Chapter 5

CONTINUITY AND DIFFERENTIABILITY

5.1 Overview

5.1.1 *Continuity of a function at a point*

Let f be a real function on a subset of the real numbers and let c be a point in the domain of f. Then f is continuous at c if

$$\lim_{x \to c} f(x) = f(c)$$

More elaborately, if the left hand limit, right hand limit and the value of the function at $x = c$ exist and are equal to each other, i.e.,

$$\lim_{x \to c^-} f(x) = f(c) = \lim_{x \to c^+} f(x)$$

then f is said to be continuous at $x = c$.

5.1.2 *Continuity in an interval*

(i) f is said to be continuous in an open interval (a, b) if it is continuous at every point in this interval.

(ii) f is said to be continuous in the closed interval $[a, b]$ if

- f is continuous in (a, b)
- $\lim_{x \to a^+} f(x) = f(a)$
- $\lim_{x \to b^-} f(x) = f(b)$

5.1.3 *Geometrical meaning of continuity*

(i) Function f will be continuous at $x = c$ if there is no break in the graph of the function at the point $(c, f(c))$.

(ii) In an interval, function is said to be continuous if there is no break in the graph of the function in the entire interval.

5.1.4 *Discontinuity*

The function f will be discontinuous at $x = a$ in any of the following cases :

(i) $\lim_{x\to a^-} f(x)$ and $\lim_{x\to a^+} f(x)$ exist but are not equal.

(ii) $\lim_{x\to a^-} f(x)$ and $\lim_{x\to a^+} f(x)$ exist and are equal but not equal to $f(a)$.

(iii) $f(a)$ is not defined.

5.1.5 *Continuity of some of the common functions*

Function $f(x)$	**Interval in which f is continuous**
1. The constant function, i.e. $f(x) = c$	
2. The identity function, i.e. $f(x) = x$	**R**
3. The polynomial function, i.e. $f(x) = a_0 x^n + a_1 x^{n-1} + ... + a_{n-1} x + a_n$	
4. $\lvert x - a \rvert$	$(-\infty, \infty)$
5. x^{-n}, n is a positive integer	$(-\infty, \infty) - \{0\}$
6. $p(x) / q(x)$, where $p(x)$ and $q(x)$ are polynomials in x	$\mathbf{R} - \{x : q(x) = 0\}$
7. $\sin x$, $\cos x$	**R**
8. $\tan x$, $\sec x$	$\mathbf{R} - \{(2n+1)\frac{\pi}{2} : n \in \mathbf{Z}\}$
9. $\cot x$, $\text{cosec } x$	$\mathbf{R} - \{(n\pi : n \in \mathbf{Z}\}$

10. e^x	**R**
11. $\log x$	$(0, \infty)$
12. The inverse trigonometric functions, i.e., $\sin^{-1} x$, $\cos^{-1} x$ etc.	In their respective domains

5.1.6 *Continuity of composite functions*

Let f and g be real valued functions such that (fog) is defined at a. If g is continuous at a and f is continuous at g (a), then (fog) is continuous at a.

5.1.7 *Differentiability*

The function defined by $f'(x) = \lim_{h \to 0} \frac{f(x+h) - f(x)}{h}$, wherever the limit exists, is defined to be the derivative of f at x. In other words, we say that a function f is differentiable at a point c in its domain if both $\lim_{h \to 0^-} \frac{f(c+h) - f(c)}{h}$, called left hand derivative, denoted by $Lf'(c)$, and $\lim_{h \to 0^+} \frac{f(c+h) - f(c)}{h}$, called right hand derivative, denoted by $Rf'(c)$, are finite and equal.

(i) The function $y = f(x)$ is said to be differentiable in an open interval (a, b) if it is differentiable at every point of (a, b)

(ii) The function $y = f(x)$ is said to be differentiable in the closed interval $[a, b]$ if $Rf'(a)$ and $Lf'(b)$ exist and $f'(x)$ exists for every point of (a, b).

