x)^{2n}$, we obtain

$$T_{r+1} = {}^{2n}C_r (1)^{2n-r} (x)^r = {}^{2n}C_r (x)^r$$

Comparing the indices of x in x^n and in T_{r+1}, we obtain

$r = n$

Therefore, the coefficient of x^n in the expansion of $(1 + x)^{2n}$ is

$$^{2n}C_n = \frac{(2n)!}{n!(2n-n)!} = \frac{(2n)!}{n!n!} = \frac{(2n)!}{(n!)^2} \qquad ...(1)$$

Assuming that x^n occurs in the $(k + 1)^{th}$ term of the expansion $(1 + x)^{2n-1}$, we obtain

$$T_{k+1} = {}^{2n-1}C_k (1)^{2n-1-k} (x)^k = {}^{2n-1}C_k (x)^k$$

Comparing the indices of x in x^n and T_{k+1}, we obtain

$k = n$

Therefore, the coefficient of x^n in the expansion of $(1 + x)^{2n-1}$ is

$$^{2n-1}C_n = \frac{(2n-1)!}{n!(2n-1-n)!} = \frac{(2n-1)!}{n!(n-1)!}$$

$$= \frac{2n.(2n-1)!}{2n.n!(n-1)!} = \frac{(2n)!}{2.n!n!} = \frac{1}{2}\left[\frac{(2n)!}{(n!)^2}\right] \qquad ...(2)$$

From (1) and (2), it is observed that

$$\frac{1}{2}\left({}^{2n}C_n\right) = {}^{2n-1}C_n$$

$$\Rightarrow {}^{2n}C_n = 2\left({}^{2n-1}C_n\right)$$

Therefore, the coefficient of x^n in the expansion of $(1 + x)^{2n}$ is twice the coefficient of x^n in the expansion of $(1 + x)^{2n-1}$.

Hence, proved.

Question 12:

Find a positive value of m for which the coefficient of x^2 in the expansion $(1 + x)^m$ is 6.

Answer

It is known that $(r + 1)^{th}$ term, (T_{r+1}), in the binomial expansion of $(a + b)^n$ is given by

$$T_{r+1} = {}^nC_r a^{n-r} b^r.$$

Assuming that x^2 occurs in the $(r + 1)^{th}$ term of the expansion $(1 + x)^m$, we obtain

$$T_{r+1} = {}^mC_r (1)^{m-r} (x)^r = {}^mC_r (x)^r$$

Comparing the indices of x in x^2 and in T_{r+1}, we obtain

$r = 2$

Therefore, the coefficient of x^2 is ${}^mC_2$.

It is given that the coefficient of x^2 in the expansion $(1 + x)^m$ is 6.

$$\therefore {}^mC_2 = 6$$

$$\Rightarrow \frac{m!}{2!(m-2)!} = 6$$

$$\Rightarrow \frac{m(m-1)(m-2)!}{2 \times (m-2)!} = 6$$

$$\Rightarrow m(m-1) = 12$$

$$\Rightarrow m^2 - m - 12 = 0$$

$$\Rightarrow m^2 - 4m + 3m - 12 = 0$$

$$\Rightarrow m(m-4) + 3(m-4) = 0$$

$$\Rightarrow (m-4)(m+3) = 0$$

$$\Rightarrow (m-4) = 0 \text{ or } (m+3) = 0$$

$$\Rightarrow m = 4 \text{ or } m = -3$$

Thus, the positive value of m, for which the coefficient of x^2 in the expansion $(1 + x)^m$ is 6, is 4.