Chapter 1

RELATIONS AND FUNCTIONS

1.1 Overview

1.1.1 *Relation*

A relation R from a non-empty set A to a non empty set B is a subset of the Cartesian product A × B. The set of all first elements of the ordered pairs in a relation R from a set A to a set B is called the domain of the relation R. The set of all second elements in a relation R from a set A to a set B is called the range of the relation R. The whole set B is called the codomain of the relation R. Note that range is always a subset of codomain.

1.1.2 *Types of Relations*

A relation R in a set A is subset of A × A. Thus empty set ϕ and A × A are two extreme relations.

(i) A relation R in a set A is called empty relation, if no element of A is related to any element of A, i.e., $R = \phi \subset A \times A$.

(ii) A relation R in a set A is called universal relation, if each element of A is related to every element of A, i.e., R = A × A.

(iii) A relation R in A is said to be reflexive if $a\text{R}a$ for all $a \in A$, R is symmetric if $a\text{R}b \Rightarrow b\text{R}a$, $\forall\ a, b \in A$ and it is said to be transitive if $a\text{R}b$ and $b\text{R}c \Rightarrow a\text{R}c$ $\forall\ a, b, c \in A$. Any relation which is reflexive, symmetric and transitive is called an equivalence relation.

☛ **Note:** An important property of an equivalence relation is that it divides the set into pairwise disjoint subsets called equivalent classes whose collection is called a partition of the set. Note that the union of all equivalence classes gives the whole set.

1.1.3 *Types of Functions*

(i) A function $f : X \to Y$ is defined to be one-one (or injective), if the images of distinct elements of X under f are distinct, i.e.,

$$x_1, x_2 \in X, f(x_1) = f(x_2) \Rightarrow x_1 = x_2.$$

(ii) A function $f : X \to Y$ is said to be onto (or surjective), if every element of Y is the image of some element of X under f, i.e., for every $y \in Y$ there exists an element $x \in X$ such that $f(x) = y$.

(iii) A function $f: \text{X} \rightarrow \text{Y}$ is said to be one-one and onto (or bijective), if f is both one-one and onto.

1.1.4 *Composition of Functions*

(i) Let $f : \text{A} \rightarrow \text{B}$ and $g : \text{B} \rightarrow \text{C}$ be two functions. Then, the composition of f and g, denoted by $g \circ f$, is defined as the function $g \circ f : \text{A} \rightarrow \text{C}$ given by

$$g \circ f (x) = g (f (x)), \ \forall \ x \in \text{A}.$$

(ii) If $f : \text{A} \rightarrow \text{B}$ and $g : \text{B} \rightarrow \text{C}$ are one-one, then $g \circ f : \text{A} \rightarrow \text{C}$ is also one-one

(iii) If $f : \text{A} \rightarrow \text{B}$ and $g : \text{B} \rightarrow \text{C}$ are onto, then $g \circ f : \text{A} \rightarrow \text{C}$ is also onto.

However, converse of above stated results (ii) and (iii) need not be true. Moreover, we have the following results in this direction.

(iv) Let $f : \text{A} \rightarrow \text{B}$ and $g : \text{B} \rightarrow \text{C}$ be the given functions such that $g \circ f$ is one-one. Then f is one-one.

(v) Let $f : \text{A} \rightarrow \text{B}$ and $g : \text{B} \rightarrow \text{C}$ be the given functions such that $g \circ f$ is onto. Then g is onto.

1.1.5 *Invertible Function*

(i) A function $f : \text{X} \rightarrow \text{Y}$ is defined to be invertible, if there exists a function $g : \text{Y} \rightarrow \text{X}$ such that $g \circ f = \text{I}_\text{X}$ and $f \circ g = \text{I}_\text{Y}$. The function g is called the inverse of f and is denoted by f^{-1}.

(ii) A function $f : \text{X} \rightarrow \text{Y}$ is invertible if and only if f is a bijective function.

(iii) If $f : \text{X} \rightarrow \text{Y}$, $g : \text{Y} \rightarrow \text{Z}$ and $h : \text{Z} \rightarrow \text{S}$ are functions, then $h \circ (g \circ f) = (h \circ g) \circ f$.

(iv) Let $f : \text{X} \rightarrow \text{Y}$ and $g : \text{Y} \rightarrow \text{Z}$ be two invertible functions. Then $g \circ f$ is also invertible with $(g \circ f)^{-1} = f^{-1} \circ g^{-1}$.

1.1.6 *Binary Operations*

(i) A binary operation $*$ on a set A is a function $* : \text{A} \times \text{A} \rightarrow \text{A}$. We denote $*(a, b)$ by $a * b$.

