NCERT Miscellaneous Solutions

Question 1:

Find a, b and n in the expansion of $(a + b)^n$ if the first three terms of the expansion are 729, 7290 and 30375, respectively.

Answer

It is known that $(r + 1)^{th}$ term, (T_{r+1}), in the binomial expansion of $(a + b)^n$ is given by

$$T_{r+1} = {}^nC_r a^{n-r} b^r.$$

The first three terms of the expansion are given as 729, 7290, and 30375 respectively. Therefore, we obtain

$$T_1 = {}^nC_0 a^{n-0} b^0 = a^n = 729 \qquad ...(1)$$

$$T_2 = {}^nC_1 a^{n-1} b^1 = na^{n-1}b = 7290 \qquad ...(2)$$

$$T_3 = {}^nC_2 a^{n-2} b^2 = \frac{n(n-1)}{2} a^{n-2} b^2 = 30375 \qquad ...(3)$$

Dividing (2) by (1), we obtain

$$\frac{na^{n-1}b}{a^n} = \frac{7290}{729}$$

$$\Rightarrow \frac{nb}{a} = 10 \qquad ...(4)$$

Dividing (3) by (2), we obtain

$$\frac{n(n-1)a^{n-2}b^2}{2na^{n-1}b} = \frac{30375}{7290}$$
$$\Rightarrow \frac{(n-1)b}{2a} = \frac{30375}{7290}$$
$$\Rightarrow \frac{(n-1)b}{a} = \frac{30375\times 2}{7290} = \frac{25}{3}$$
$$\Rightarrow \frac{nb}{a} - \frac{b}{a} = \frac{25}{3}$$
$$\Rightarrow 10 - \frac{b}{a} = \frac{25}{3} \qquad [\text{Using } (4)]$$
$$\Rightarrow \frac{b}{a} = 10 - \frac{25}{3} = \frac{5}{3} \qquad ...(5)$$

From (4) and (5), we obtain

$$n \cdot \frac{5}{3} = 10$$
$$\Rightarrow n = 6$$

Substituting n = 6 in equation (1), we obtain

$a^6 = 729$

$$\Rightarrow a = \sqrt[6]{729} = 3$$

From (5), we obtain

$$\frac{b}{3} = \frac{5}{3} \Rightarrow b = 5$$

Thus, a = 3, b = 5, and n = 6.

Question 2:

Find a if the coefficients of x^2 and x^3 in the expansion of $(3 + ax)^9$ are equal.

Answer

It is known that $(r + 1)^{th}$ term, (T_{r+1}), in the binomial expansion of $(a + b)^n$ is given by

$$T_{r+1} = {}^nC_r a^{n-r} b^r.$$

Assuming that x^2 occurs in the $(r + 1)^{th}$ term in the expansion of $(3 + ax)^9$, we obtain

$$T_{r+1} = {}^9C_r (3)^{9-r} (ax)^r = {}^9C_r (3)^{9-r} a^r x^r$$

Comparing the indices of x in x^2 and in T_{r+1}, we obtain

$r = 2$

Thus, the coefficient of x^2 is

$$^9C_2(3)^{9-2}a^2 = \frac{9!}{2!7!}(3)^7 a^2 = 36(3)^7 a^2$$

Assuming that x^3 occurs in the $(k + 1)^{th}$ term in the expansion of $(3 + ax)^9$, we obtain

$$T_{k+1} = {}^9C_k(3)^{9-k}(ax)^k = {}^9C_k(3)^{9-k}a^k x^k$$

Comparing the indices of x in x^3 and in T_{k+1}, we obtain

$k = 3$

Thus, the coefficient of x^3 is

$$^9C_3(3)^{9-3}a^3 = \frac{9!}{3!6!}(3)^6 a^3 = 84(3)^6 a^3$$

It is given that the coefficients of x^2 and x^3 are the same.

$$84(3)^6 a^3 = 36(3)^7 a^2$$

$$\Rightarrow 84a = 36 \times 3$$

$$\Rightarrow a = \frac{36 \times 3}{84} = \frac{104}{84}$$

$$\Rightarrow a = \frac{9}{7}$$

Thus, the required value of a is $\frac{9}{7}$.

Question 3:

Find the coefficient of x^5 in the product $(1 + 2x)^6 (1 - x)^7$ using binomial theorem.

