

**Exercise 9.4****Question 1:**

Find the sum to  $n$  terms of the series  $1 \times 2 + 2 \times 3 + 3 \times 4 + 4 \times 5 + \dots$

Answer

The given series is  $1 \times 2 + 2 \times 3 + 3 \times 4 + 4 \times 5 + \dots$

$n^{\text{th}}$  term,  $a_n = n(n + 1)$

$$\begin{aligned}\therefore S_n &= \sum_{k=1}^n a_k = \sum_{k=1}^n k(k+1) \\ &= \sum_{k=1}^n k^2 + \sum_{k=1}^n k \\ &= \frac{n(n+1)(2n+1)}{6} + \frac{n(n+1)}{2} \\ &= \frac{n(n+1)}{2} \left( \frac{2n+1}{3} + 1 \right) \\ &= \frac{n(n+1)}{2} \left( \frac{2n+4}{3} \right) \\ &= \frac{n(n+1)(n+2)}{3}\end{aligned}$$

**Question 2:**

Find the sum to  $n$  terms of the series  $1 \times 2 \times 3 + 2 \times 3 \times 4 + 3 \times 4 \times 5 + \dots$

Answer

The given series is  $1 \times 2 \times 3 + 2 \times 3 \times 4 + 3 \times 4 \times 5 + \dots$

$n^{\text{th}}$  term,  $a_n = n(n + 1)(n + 2)$

$$= (n^2 + n)(n + 2)$$

$$= n^3 + 3n^2 + 2n$$

$$\begin{aligned}
 \therefore S_n &= \sum_{k=1}^n a_k \\
 &= \sum_{k=1}^n k^3 + 3 \sum_{k=1}^n k^2 + 2 \sum_{k=1}^n k \\
 &= \left[ \frac{n(n+1)}{2} \right]^2 + \frac{3n(n+1)(2n+1)}{6} + \frac{2n(n+1)}{2} \\
 &= \left[ \frac{n(n+1)}{2} \right]^2 + \frac{n(n+1)(2n+1)}{2} + n(n+1) \\
 &= \frac{n(n+1)}{2} \left[ \frac{n(n+1)}{2} + 2n+1+2 \right] \\
 &= \frac{n(n+1)}{2} \left[ \frac{n^2 + n + 4n + 6}{2} \right] \\
 &= \frac{n(n+1)}{4} (n^2 + 5n + 6) \\
 &= \frac{n(n+1)}{4} (n^2 + 2n + 3n + 6) \\
 &= \frac{n(n+1) [n(n+2) + 3(n+2)]}{4} \\
 &= \frac{n(n+1)(n+2)(n+3)}{4}
 \end{aligned}$$

**Question 3:**

Find the sum to  $n$  terms of the series  $3 \times 1^2 + 5 \times 2^2 + 7 \times 3^2 + \dots$

Answer

The given series is  $3 \times 1^2 + 5 \times 2^2 + 7 \times 3^2 + \dots$

$n^{\text{th}}$  term,  $a_n = (2n + 1) n^2 = 2n^3 + n^2$

$$\begin{aligned}
 \therefore S_n &= \sum_{k=1}^n a_k \\
 &= \sum_{k=1}^n (2k^3 + k^2) = 2 \sum_{k=1}^n k^3 + \sum_{k=1}^n k^2 \\
 &= 2 \left[ \frac{n(n+1)}{2} \right]^2 + \frac{n(n+1)(2n+1)}{6} \\
 &= \frac{n^2(n+1)^2}{2} + \frac{n(n+1)(2n+1)}{6} \\
 &= \frac{n(n+1)}{2} \left[ n(n+1) + \frac{2n+1}{3} \right] \\
 &= \frac{n(n+1)}{2} \left[ \frac{3n^2 + 3n + 2n + 1}{3} \right] \\
 &= \frac{n(n+1)}{2} \left[ \frac{3n^2 + 5n + 1}{3} \right] \\
 &= \frac{n(n+1)(3n^2 + 5n + 1)}{6}
 \end{aligned}$$

**Question 4:**

Find the sum to  $n$  terms of the series  $\frac{1}{1 \times 2} + \frac{1}{2 \times 3} + \frac{1}{3 \times 4} + \dots$