(ii) A binary operation $*$ on the set X is called commutative, if $a * b = b * a$ for every $a, b \in \text{X}$.

(iii) A binary operation $* : \text{A} \times \text{A} \rightarrow \text{A}$ is said to be associative if $(a * b) * c = a * (b * c)$, for every $a, b, c \in \text{A}$.

(iv) Given a binary operation $* : \text{A} \times \text{A} \rightarrow \text{A}$, an element $e \in \text{A}$, if it exists, is called identity for the operation $*$, if $a * e = a = e * a$, $\forall \ a \in \text{A}$.

(v) Given a binary operation $*$: A × A → A, with the identity element e in A, an element $a \in$ A, is said to be invertible with respect to the operation $*$, if there exists an element b in A such that $a * b = e = b * a$ and b is called the inverse of a and is denoted by a^{-1}.

1.2 Solved Examples

Short Answer (S.A.)

Example 1 Let A = {0, 1, 2, 3} and define a relation R on A as follows:

R = {(0, 0), (0, 1), (0, 3), (1, 0), (1, 1), (2, 2), (3, 0), (3, 3)}.

Is R reflexive? symmetric? transitive?

Solution R is reflexive and symmetric, but not transitive since for (1, 0) $\in$ R and (0, 3) $\in$ R whereas (1, 3) $\notin$ R.

Example 2 For the set A = {1, 2, 3}, define a relation R in the set A as follows:

R = {(1, 1), (2, 2), (3, 3), (1, 3)}.

Write the ordered pairs to be added to R to make it the smallest equivalence relation.

Solution (3, 1) is the single ordered pair which needs to be added to R to make it the smallest equivalence relation.

Example 3 Let R be the equivalence relation in the set **Z** of integers given by R = {(a, b) : 2 divides $a - b$}. Write the equivalence class [0].

Solution [0] = {0, ± 2, ± 4, ± 6,...}

Example 4 Let the function f : **R** → **R** be defined by $f(x) = 4x - 1$, $\forall\ x \in$ **R**. Then, show that f is one-one.

Solution For any two elements $x_1, x_2 \in$ **R** such that $f(x_1) = f(x_2)$, we have

$$4x_1 - 1 = 4x_2 - 1$$

$$\Rightarrow \quad 4x_1 = 4x_2, \text{ i.e., } x_1 = x_2$$

Hence f is one-one.

Example 5 If f = {(5, 2), (6, 3)}, g = {(2, 5), (3, 6)}, write $f \circ g$.

Solution $f \circ g$ = {(2, 2), (3, 3)}

Example 6 Let f : **R** → **R** be the function defined by $f(x) = 4x - 3\ \forall\ x \in$ **R**. Then write f^{-1}.

Solution Given that $f(x) = 4x - 3 = y$ (say), then

$$4x = y + 3$$

$$\Rightarrow \quad x = \frac{y+3}{4}$$

Hence $\quad f^{-1}(y) = \dfrac{y+3}{4} \quad \Rightarrow \quad f^{-1}(x) = \dfrac{x+3}{4}$

Example 7 Is the binary operation $*$ defined on **Z** (set of integer) by $m * n = m - n + mn \ \forall \ m, n \in \mathbf{Z}$ commutative?

Solution No. Since for $1, 2 \in \mathbf{Z}$, $1 * 2 = 1 - 2 + 1.2 = 1$ while $2 * 1 = 2 - 1 + 2.1 = 3$ so that $1 * 2 \neq 2 * 1$.

Example 8 If $f = \{(5, 2), (6, 3)\}$ and $g = \{(2, 5), (3, 6)\}$, write the range of f and g.

Solution The range of $f = \{2, 3\}$ and the range of $g = \{5, 6\}$.

Example 9 If A = {1, 2, 3} and f, g are relations corresponding to the subset of A × A indicated against them, which of f, g is a function? Why?

$$f = \{(1, 3), (2, 3), (3, 2)\}$$

$$g = \{(1, 2), (1, 3), (3, 1)\}$$

Solution f is a function since each element of A in the first place in the ordered pairs is related to only one element of A in the second place while g is not a function because 1 is related to more than one element of A, namely, 2 and 3.

Example 10 If A = $\{a, b, c, d\}$ and $f = \{(a, b), (b, d), (c, a), (d, c)\}$, show that f is one-one from A onto A. Find f^{-1}.

Solution f is one-one since each element of A is assigned to distinct element of the set A. Also, f is onto since f (A) = A. Moreover, $f^{-1} = \{(b, a), (d, b), (a, c), (c, d)\}$.

Example 11 In the set **N** of natural numbers, define the binary operation $*$ by $m * n = g.c.d\ (m, n)$, $m, n \in \mathbf{N}$. Is the operation $*$ commutative and associative?