(iii) Every differentiable function is continuous, but the converse is not true

5.1.8 *Algebra of derivatives*

If u, v are functions of x, then

(i) $\frac{d(u \pm v)}{dx} = \frac{du}{dx} \pm \frac{dv}{dx}$

(ii) $\frac{d}{dx}(uv) = u\frac{dv}{dx} + v\frac{du}{dx}$

(iii) $\frac{d}{dx}\left(\frac{u}{v}\right) = \frac{v\frac{du}{dx} - u\frac{dv}{dx}}{v^2}$

5.1.9 Chain rule is a rule to differentiate composition of functions. Let $f = vou$. If $t = u(x)$ and both $\frac{dt}{dx}$ and $\frac{dv}{dt}$ exist then $\frac{df}{dx} = \frac{dv}{dt} \cdot \frac{dt}{dx}$

5.1.10 Following are some of the standard derivatives (in appropriate domains)

1. $\frac{d}{dx}(\sin^{-1} x) = \frac{1}{\sqrt{1-x^2}}$
2. $\frac{d}{dx}(\cos^{-1} x) = \frac{-1}{\sqrt{1-x^2}}$
3. $\frac{d}{dx}(\tan^{-1} x) = \frac{1}{1+x^2}$
4. $\frac{d}{dx}(\cot^{-1} x) = \frac{-1}{1+x^2}$
5. $\frac{d}{dx}(\sec^{-1} x) = \frac{1}{|x|\sqrt{x^2-1}}, |x| > 1$
6. $\frac{d}{dx}(\text{cosec}^{-1} x) = \frac{-1}{|x|\sqrt{x^2-1}}, |x| > 1$

5.1.11 *Exponential and logarithmic functions*

(i) The exponential function with positive base $b > 1$ is the function $y = f(x) = b^x$. Its domain is **R**, the set of all real numbers and range is the set of all positive real numbers. Exponential function with base 10 is called the common exponential function and with base e is called the natural exponential function.

(ii) Let $b > 1$ be a real number. Then we say logarithm of a to base b is x if $b^x = a$, Logarithm of a to the base b is denoted by $\log_b a$. If the base $b = 10$, we say it is common logarithm and if $b = e$, then we say it is natural logarithms. $\log x$ denotes the logarithm function to base e. The domain of logarithm function is $\mathbf{R}^+$, the set of all positive real numbers and the range is the set of all real numbers.

(iii) The properties of logarithmic function to any base $b > 1$ are listed below:

1. $\log_b (xy) = \log_b x + \log_b y$

2. $\log_b \left(\frac{x}{y}\right) = \log_b x - \log_b y$

3. $\log_b x^n = n \log_b x$

4. $\log_b x = \dfrac{\log_c x}{\log_c b}$, where $c > 1$

5. $\log_b x = \dfrac{1}{\log_x b}$

6. $\log_b b = 1$ and $\log_b 1 = 0$

(iv) The derivative of e^x w.r.t., x is e^x , i.e. $\dfrac{d}{dx}(e^x) = e^x$. The derivative of log$x$ w.r.t., x is $\dfrac{1}{x}$; i.e. $\dfrac{d}{dx}(\log x) = \dfrac{1}{x}$.

5.1.12 Logarithmic differentiation is a powerful technique to differentiate functions of the form $f(x) = (u(x))^{v(x)}$, where both f and u need to be positive functions for this technique to make sense.

5.1.13 Differentiation of a function with respect to another function

Let $u = f(x)$ and $v = g(x)$ be two functions of x, then to find derivative of $f(x)$ w.r.t. to $g(x)$, i.e., to find $\dfrac{du}{dv}$, we use the formula

$$\frac{du}{dv} = \frac{\frac{du}{dx}}{\frac{dv}{dx}} .$$

5.1.14 *Second order derivative*

$\dfrac{d}{dx}\left(\dfrac{dy}{dx}\right) = \dfrac{d^2y}{dx^2}$ is called the second order derivative of y w.r.t. x. It is denoted by y'' or y_2 , if $y = f(x)$.

5.1.15 *Rolle's Theorem*

Let $f : [a, b] \rightarrow \mathbf{R}$ be continuous on $[a, b]$ and differentiable on (a, b), such that $f(a) = f(b)$, where a and b are some real numbers. Then there exists at least one point c in (a, b) such that $f'(c) = 0$.

Geometrically Rolle's theorem ensures that there is at least one point on the curve $y = f(x)$ at which tangent is parallel to x-axis (abscissa of the point lying in (a, b)).

5.1.16 *Mean Value Theorem (Lagrange)*

Let $f : [a, b] \to \mathbf{R}$ be a continuous function on $[a, b]$ and differentiable on (a, b). Then there exists at least one point c in (a, b) such that $f'(c) = \dfrac{f(b) - f(a)}{b - a}$.

Geometrically, Mean Value Theorem states that there exists at least one point c in (a, b) such that the tangent at the point $(c, f(c))$ is parallel to the secant joining the points $(a, f(a))$ and $(b, f(b))$.