Answer

Using Binomial Theorem, the expressions, $(1 + 2x)^6$ and $(1 - x)^7$, can be expanded as

$$\begin{aligned}(1+2x)^6 &= {}^6C_0 + {}^6C_1(2x) + {}^6C_2(2x)^2 + {}^6C_3(2x)^3 + {}^6C_4(2x)^4 \\ &\quad + {}^6C_5(2x)^5 + {}^6C_6(2x)^6 \\ &= 1 + 6(2x) + 15(2x)^2 + 20(2x)^3 + 15(2x)^4 + 6(2x)^5 + (2x)^6 \\ &= 1 + 12x + 60x^2 + 160x^3 + 240x^4 + 192x^5 + 64x^6\end{aligned}$$

$$(1-x)^7 = {}^7C_0 - {}^7C_1(x) + {}^7C_2(x)^2 - {}^7C_3(x)^3 + {}^7C_4(x)^4$$
$$- {}^7C_5(x)^5 + {}^7C_6(x)^6 - {}^7C_7(x)^7$$
$$= 1 - 7x + 21x^2 - 35x^3 + 35x^4 - 21x^5 + 7x^6 - x^7$$
$$\therefore (1+2x)^6(1-x)^7$$
$$= \left(1+12x+60x^2+160x^3+240x^4+192x^5+64x^6\right)\left(1-7x+21x^2-35x^3+35x^4-21x^5+7x^6-x^7\right)$$

The complete multiplication of the two brackets is not required to be carried out. Only those terms, which involve x^5, are required.

The terms containing x^5 are

$$1\left(-21x^5\right)+\left(12x\right)\left(35x^4\right)+\left(60x^2\right)\left(-35x^3\right)+\left(160x^3\right)\left(21x^2\right)+\left(240x^4\right)\left(-7x\right)+\left(192x^5\right)(1)$$
$$= 171x^5$$

Thus, the coefficient of x^5 in the given product is 171.

Question 4:

If a and b are distinct integers, prove that $a - b$ is a factor of $a^n - b^n$, whenever n is a positive integer.

[**Hint:** write $a^n = (a - b + b)^n$ and expand]

Answer

In order to prove that $(a - b)$ is a factor of $(a^n - b^n)$, it has to be proved that

$a^n - b^n = k(a - b)$, where k is some natural number

It can be written that, $a = a - b + b$

$$\therefore a^n = (a-b+b)^n = \left[(a-b)+b\right]^n$$
$$= {}^nC_0(a-b)^n + {}^nC_1(a-b)^{n-1}b + \ldots + {}^nC_{n-1}(a-b)b^{n-1} + {}^nC_nb^n$$
$$= (a-b)^n + {}^nC_1(a-b)^{n-1}b + \ldots + {}^nC_{n-1}(a-b)b^{n-1} + b^n$$
$$\Rightarrow a^n - b^n = (a-b)\left[(a-b)^{n-1} + {}^nC_1(a-b)^{n-2}b + \ldots + {}^nC_{n-1}b^{n-1}\right]$$
$$\Rightarrow a^n - b^n = k(a-b)$$
$$\text{where, } k = \left[(a-b)^{n-1} + {}^nC_1(a-b)^{n-2}b + \ldots + {}^nC_{n-1}b^{n-1}\right] \text{ is a natural number}$$

This shows that $(a - b)$ is a factor of $(a^n - b^n)$, where n is a positive integer.

Question 5:

Evaluate $\left(\sqrt{3}+\sqrt{2}\right)^6-\left(\sqrt{3}-\sqrt{2}\right)^6$.

Answer

Firstly, the expression $(a + b)^6 - (a - b)^6$ is simplified by using Binomial Theorem.

This can be done as

$$\begin{aligned}(a+b)^6 &= {}^6C_0a^6 + {}^6C_1a^5b + {}^6C_2a^4b^2 + {}^6C_3a^3b^3 + {}^6C_4a^2b^4 + {}^6C_5a^1b^5 + {}^6C_6b^6\\ &= a^6+6a^5b+15a^4b^2+20a^3b^3+15a^2b^4+6ab^5+b^6\\ (a-b)^6 &= {}^6C_0a^6 - {}^6C_1a^5b + {}^6C_2a^4b^2 - {}^6C_3a^3b^3 + {}^6C_4a^2b^4 - {}^6C_5a^1b^5 + {}^6C_6b^6\\ &= a^6-6a^5b+15a^4b^2-20a^3b^3+15a^2b^4-6ab^5+b^6\end{aligned}$$