Solution The operation is clearly commutative since

$$m * n = g.c.d\ (m, n) = g.c.d\ (n, m) = n * m \quad \forall \ m, n \in \mathbf{N}.$$

It is also associative because for $l, m, n \in \mathbf{N}$, we have

$$l * (m * n) = g.\ c.\ d\ (l, g.c.d\ (m, n))$$
$$= g.c.d.\ (g.\ c.\ d\ (l, m), n)$$
$$= (l * m) * n.$$

Long Answer (L.A.)

Example 12 In the set of natural numbers **N**, define a relation R as follows: $\forall\, n, m \in \mathbf{N}$, $n\text{R}m$ if on division by 5 each of the integers n and m leaves the remainder less than 5, i.e. one of the numbers 0, 1, 2, 3 and 4. Show that R is equivalence relation. Also, obtain the pairwise disjoint subsets determined by R.

Solution R is reflexive since for each $a \in \mathbf{N}$, $a\text{R}a$. R is symmetric since if $a\text{R}b$, then $b\text{R}a$ for $a, b \in \mathbf{N}$. Also, R is transitive since for $a, b, c \in \mathbf{N}$, if $a\text{R}b$ and $b\text{R}c$, then $a\text{R}c$. Hence R is an equivalence relation in **N** which will partition the set **N** into the pairwise disjoint subsets. The equivalent classes are as mentioned below:

$$A_0 = \{5, 10, 15, 20 \ldots\}$$
$$A_1 = \{1, 6, 11, 16, 21 \ldots\}$$
$$A_2 = \{2, 7, 12, 17, 22, \ldots\}$$
$$A_3 = \{3, 8, 13, 18, 23, \ldots\}$$
$$A_4 = \{4, 9, 14, 19, 24, \ldots\}$$

It is evident that the above five sets are pairwise disjoint and

$$A_0 \cup A_1 \cup A_2 \cup A_3 \cup A_4 = \bigcup_{i=0}^{4} A_i = \mathbf{N}.$$

Example 13 Show that the function $f : \mathbf{R} \to \mathbf{R}$ defined by $f(x) = \dfrac{x}{x^2+1}, \forall\, x \in \mathbf{R}$, is neither one-one nor onto.

Solution For $x_1, x_2 \in \mathbf{R}$, consider

$$f(x_1) = f(x_2)$$

$$\Rightarrow \frac{x_1}{x_1^2+1} = \frac{x_2}{x_2^2+1}$$

$$\Rightarrow x_1 x_2^2 + x_1 = x_2 x_1^2 + x_2$$

$$\Rightarrow x_1 x_2 (x_2 - x_1) = x_2 - x_1$$

$$\Rightarrow x_1 = x_2 \text{ or } x_1 x_2 = 1$$

We note that there are point, x_1 and x_2 with $x_1 \neq x_2$ and $f(x_1) = f(x_2)$, for instance, if we take $x_1 = 2$ and $x_2 = \dfrac{1}{2}$, then we have $f(x_1) = \dfrac{2}{5}$ and $f(x_2) = \dfrac{2}{5}$ but $2 \neq \dfrac{1}{2}$. Hence f is not one-one. Also, f is not onto for if so then for $1 \in \mathbf{R}\ \exists\ x \in \mathbf{R}$ such that $f(x) = 1$

which gives $\frac{x}{x^2+1}=1$. But there is no such x in the domain **R**, since the equation $x^2 - x + 1 = 0$ does not give any real value of x.

Example 14 Let f, g : **R** $\to$ **R** be two functions defined as $f(x) = |x| + x$ and $g(x) = |x| - x \quad \forall\ x \in$ **R**. Then, find $f \circ g$ and $g \circ f$.

Solution Here $f(x) = |x| + x$ which can be redefined as

$$f(x) = \begin{cases} 2x \text{ if } x \geq 0 \\ 0 \text{ if } x < 0 \end{cases}$$

Similarly, the function g defined by $g(x) = |x| - x$ may be redefined as

$$g(x) = \begin{cases} 0 \text{ if } x \geq 0 \\ -2x \text{ if } x < 0 \end{cases}$$

Therefore, $g \circ f$ gets defined as :

For $x \geq 0$, $(g \circ f)(x) = g(f(x) = g(2x) = 0$

and for $x < 0$, $(g \circ f)(x) = g(f(x) = g(0) = 0$.

Consequently, we have $(g \circ f)(x) = 0,\ \forall\ x \in$ **R**.