5.2 Solved Examples

Short Answer (S.A.)

Example 1 Find the value of the constant k so that the function f defined below is continuous at $x = 0$, where $f(x) = \begin{cases} \dfrac{1-\cos 4x}{8x^2}, & x \neq 0 \\ k, & x = 0 \end{cases}$.

Solution It is given that the function f is continuous at $x = 0$. Therefore, $\lim_{x \to 0} f(x) = f(0)$

$$\Rightarrow \quad \lim_{x \to 0} \frac{1-\cos 4x}{8x^2} = k$$

$$\Rightarrow \quad \lim_{x \to 0} \frac{2\sin^2 2x}{8x^2} = k$$

$$\Rightarrow \quad \lim_{x \to 0} \left(\frac{\sin 2x}{2x}\right)^2 = k$$

$$\Rightarrow \quad k = 1$$

Thus, f is continuous at $x = 0$ if $k = 1$.

Example 2 Discuss the continuity of the function $f(x) = \sin x \,.\, \cos x$.

Solution Since $\sin x$ and $\cos x$ are continuous functions and product of two continuous function is a continuous function, therefore $f(x) = \sin x \,.\, \cos x$ is a continuous function.

Example 3 If $f(x)=\begin{cases}\dfrac{x^3+x^2-16x+20}{(x-2)^2}, & x\neq 2\\ k, & x=2\end{cases}$ is continuous at $x = 2$, find the value of k.

Solution Given $f(2) = k$.

Now, $\lim\limits_{x\to 2^-} f(x)=\lim\limits_{x\to 2^+} f(x)=\lim\limits_{x\to 2}\dfrac{x^3+x^2-16x+20}{(x-2)^2}$

$$=\lim_{x\to 2}\frac{(x+5)(x-2)^2}{(x-2)^2}=\lim_{x\to 2}(x+5)=7$$

As f is continuous at $x = 2$, we have

$$\lim_{x\to 2} f(x)=f(2)$$

$\Rightarrow$ $k = 7$.

Example 4 Show that the function f defined by

$$f(x)=\begin{cases}x\sin\dfrac{1}{x}, & x\neq 0\\ 0, & x=0\end{cases}$$

is continuous at $x = 0$.

Solution Left hand limit at $x = 0$ is given by

$$\lim_{x\to 0^-} f(x)=\lim_{x\to 0^-} x\sin\frac{1}{x} = 0 \qquad \left[\text{since, } -1<\sin\frac{1}{x}<1\right]$$

Similarly $\lim\limits_{x\to 0^+} f(x)=\lim\limits_{x\to 0^+} x\sin\dfrac{1}{x}=0$. Moreover $f(0) = 0$.

Thus $\lim\limits_{x\to 0^-} f(x)=\lim\limits_{x\to 0^+} f(x)=f(0)$. Hence f is continuous at $x = 0$

Example 5 Given $f(x) = \dfrac{1}{x-1}$. Find the points of discontinuity of the composite function $y = f[f(x)]$.

Solution We know that $f(x) = \dfrac{1}{x-1}$ is discontinuous at $x = 1$

Now, for $x\neq 1$,

$$f(f(x)) = f\left(\frac{1}{x-1}\right) = \frac{1}{\frac{1}{x-1}-1} = \frac{x-1}{2-x},$$

which is discontinuous at $x = 2$.

Hence, the points of discontinuity are $x = 1$ and $x = 2$.

Example 6 Let $f(x) = x|x|$, for all $x \in \mathbf{R}$. Discuss the derivability of $f(x)$ at $x = 0$

Solution We may rewrite f as $f(x) = \begin{cases} x^2, \text{ if } x \geq 0 \\ -x^2, \text{if } x < 0 \end{cases}$

Now $Lf'(0) = \lim\limits_{h\to 0^-} \frac{f(0+h)-f(0)}{h} = \lim\limits_{h\to 0^-} \frac{-h^2-0}{h} = \lim\limits_{h\to 0^-} -h = 0$

Now $Rf'(0) = \lim\limits_{h\to 0^+} \frac{f(0+h)-f(0)}{h} = \lim\limits_{h\to 0^+} \frac{h^2-0}{h} = \lim\limits_{h\to 0^-} h = 0$

Since the left hand derivative and right hand derivative both are equal, hence f is differentiable at $x = 0$.