$$\therefore (a+b)^6-(a-b)^6 = 2\left[6a^5b+20a^3b^3+6ab^5\right]$$

Putting $a=\sqrt{3}$ and $b=\sqrt{2}$, we obtain

$$\begin{aligned}\left(\sqrt{3}+\sqrt{2}\right)^6-\left(\sqrt{3}-\sqrt{2}\right)^6 &= 2\left[6\left(\sqrt{3}\right)^5\left(\sqrt{2}\right)+20\left(\sqrt{3}\right)^3\left(\sqrt{2}\right)^3+6\left(\sqrt{3}\right)\left(\sqrt{2}\right)^5\right]\\ &= 2\left[54\sqrt{6}+120\sqrt{6}+24\sqrt{6}\right]\\ &= 2\times198\sqrt{6}\\ &= 396\sqrt{6}\end{aligned}$$

Question 6:

Find the value of $\left(a^2+\sqrt{a^2-1}\right)^4+\left(a^2-\sqrt{a^2-1}\right)^4$.

Answer

Firstly, the expression $(x + y)^4 + (x - y)^4$ is simplified by using Binomial Theorem.

This can be done as

$$(x+y)^4 = {}^4C_0x^4 + {}^4C_1x^3y + {}^4C_2x^2y^2 + {}^4C_3xy^3 + {}^4C_4y^4$$
$$= x^4 + 4x^3y + 6x^2y^2 + 4xy^3 + y^4$$
$$(x-y)^4 = {}^4C_0x^4 - {}^4C_1x^3y + {}^4C_2x^2y^2 - {}^4C_3xy^3 + {}^4C_4y^4$$
$$= x^4 - 4x^3y + 6x^2y^2 - 4xy^3 + y^4$$
$$\therefore (x+y)^4 + (x-y)^4 = 2\left(x^4 + 6x^2y^2 + y^4\right)$$

Putting $x = a^2$ and $y = \sqrt{a^2-1}$, we obtain

$$\left(a^2+\sqrt{a^2-1}\right)^4 + \left(a^2-\sqrt{a^2-1}\right)^4 = 2\left[\left(a^2\right)^4 + 6\left(a^2\right)^2\left(\sqrt{a^2-1}\right)^2 + \left(\sqrt{a^2-1}\right)^4\right]$$
$$= 2\left[a^8 + 6a^4\left(a^2-1\right) + \left(a^2-1\right)^2\right]$$
$$= 2\left[a^8 + 6a^6 - 6a^4 + a^4 - 2a^2 + 1\right]$$
$$= 2\left[a^8 + 6a^6 - 5a^4 - 2a^2 + 1\right]$$
$$= 2a^8 + 12a^6 - 10a^4 - 4a^2 + 2$$

Question 7:

Find an approximation of $(0.99)^5$ using the first three terms of its expansion.

Answer

0.99 = 1 – 0.01

$$\therefore (0.99)^5 = (1-0.01)^5$$
$$= {}^5C_0(1)^5 - {}^5C_1(1)^4(0.01) + {}^5C_2(1)^3(0.01)^2 \qquad \text{(Approximately)}$$
$$= 1 - 5(0.01) + 10(0.01)^2$$
$$= 1 - 0.05 + 0.001$$
$$= 1.001 - 0.05$$
$$= 0.951$$

Thus, the value of $(0.99)^5$ is approximately 0.951.

Question 8:

Find *n*, if the ratio of the fifth term from the beginning to the fifth term from the end in the expansion of $\left(\sqrt[4]{2}+\frac{1}{\sqrt[4]{3}}\right)^n$ is $\sqrt{6}:1$

Answer

In the expansion, $(a+b)^n = {}^nC_0a^n + {}^nC_1a^{n-1}b + {}^nC_2a^{n-2}b^2 + \ldots + {}^nC_{n-1}ab^{n-1} + {}^nC_nb^n$,

Fifth term from the beginning $= {}^nC_4a^{n-4}b^4$

Fifth term from the end $= {}^nC_{n-4}a^4b^{n-4}$

Therefore, it is evident that in the expansion of $\left(\sqrt[4]{2}+\frac{1}{\sqrt[4]{3}}\right)^n$, the fifth term from the beginning is ${}^nC_4\left(\sqrt[4]{2}\right)^{n-4}\left(\frac{1}{\sqrt[4]{3}}\right)^4$ and the fifth term from the end is ${}^nC_{n-4}\left(\sqrt[4]{2}\right)^4\left(\frac{1}{\sqrt[4]{3}}\right)^{n-4}$.