Similarly, $f \circ g$ gets defined as:

For $x \geq 0$, $(f \circ g)(x) = f(g(x) = f(0) = 0$,

and for $x < 0$, $(f \circ g)(x) = f(g(x)) = f(-2x) = -4x$.

i.e. $$(f \circ g)(x) = \begin{cases} 0, x > 0 \\ -4x, x < 0 \end{cases}$$

Example 15 Let **R** be the set of real numbers and f : **R** $\to$ **R** be the function defined by $f(x) = 4x + 5$. Show that f is invertible and find f^{-1}.

Solution Here the function f : **R** $\to$ **R** is defined as $f(x) = 4x + 5 = y$ (say). Then

$$4x = y - 5 \qquad \text{or} \qquad x = \frac{y-5}{4}.$$

This leads to a function $g : \mathbf{R} \to \mathbf{R}$ defined as

$$g(y) = \frac{y-5}{4}.$$

Therefore, $(g \circ f)(x) = g(f(x)) = g(4x + 5)$

$$= \frac{4x+5-5}{4} = x$$

or $g \circ f = I_R$

Similarly $(f \circ g)(y) = f(g(y))$

$$= f\left(\frac{y-5}{4}\right)$$

$$= 4\left(\frac{y-5}{4}\right) + 5 = y$$

or $f \circ g = I_R$.

Hence f is invertible and $f^{-1} = g$ which is given by

$$f^{-1}(x) = \frac{x-5}{4}$$

Example 16 Let $*$ be a binary operation defined on $\mathbf{Q}$. Find which of the following binary operations are associative

(i) $a * b = a - b$ for $a, b \in \mathbf{Q}$.

(ii) $a * b = \frac{ab}{4}$ for $a, b \in \mathbf{Q}$.

(iii) $a * b = a - b + ab$ for $a, b \in \mathbf{Q}$.

(iv) $a * b = ab^2$ for $a, b \in \mathbf{Q}$.

Solution

(i) $*$ is not associative for if we take $a = 1$, $b = 2$ and $c = 3$, then
$(a * b) * c = (1 * 2) * 3 = (1 - 2) * 3 = -1 - 3 = -4$ and
$a * (b * c) = 1 * (2 * 3) = 1 * (2 - 3) = 1 - (-1) = 2.$

Thus $(a * b) * c \neq a * (b * c)$ and hence $*$ is not associative.

(ii) $*$ is associative since **Q** is associative with respect to multiplication.

(iii) $*$ is not associative for if we take $a = 2$, $b = 3$ and $c = 4$, then

$(a * b) * c = (2 * 3) * 4 = (2 - 3 + 6) * 4 = 5 * 4 = 5 - 4 + 20 = 21$, and

$a * (b * c) = 2 * (3 * 4) = 2 * (3 - 4 + 12) = 2 * 11 = 2 - 11 + 22 = 13$

Thus $(a * b) * c \neq a * (b * c)$ and hence $*$ is not associative.

(iv) $*$ is not associative for if we take $a = 1$, $b = 2$ and $c = 3$, then $(a * b) * c = (1 * 2) * 3 = 4 * 3 = 4 \times 9 = 36$ and $a * (b * c) = 1 * (2 * 3) = 1 * 18 = 1 \times 18^2 = 324$.

Thus $(a * b) * c \neq a * (b * c)$ and hence $*$ is not associative.

Objective Type Questions

Choose the correct answer from the given four options in each of the Examples 17 to 25.

Example 17 Let R be a relation on the set **N** of natural numbers defined by $n\text{R}m$ if n divides m. Then R is

(A) Reflexive and symmetric (B) Transitive and symmetric
(C) Equivalence (D) Reflexive, transitive but not symmetric

Solution The correct choice is (D).

Since n divides n, $\forall\ n \in$ **N**, R is reflexive. R is not symmetric since for 3, 6 $\in$ **N**, $3 \text{ R } 6 \neq 6 \text{ R } 3$. R is transitive since for n, m, r whenever n/m and $m/r \Rightarrow n/r$, i.e., n divides m and m divides r, then n will devide r.

Example 18 Let L denote the set of all straight lines in a plane. Let a relation R be defined by $l\text{R}m$ if and only if l is perpendicular to m $\forall\ l, m \in$ L. Then R is

(A) reflexive (B) symmetric
(C) transitive (D) none of these

Solution The correct choice is (B).

Example 19 Let **N** be the set of natural numbers and the function f : **N** $\to$ **N** be defined by $f(n) = 2n + 3$ $\forall\ n \in$ **N**. Then f is

(A) surjective (B) injective
(C) bijective (D) none of these

Solution (B) is the correct option.

Example 20 Set A has 3 elements and the set B has 4 elements. Then the number of injective mappings that can be defined from A to B is

(A) 144 (B) 12
(C) 24 (D) 64

Solution The correct choice is (C). The total number of injective mappings from the set containing 3 elements into the set containing 4 elements is ${}^4P_3 = 4! = 24$.