Answer

The given series is  $\frac{1}{1 \times 2} + \frac{1}{2 \times 3} + \frac{1}{3 \times 4} + \dots$

$$n^{\text{th}} \text{ term, } a_n = \frac{1}{n(n+1)} = \frac{1}{n} - \frac{1}{n+1} \quad (\text{By partial fractions})$$

$$a_1 = \frac{1}{1} - \frac{1}{2}$$

$$a_2 = \frac{1}{2} - \frac{1}{3}$$

$$a_3 = \frac{1}{3} - \frac{1}{4} \dots$$

$$a_n = \frac{1}{n} - \frac{1}{n+1}$$

Adding the above terms column wise, we obtain

$$a_1 + a_2 + \dots + a_n = \left[ \frac{1}{1} + \frac{1}{2} + \frac{1}{3} + \dots + \frac{1}{n} \right] - \left[ \frac{1}{2} + \frac{1}{3} + \frac{1}{4} + \dots + \frac{1}{n+1} \right]$$

$$\therefore S_n = 1 - \frac{1}{n+1} = \frac{n+1-1}{n+1} = \frac{n}{n+1}$$

#### Question 5:

Find the sum to  $n$  terms of the series  $5^2 + 6^2 + 7^2 + \dots + 20^2$

Answer

The given series is  $5^2 + 6^2 + 7^2 + \dots + 20^2$

$n^{\text{th}}$  term,  $a_n = (n + 4)^2 = n^2 + 8n + 16$

$$\begin{aligned} \therefore S_n &= \sum_{k=1}^n a_k = \sum_{k=1}^n (k^2 + 8k + 16) \\ &= \sum_{k=1}^n k^2 + 8 \sum_{k=1}^n k + \sum_{k=1}^n 16 \\ &= \frac{n(n+1)(2n+1)}{6} + \frac{8n(n+1)}{2} + 16n \end{aligned}$$

16<sup>th</sup> term is  $(16 + 4)^2 = 20^2$

$$\begin{aligned}
 \therefore S_{16} &= \frac{16(16+1)(2 \times 16+1)}{6} + \frac{8 \times 16 \times (16+1)}{2} + 16 \times 16 \\
 &= \frac{(16)(17)(33)}{6} + \frac{(8) \times 16 \times (16+1)}{2} + 16 \times 16 \\
 &= \frac{(16)(17)(33)}{6} + \frac{(8)(16)(17)}{2} + 256 \\
 &= 1496 + 1088 + 256 \\
 &= 2840 \\
 \therefore 5^2 + 6^2 + 7^2 + \dots + 20^2 &= 2840
 \end{aligned}$$

**Question 6:**

Find the sum to  $n$  terms of the series  $3 \times 8 + 6 \times 11 + 9 \times 14 + \dots$

Answer

The given series is  $3 \times 8 + 6 \times 11 + 9 \times 14 + \dots$

$$a_n = (n^{\text{th}} \text{ term of } 3, 6, 9 \dots) \times (n^{\text{th}} \text{ term of } 8, 11, 14, \dots)$$

$$= (3n)(3n + 5)$$

$$= 9n^2 + 15n$$

$$\begin{aligned}
 \therefore S_n &= \sum_{k=1}^n a_k = \sum_{k=1}^n (9k^2 + 15k) \\
 &= 9 \sum_{k=1}^n k^2 + 15 \sum_{k=1}^n k \\
 &= 9 \times \frac{n(n+1)(2n+1)}{6} + 15 \times \frac{n(n+1)}{2} \\
 &= \frac{3n(n+1)(2n+1)}{2} + \frac{15n(n+1)}{2} \\
 &= \frac{3n(n+1)}{2} (2n+1+5) \\
 &= \frac{3n(n+1)}{2} (2n+6) \\
 &= 3n(n+1)(n+3)
 \end{aligned}$$

**Question 7:**

Find the sum to  $n$  terms of the series  $1^2 + (1^2 + 2^2) + (1^2 + 2^2 + 3^2) + \dots$

Answer

The given series is  $1^2 + (1^2 + 2^2) + (1^2 + 2^2 + 3^2) + \dots$

$$a_n = (1^2 + 2^2 + 3^2 + \dots + n^2)$$

$$= \frac{n(n+1)(2n+1)}{6}$$

$$= \frac{n(2n^2 + 3n + 1)}{6} = \frac{2^3 + 3n^2 + n}{6}$$

$$= \frac{1}{3}n^3 + \frac{1}{2}n^2 + \frac{1}{6}n$$

$$\therefore S_n = \sum_{k=1}^n a_k$$

$$= \sum_{k=1}^n \left( \frac{1}{3}k^3 + \frac{1}{2}k^2 + \frac{1}{6}k \right)$$

$$= \frac{1}{3} \sum_{k=1}^n k^3 + \frac{1}{2} \sum_{k=1}^n k^2 + \frac{1}{6} \sum_{k=1}^n k$$