$${}^nC_4\left(\sqrt[4]{2}\right)^{n-4}\left(\frac{1}{\sqrt[4]{3}}\right)^4 = {}^nC_4\frac{\left(\sqrt[4]{2}\right)^n}{\left(\sqrt[4]{2}\right)^4}\cdot\frac{1}{3} = {}^nC_4\frac{\left(\sqrt[4]{2}\right)^n}{2}\cdot\frac{1}{3} = \frac{n!}{6.4!(n-4)!}\left(\sqrt[4]{2}\right)^n \qquad \ldots(1)$$

$${}^nC_{n-4}\left(\sqrt[4]{2}\right)^4\left(\frac{1}{\sqrt[4]{3}}\right)^{n-4} = {}^nC_{n-4}.2\cdot\frac{\left(\sqrt[4]{3}\right)^4}{\left(\sqrt[4]{3}\right)^n} = {}^nC_{n-4}.2\cdot\frac{3}{\left(\sqrt[4]{3}\right)^n} = \frac{6n!}{(n-4)!4!}\cdot\frac{1}{\left(\sqrt[4]{3}\right)^n} \qquad \ldots(2)$$

It is given that the ratio of the fifth term from the beginning to the fifth term from the end is $\sqrt{6}:1$. Therefore, from (1) and (2), we obtain

$$\frac{n!}{6.4!(n-4)!}\left(\sqrt[4]{2}\right)^n : \frac{6n!}{(n-4)!4!}\cdot\frac{1}{\left(\sqrt[4]{3}\right)^n} = \sqrt{6}:1$$

$$\Rightarrow \frac{\left(\sqrt[4]{2}\right)^n}{6} : \frac{6}{\left(\sqrt[4]{3}\right)^n} = \sqrt{6}:1$$

$$\Rightarrow \frac{\left(\sqrt[4]{2}\right)^n}{6} \times \frac{\left(\sqrt[4]{3}\right)^n}{6} = \sqrt{6}$$

$$\Rightarrow \left(\sqrt[4]{6}\right)^n = 36\sqrt{6}$$

$$\Rightarrow 6^{\frac{n}{4}} = 6^{\frac{5}{2}}$$

$$\Rightarrow \frac{n}{4} = \frac{5}{2}$$

$$\Rightarrow n = 4\times\frac{5}{2} = 10$$

Thus, the value of *n* is 10.

Question 9:

Expand using Binomial Theorem $\left(1+\frac{x}{2}-\frac{2}{x}\right)^4, x \neq 0$.

Answer

Using Binomial Theorem, the given expression $\left(1+\frac{x}{2}-\frac{2}{x}\right)^4$ can be expanded as

$$\left[\left(1+\frac{x}{2}\right)-\frac{2}{x}\right]^4$$

$$= {}^4C_0\left(1+\frac{x}{2}\right)^4 - {}^4C_1\left(1+\frac{x}{2}\right)^3\left(\frac{2}{x}\right) + {}^4C_2\left(1+\frac{x}{2}\right)^2\left(\frac{2}{x}\right)^2 - {}^4C_3\left(1+\frac{x}{2}\right)\left(\frac{2}{x}\right)^3 + {}^4C_4\left(\frac{2}{x}\right)^4$$

$$=\left(1+\frac{x}{2}\right)^4 - 4\left(1+\frac{x}{2}\right)^3\left(\frac{2}{x}\right) + 6\left(1+x+\frac{x^2}{4}\right)\left(\frac{4}{x^2}\right) - 4\left(1+\frac{x}{2}\right)\left(\frac{8}{x^3}\right) + \frac{16}{x^4}$$

$$=\left(1+\frac{x}{2}\right)^4 - \frac{8}{x}\left(1+\frac{x}{2}\right)^3 + \frac{24}{x^2} + \frac{24}{x} + 6 - \frac{32}{x^3} - \frac{16}{x^2} + \frac{16}{x^4}$$

$$=\left(1+\frac{x}{2}\right)^4 - \frac{8}{x}\left(1+\frac{x}{2}\right)^3 + \frac{8}{x^2} + \frac{24}{x} + 6 - \frac{32}{x^3} + \frac{16}{x^4} \qquad \ldots(1)$$