Example 21 Let $f : \mathbf{R} \to \mathbf{R}$ be defined by $f(x) = \sin x$ and $g : \mathbf{R} \to \mathbf{R}$ be defined by $g(x) = x^2$, then $f \, o \, g$ is

(A) $x^2 \sin x$ (B) $(\sin x)^2$
(C) $\sin x^2$ (D) $\dfrac{\sin x}{x^2}$

Solution (C) is the correct choice.

Example 22 Let $f : \mathbf{R} \to \mathbf{R}$ be defined by $f(x) = 3x - 4$. Then $f^{-1}(x)$ is given by

(A) $\dfrac{x+4}{3}$ (B) $\dfrac{x}{3} - 4$
(C) $3x + 4$ (D) None of these

Solution (A) is the correct choice.

Example 23 Let $f : \mathbf{R} \to \mathbf{R}$ be defined by $f(x) = x^2 + 1$. Then, pre-images of 17 and – 3, respectively, are

(A) $\phi, \{4, -4\}$ (B) $\{3, -3\}, \phi$
(C) $\{4, -4\}, \phi$ (D) $\{4, -4, \{2, -2\}$

Solution (C) is the correct choice since for $f^{-1}(17) = x \Rightarrow f(x) = 17$ or $x^2 + 1 = 17$ $\Rightarrow x = \pm 4$ or $f^{-1}(17) = \{4, -4\}$ and for $f^{-1}(-3) = x \Rightarrow f(x) = -3 \Rightarrow x^2 + 1 = -3 \Rightarrow x^2 = -4$ and hence $f^{-1}(-3) = \phi$.

Example 24 For real numbers x and y, define xRy if and only if $x - y + \sqrt{2}$ is an irrational number. Then the relation R is

(A) reflexive (B) symmetric
(C) transitive (D) none of these

Solution (A) is the correct choice.

Fill in the blanks in each of the Examples 25 to 30.

Example 25 Consider the set A = {1, 2, 3} and R be the smallest equivalence relation on A, then R = ________

Solution R = {(1, 1), (2, 2), (3, 3)}.

Example 26 The domain of the function f : **R** → **R** defined by $f(x) = \sqrt{x^2 - 3x + 2}$ is ________.

Solution Here $x^2 - 3x + 2 \geq 0$

$\Rightarrow \quad (x-1)(x-2) \geq 0$

$\Rightarrow \quad x \leq 1$ or $x \geq 2$

Hence the domain of $f = (-\infty, 1] \cup [2, \infty)$

Example 27 Consider the set A containing n elements. Then, the total number of injective functions from A onto itself is ________.

Solution $n!$

Example 28 Let **Z** be the set of integers and R be the relation defined in **Z** such that aRb if $a - b$ is divisible by 3. Then R partitions the set **Z** into ________ pairwise disjoint subsets.

Solution Three.

Example 29 Let **R** be the set of real numbers and $*$ be the binary operation defined on **R** as $a * b = a + b - ab \;\; \forall \; a, b \in$ **R**. Then, the identity element with respect to the binary operation $*$ is _______.

Solution 0 is the identity element with respect to the binary operation $*$.

State **True** or **False** for the statements in each of the Examples 30 to 34.

Example 30 Consider the set A = {1, 2, 3} and the relation R = {(1, 2), (1, 3)}. R is a transitive relation.

Solution True.

Example 31 Let A be a finite set. Then, each injective function from A into itself is not surjective.

Solution False.

Example 32 For sets A, B and C, let f : A → B, g : B → C be functions such that $g \circ f$ is injective. Then both f and g are injective functions.

Solution False.

Example 33 For sets A, B and C, let f : A → B, g : B → C be functions such that $g \circ f$ is surjective. Then g is surjective

Solution True.

Example 34 Let **N** be the set of natural numbers. Then, the binary operation $*$ in **N** defined as $a * b = a + b$, $\forall\ a, b \in$ **N** has identity element.

Solution False.

1.3 EXERCISE

Short Answer (S.A.)

1. Let A = {a, b, c} and the relation R be defined on A as follows:

$$R = \{(a, a), (b, c), (a, b)\}.$$

Then, write minimum number of ordered pairs to be added in **R** to make **R** reflexive and transitive.

2. Let D be the domain of the real valued function f defined by $f(x) = \sqrt{25 - x^2}$. Then, write D.

3. Let $f, g : \mathbf{R} \rightarrow \mathbf{R}$ be defined by $f(x) = 2x + 1$ and $g(x) = x^2 - 2$, $\forall x \in$ **R**, respectively. Then, find $g\ o\ f$.