$$= \frac{1}{3} \frac{n^2(n+1)^2}{(2)^2} + \frac{1}{2} \times \frac{n(n+1)(2n+1)}{6} + \frac{1}{6} \times \frac{n(n+1)}{2}$$

$$= \frac{n(n+1)}{6} \left[ \frac{n(n+1)}{2} + \frac{(2n+1)}{2} + \frac{1}{2} \right]$$

$$= \frac{n(n+1)}{6} \left[ \frac{n^2 + n + 2n + 1 + 1}{2} \right]$$

$$= \frac{n(n+1)}{6} \left[ \frac{n^2 + n + 2n + 2}{2} \right]$$

$$= \frac{n(n+1)}{6} \left[ \frac{n(n+1) + 2(n+1)}{2} \right]$$

$$= \frac{n(n+1)}{6} \left[ \frac{(n+1)(n+2)}{2} \right]$$

$$= \frac{n(n+1)^2(n+2)}{12}$$

**Question 8:**

Find the sum to  $n$  terms of the series whose  $n^{\text{th}}$  term is given by  $n(n+1)(n+4)$ .

Answer

$$a_n = n(n+1)(n+4) = n(n^2 + 5n + 4) = n^3 + 5n^2 + 4n$$

$$\begin{aligned} \therefore S_n &= \sum_{k=1}^n a_k = \sum_{k=1}^n k^3 + 5 \sum_{k=1}^n k^2 + 4 \sum_{k=1}^n k \\ &= \frac{n^2(n+1)^2}{4} + \frac{5n(n+1)(2n+1)}{6} + \frac{4n(n+1)}{2} \\ &= \frac{n(n+1)}{2} \left[ \frac{n(n+1)}{2} + \frac{5(2n+1)}{3} + 4 \right] \\ &= \frac{n(n+1)}{2} \left[ \frac{3n^2 + 3n + 20n + 10 + 24}{6} \right] \\ &= \frac{n(n+1)}{2} \left[ \frac{3n^2 + 23n + 34}{6} \right] \\ &= \frac{n(n+1)(3n^2 + 23n + 34)}{12} \end{aligned}$$

**Question 9:**

Find the sum to  $n$  terms of the series whose  $n^{\text{th}}$  term is given by  $n^2 + 2^n$

Answer

$$a_n = n^2 + 2^n$$

$$\therefore S_n = \sum_{k=1}^n k^2 + 2^k = \sum_{k=1}^n k^2 + \sum_{k=1}^n 2^k \quad (1)$$

$$\sum_{k=1}^n 2^k = 2^1 + 2^2 + 2^3 + \dots$$

Consider

The above series  $2, 2^2, 2^3, \dots$  is a G.P. with both the first term and common ratio equal to 2.

$$\therefore \sum_{k=1}^n 2^k = \frac{(2) \left[ (2)^n - 1 \right]}{2 - 1} = 2(2^n - 1) \quad (2)$$

Therefore, from (1) and (2), we obtain

$$S_n = \sum_{k=1}^n k^2 + 2(2^n - 1) = \frac{n(n+1)(2n+1)}{6} + 2(2^n - 1)$$

**Question 10:**

Find the sum to  $n$  terms of the series whose  $n^{\text{th}}$  term is given by  $(2n - 1)^2$

Answer

$$a_n = (2n - 1)^2 = 4n^2 - 4n + 1$$

$$\begin{aligned}\therefore S_n &= \sum_{k=1}^n a_k = \sum_{k=1}^n (4k^2 - 4k + 1) \\ &= 4 \sum_{k=1}^n k^2 - 4 \sum_{k=1}^n k + \sum_{k=1}^n 1 \\ &= \frac{4n(n+1)(2n+1)}{6} - \frac{4n(n+1)}{2} + n \\ &= \frac{2n(n+1)(2n+1)}{3} - 2n(n+1) + n \\ &= n \left[ \frac{2(2n^2 + 3n + 1)}{3} - 2(n+1) + 1 \right] \\ &= n \left[ \frac{4n^2 + 6n + 2 - 6n - 6 + 3}{3} \right] \\ &= n \left[ \frac{4n^2 - 1}{3} \right] \\ &= \frac{n(2n+1)(2n-1)}{3}\end{aligned}$$