Again by using Binomial Theorem, we obtain

$$\left(1+\frac{x}{2}\right)^4 = {}^4C_0(1)^4 + {}^4C_1(1)^3\left(\frac{x}{2}\right) + {}^4C_2(1)^2\left(\frac{x}{2}\right)^2 + {}^4C_3(1)^1\left(\frac{x}{2}\right)^3 + {}^4C_4\left(\frac{x}{2}\right)^4$$

$$=1+4\times\frac{x}{2}+6\times\frac{x^2}{4}+4\times\frac{x^3}{8}+\frac{x^4}{16}$$

$$=1+2x+\frac{3x^2}{2}+\frac{x^3}{2}+\frac{x^4}{16} \qquad \ldots(2)$$

$$\left(1+\frac{x}{2}\right)^3 = {}^3C_0(1)^3 + {}^3C_1(1)^2\left(\frac{x}{2}\right) + {}^3C_2(1)\left(\frac{x}{2}\right)^2 + {}^3C_3\left(\frac{x}{2}\right)^3$$

$$=1+\frac{3x}{2}+\frac{3x^2}{4}+\frac{x^3}{8} \qquad \ldots(3)$$

From (1), (2), and (3), we obtain

$$\left[\left(1+\frac{x}{2}\right)-\frac{2}{x}\right]^4$$

$$=1+2x+\frac{3x^2}{2}+\frac{x^3}{2}+\frac{x^4}{16}-\frac{8}{x}\left(1+\frac{3x}{2}+\frac{3x^2}{4}+\frac{x^3}{8}\right)+\frac{8}{x^2}+\frac{24}{x}+6-\frac{32}{x^3}+\frac{16}{x^4}$$

$$=1+2x+\frac{3}{2}x^2+\frac{x^3}{2}+\frac{x^4}{16}-\frac{8}{x}-12-6x-x^2+\frac{8}{x^2}+\frac{24}{x}+6-\frac{32}{x^3}+\frac{16}{x^4}$$

$$=\frac{16}{x}+\frac{8}{x^2}-\frac{32}{x^3}+\frac{16}{x^4}-4x+\frac{x^2}{2}+\frac{x^3}{2}+\frac{x^4}{16}-5$$

Question 10:

Find the expansion of $\left(3x^2 - 2ax + 3a^2\right)^3$ using binomial theorem.

Answer

Using Binomial Theorem, the given expression $\left(3x^2 - 2ax + 3a^2\right)^3$ can be expanded as

$$\left[\left(3x^2 - 2ax\right) + 3a^2\right]^3$$
$$= {}^3C_0\left(3x^2 - 2ax\right)^3 + {}^3C_1\left(3x^2 - 2ax\right)^2\left(3a^2\right) + {}^3C_2\left(3x^2 - 2ax\right)\left(3a^2\right)^2 + {}^3C_3\left(3a^2\right)^3$$
$$= \left(3x^2 - 2ax\right)^3 + 3\left(9x^4 - 12ax^3 + 4a^2x^2\right)\left(3a^2\right) + 3\left(3x^2 - 2ax\right)\left(9a^4\right) + 27a^6$$
$$= \left(3x^2 - 2ax\right)^3 + 81a^2x^4 - 108a^3x^3 + 36a^4x^2 + 81a^4x^2 - 54a^5x + 27a^6$$
$$= \left(3x^2 - 2ax\right)^3 + 81a^2x^4 - 108a^3x^3 + 117a^4x^2 - 54a^5x + 27a^6 \qquad \ldots(1)$$

Again by using Binomial Theorem, we obtain

$$\left(3x^2 - 2ax\right)^3$$
$$= {}^3C_0\left(3x^2\right)^3 - {}^3C_1\left(3x^2\right)^2(2ax) + {}^3C_2\left(3x^2\right)(2ax)^2 - {}^3C_3(2ax)^3$$
$$= 27x^6 - 3\left(9x^4\right)(2ax) + 3\left(3x^2\right)\left(4a^2x^2\right) - 8a^3x^3$$
$$= 27x^6 - 54ax^5 + 36a^2x^4 - 8a^3x^3 \qquad \ldots(2)$$

From (1) and (2), we obtain

$$\left(3x^2 - 2ax + 3a^2\right)^3$$
$$= 27x^6 - 54ax^5 + 36a^2x^4 - 8a^3x^3 + 81a^2x^4 - 108a^3x^3 + 117a^4x^2 - 54a^5x + 27a^6$$
$$= 27x^6 - 54ax^5 + 117a^2x^4 - 116a^3x^3 + 117a^4x^2 - 54a^5x + 27a^6$$