4. Let $f : \mathbf{R} \rightarrow \mathbf{R}$ be the function defined by $f(x) = 2x - 3\ \forall x \in$ R. write f^{-1}.

5. If A = {a, b, c, d} and the function $f = \{(a, b), (b, d), (c, a), (d, c)\}$, write f^{-1}.

6. If $f : \mathbf{R} \rightarrow \mathbf{R}$ is defined by $f(x) = x^2 - 3x + 2$, write $f(f(x))$.

7. Is $g = \{(1, 1), (2, 3), (3, 5), (4, 7)\}$ a function? If g is described by $g(x) = \alpha x + \beta$, then what value should be assigned to α and β.

8. Are the following set of ordered pairs functions? If so, examine whether the mapping is injective or surjective.

(i) {(x, y): x is a person, y is the mother of x}.

(ii){(a, b): a is a person, b is an ancestor of a}.

9. If the mappings f and g are given by

$f = \{(1, 2), (3, 5), (4, 1)\}$ and $g = \{(2, 3), (5, 1), (1, 3)\}$, write $f\ o\ g$.

10. Let **C** be the set of complex numbers. Prove that the mapping $f : \mathbf{C} \rightarrow \mathbf{R}$ given by $f(z) = |z|$, $\forall\ z \in$ **C**, is neither one-one nor onto.

11. Let the function $f : \mathbf{R} \rightarrow \mathbf{R}$ be defined by $f(x) = \cos x$, $\forall\ x \in$ **R**. Show that f is neither one-one nor onto.

12. Let X = {1, 2, 3}and Y = {4, 5}. Find whether the following subsets of X ×Y are functions from X to Y or not.

(i) $f = \{(1, 4), (1, 5), (2, 4), (3, 5)\}$ (ii) $g = \{(1, 4), (2, 4), (3, 4)\}$

(iii) $h = \{(1,4), (2, 5), (3, 5)\}$ (iv) $k = \{(1,4), (2, 5)\}$.

13. If functions $f : A \rightarrow B$ and $g : B \rightarrow A$ satisfy $g\ o\ f = I_A$, then show that f is one-one and g is onto.

14. Let $f : \mathbf{R} \to \mathbf{R}$ be the function defined by $f(x) = \dfrac{1}{2 - \cos x}$ $\forall\, x \in \mathbf{R}$. Then, find the range of f.

15. Let n be a fixed positive integer. Define a relation R in $\mathbf{Z}$ as follows: $\forall\, a, b \in \mathbf{Z}$, $a\mathrm{R}b$ if and only if $a - b$ is divisible by n. Show that R is an equivalance relation.

Long Answer (L.A.)

16. If A = {1, 2, 3, 4 }, define relations on A which have properties of being:

(a) reflexive, transitive but not symmetric
(b) symmetric but neither reflexive nor transitive
(c) reflexive, symmetric and transitive.

17. Let R be relation defined on the set of natural number **N** as follows:

R = $\{(x, y): x \in \mathbf{N}, y \in \mathbf{N}, 2x + y = 41\}$. Find the domain and range of the relation R. Also verify whether R is reflexive, symmetric and transitive.

18. Given A = {2, 3, 4}, B = {2, 5, 6, 7}. Construct an example of each of the following:

(a) an injective mapping from A to B
(b) a mapping from A to B which is not injective
(c) a mapping from B to A.

19. Give an example of a map

(i) which is one-one but not onto
(ii) which is not one-one but onto
(iii) which is neither one-one nor onto.

20. Let A = $\mathbf{R} - \{3\}$, B = $\mathbf{R} - \{1\}$. Let $f : \mathrm{A} \to \mathrm{B}$ be defined by $f(x) = \dfrac{x-2}{x-3}$ $\forall\, x \in \mathrm{A}$. Then show that f is bijective.

21. Let A = [–1, 1]. Then, discuss whether the following functions defined on A are one-one, onto or bijective:

(i) $f(x) = \dfrac{x}{2}$ (ii) $g(x) = |x|$

(iii) $h(x) = x|x|$ (iv) $k(x) = x^2$.

22. Each of the following defines a relation on **N**:

(i) x is greater than y, $x, y \in \mathbf{N}$
(ii) $x + y = 10$, $x, y \in \mathbf{N}$

(iii) $x\,y$ is square of an integer $x, y \in$ **N**

(iv) $x + 4y = 10$ $x, y \in$ **N**.

Determine which of the above relations are reflexive, symmetric and transitive.

23. Let A = {1, 2, 3, ... 9} and R be the relation in A ×A defined by (a, b) R (c, d) if $a + d = b + c$ for $(a, b), (c, d)$ in A ×A. Prove that R is an equivalence relation and also obtain the equivalent class [(2, 5)].

24. Using the definition, prove that the function f : A → B is invertible if and only if f is both one-one and onto.

25. Functions f, g : **R** → **R** are defined, respectively, by $f(x) = x^2 + 3x + 1$, $g(x) = 2x - 3$, find

(i) $f \circ g$ (ii) $g \circ f$ (iii) $f \circ f$ (iv) $g \circ g$

26. Let $*$ be the binary operation defined on **Q**. Find which of the following binary operations are commutative

(i) $a * b = a - b \; \forall \; a, b \in \mathbf{Q}$ (ii) $a * b = a^2 + b^2 \; \forall \; a, b \in \mathbf{Q}$

(iii) $a * b = a + ab \; \forall \; a, b \in \mathbf{Q}$ (iv) $a * b = (a - b)^2 \; \forall \; a, b \in \mathbf{Q}$

27. Let $*$ be binary operation defined on **R** by $a * b = 1 + ab$, $\forall \; a, b \in \mathbf{R}$. Then the operation $*$ is

(i) commutative but not associative

(ii) associative but not commutative

(iii) neither commutative nor associative

(iv) both commutative and associative

Objective Type Questions

Choose the correct answer out of the given four options in each of the Exercises from 28 to 47 (M.C.Q.).

28. Let T be the set of all triangles in the Euclidean plane, and let a relation R on T be defined as aRb if a is congruent to b $\forall \; a, b \in$ T. Then R is

(A) reflexive but not transitive (B) transitive but not symmetric

(C) equivalence (D) none of these

29. Consider the non-empty set consisting of children in a family and a relation R defined as aRb if a is brother of b. Then R is

(A) symmetric but not transitive (B) transitive but not symmetric

(C) neither symmetric nor transitive (D) both symmetric and transitive

30. The maximum number of equivalence relations on the set A = {1, 2, 3} are

(A) 1 (B) 2

(C) 3 (D) 5

31. If a relation R on the set {1, 2, 3} be defined by R = {(1, 2)}, then R is

(A) reflexive (B) transitive

(C) symmetric (D) none of these

32. Let us define a relation R in **R** as aRb if $a \geq b$. Then R is

(A) an equivalence relation (B) reflexive, transitive but not symmetric

(C) symmetric, transitive but not reflexive (D) neither transitive nor reflexive but symmetric.

33. Let A = {1, 2, 3} and consider the relation

R = {1, 1), (2, 2), (3, 3), (1, 2), (2, 3), (1,3)}.

Then R is

(A) reflexive but not symmetric (B) reflexive but not transitive

(C) symmetric and transitive (D) neither symmetric, nor transitive

34. The identity element for the binary operation $*$ defined on Q ~ {0} as

$a * b = \dfrac{ab}{2} \; \forall \, a, b \in$ Q ~ {0} is

(A) 1 (B) 0

(C) 2 (D) none of these

35. If the set A contains 5 elements and the set B contains 6 elements, then the number of one-one and onto mappings from A to B is

(A) 720 (B) 120

(C) 0 (D) none of these

36. Let A = {1, 2, 3, ...n} and B = {a, b}. Then the number of surjections from A into B is

(A) ${}^nP_2$ (B) $2^n - 2$

(C) $2^n - 1$ (D) None of these

37. Let $f : \mathbf{R} \to \mathbf{R}$ be defined by $f(x) = \frac{1}{x}$ $\forall x \in \mathbf{R}$. Then f is

(A) one-one (B) onto

(C) bijective (D) f is not defined

38. Let $f : \mathbf{R} \to \mathbf{R}$ be defined by $f(x) = 3x^2 - 5$ and $g : \mathbf{R} \to \mathbf{R}$ by $g(x) = \frac{x}{x^2+1}$.

Then $g \circ f$ is

(A) $\frac{3x^2 - 5}{9x^4 - 30x^2 + 26}$ (B) $\frac{3x^2 - 5}{9x^4 - 6x^2 + 26}$

(C) $\frac{3x^2}{x^4 + 2x^2 - 4}$ (D) $\frac{3x^2}{9x^4 + 30x^2 - 2}$

39. Which of the following functions from **Z** into **Z** are bijections?

(A) $f(x) = x^3$ (B) $f(x) = x + 2$

(C) $f(x) = 2x + 1$ (D) $f(x) = x^2 + 1$

40. Let $f : \mathbf{R} \to \mathbf{R}$ be the functions defined by $f(x) = x^3 + 5$. Then $f^{-1}(x)$ is

(A) $(x+5)^{\frac{1}{3}}$ (B) $(x-5)^{\frac{1}{3}}$

(C) $(5-x)^{\frac{1}{3}}$ (D) $5 - x$

41. Let $f : A \to B$ and $g : B \to C$ be the bijective functions. Then $(g \circ f)^{-1}$ is

(A) $f^{-1} \circ g^{-1}$ (B) $f \circ g$

(C) $g^{-1} \circ f^{-1}$ (D) $g \circ f$

42. Let $f : \mathbf{R} - \left\{\frac{3}{5}\right\} \to \mathbf{R}$ be defined by $f(x) = \frac{3x+2}{5x-3}$. Then

(A) $f^{-1}(x) = f(x)$ (B) $f^{-1}(x) = -f(x)$

(C) $(f \circ f)\, x = -x$ (D) $f^{-1}(x) = \frac{1}{19} f(x)$

43. Let $f : [0, 1] \to [0, 1]$ be defined by $f(x) = \begin{cases} x, & \text{if } x \text{ is rational} \\ 1-x, & \text{if } x \text{ is irrational} \end{cases}$

Then $(f \, o \, f) \, x$ is

(A) constant (B) $1 + x$

(C) x (D) none of these

44. Let $f : [2, \infty) \rightarrow \mathbf{R}$ be the function defined by $f(x) = x^2 - 4x + 5$, then the range of f is

(A) $\mathbf{R}$ (B) $[1, \infty)$

(C) $[4, \infty)$ (B) $[5, \infty)$

45. Let $f : \mathbf{N} \rightarrow \mathbf{R}$ be the function defined by $f(x) = \frac{2x-1}{2}$ and $g : \mathbf{Q} \rightarrow \mathbf{R}$ be another function defined by $g(x) = x + 2$. Then $(g \, o \, f) \, \frac{3}{2}$ is

(A) 1 (B) 1

(C) $\frac{7}{2}$ (B) none of these

46. Let $f : \mathbf{R} \rightarrow \mathbf{R}$ be defined by

$$f(x) = \begin{cases} 2x : x > 3 \\ x^2 : 1 < x \le 3 \\ 3x : x \le 1 \end{cases}$$

Then $f(-1) + f(2) + f(4)$ is

(A) 9 (B) 14

(C) 5 (D) none of these

47. Let $f : \mathbf{R} \rightarrow \mathbf{R}$ be given by $f(x) = \tan x$. Then $f^{-1}(1)$ is

(A) $\frac{\pi}{4}$ (B) $\{n\pi + \frac{\pi}{4} : n \in \mathrm{Z}\}$

(C) does not exist (D) none of these

Fill in the blanks in each of the Exercises 48 to 52.

48. Let the relation R be defined in $\mathbf{N}$ by $a\mathrm{R}b$ if $2a + 3b = 30$. Then R = ______.

49. Let the relation R be defined on the set A = {1, 2, 3, 4, 5} by R = $\{(a, b) : |a^2 - b^2| < 8\}$. Then R is given by ______.

50. Let f = {(1, 2), (3, 5), (4, 1)} and g = {(2, 3), (5, 1), (1, 3)}. Then $g \, o \, f$ = ______ and $f \, o \, g$ = ______.

51. Let $f : \mathbf{R} \rightarrow \mathbf{R}$ be defined by $f(x) = \frac{x}{\sqrt{1+x^2}}$. Then $(f \circ f \circ f)(x)$ = ————

52. If $f(x) = (4 - (x-7)^3\}$, then $f^{-1}(x)$ = ————.

State **True** or **False** for the statements in each of the Exercises 53 to 63.

53. Let R = {(3, 1), (1, 3), (3, 3)} be a relation defined on the set A = {1, 2, 3}. Then R is symmetric, transitive but not reflexive.

54. Let $f : \mathbf{R} \rightarrow \mathbf{R}$ be the function defined by $f(x) = \sin(3x+2)$ $\forall x \in$ R. Then f is invertible.

55. Every relation which is symmetric and transitive is also reflexive.

56. An integer m is said to be related to another integer n if m is a integral multiple of n. This relation in **Z** is reflexive, symmetric and transitive.

57. Let A = {0, 1} and **N** be the set of natural numbers. Then the mapping $f : \mathbf{N} \rightarrow$ A defined by $f(2n-1) = 0$, $f(2n) = 1$, $\forall n \in \mathbf{N}$, is onto.

58. The relation R on the set A = {1, 2, 3} defined as R = {{1, 1), (1, 2), (2, 1), (3, 3)} is reflexive, symmetric and transitive.

59. The composition of functions is commutative.

60. The composition of functions is associative.

61. Every function is invertible.

62. A binary operation on a set has always the identity element.