Example 7 Differentiate $\sqrt{\tan\sqrt{x}}$ w.r.t. x

Solution Let $y = \sqrt{\tan\sqrt{x}}$. Using chain rule, we have

$$\frac{dy}{dx} = \frac{1}{2\sqrt{\tan\sqrt{x}}} \cdot \frac{d}{dx}(\tan\sqrt{x})$$

$$= \frac{1}{2\sqrt{\tan\sqrt{x}}} \cdot \sec^2\sqrt{x}\, \frac{d}{dx}(\sqrt{x})$$

$$= \frac{1}{2\sqrt{\tan\sqrt{x}}} (\sec^2\sqrt{x}) \left(\frac{1}{2\sqrt{x}}\right)$$

$$= \frac{(\sec^2\sqrt{x})}{4\sqrt{x}\sqrt{\tan\sqrt{x}}}.$$

Example 8 If $y = \tan(x + y)$, find $\frac{dy}{dx}$.

Solution Given $y = \tan(x + y)$. differentiating both sides w.r.t. x, we have

$$\frac{dy}{dx} = \sec^2(x+y)\frac{d}{dx}(x+y)$$

$$= \sec^2(x+y)\left(1+\frac{dy}{dx}\right)$$

or $\quad [1 - \sec^2(x+y)]\frac{dy}{dx} = \sec^2(x+y)$

Therefore, $\quad \dfrac{dy}{dx} = \dfrac{\sec^2(x+y)}{1-\sec^2(x+y)} = -\operatorname{cosec}^2(x+y).$

Example 9 If $e^x + e^y = e^{x+y}$, prove that

$$\frac{dy}{dx} = -e^{y-x}.$$

Solution Given that $e^x + e^y = e^{x+y}$. Differentiating both sides w.r.t. x, we have

$$e^x + e^y\frac{dy}{dx} = e^{x+y}\left(1+\frac{dy}{dx}\right)$$

or $\quad (e^y - e^{x+y})\dfrac{dy}{dx} = e^{x+y} - e^x,$

which implies that $\dfrac{dy}{dx} = \dfrac{e^{x+y}-e^x}{e^y-e^{x+y}} = \dfrac{e^x+e^y-e^x}{e^y-e^x-e^y} = -e^{y-x}.$

Example 10 Find $\dfrac{dy}{dx}$, if $y = \tan^{-1}\left(\dfrac{3x-x^3}{1-3x^2}\right), -\dfrac{1}{\sqrt{3}} < x < \dfrac{1}{\sqrt{3}}$.

Solution Put $x = \tan\theta$, where $\dfrac{-\pi}{6} < \theta < \dfrac{\pi}{6}$.

Therefore, $\quad y = \tan^{-1}\left(\dfrac{3\tan\theta - \tan^3\theta}{1-3\tan^2\theta}\right)$

$= \tan^{-1}(\tan 3\theta)$

$= 3\theta \qquad$ (because $\dfrac{-\pi}{2} < 3\theta < \dfrac{\pi}{2}$)

$= 3\tan^{-1}x$

Hence, $\dfrac{dy}{dx} = \dfrac{3}{1+x^2}$.

Example 11 If $y = \sin^{-1}\left\{x\sqrt{1-x} - \sqrt{x}\sqrt{1-x^2}\right\}$ and $0 < x < 1$, then find $\dfrac{dy}{dx}$.

Solution We have $y = \sin^{-1}\left\{x\sqrt{1-x} - \sqrt{x}\sqrt{1-x^2}\right\}$, where $0 < x < 1$.

Put $x = \sin A$ and $\sqrt{x} = \sin B$

Therefore, $y = \sin^{-1}\left\{\sin A\sqrt{1-\sin^2 B} - \sin B\sqrt{1-\sin^2 A}\right\}$

$$= \sin^{-1}\left\{\sin A\cos B - \sin B\cos A\right\}$$

$$= \sin^{-1}\left\{\sin(A-B)\right\} = A - B$$

Thus $y = \sin^{-1} x - \sin^{-1}\sqrt{x}$

Differentiating w.r.t. x, we get

$$\frac{dy}{dx} = \frac{1}{\sqrt{1-x^2}} - \frac{1}{\sqrt{1-\left(\sqrt{x}\right)^2}} \cdot \frac{d}{dx}\left(\sqrt{x}\right)$$

$$= \frac{1}{\sqrt{1-x^2}} - \frac{1}{2\sqrt{x}\sqrt{1-x}}.$$

Example 12 If $x = a\sec^3\theta$ and $y = a\tan^3\theta$, find $\dfrac{dy}{dx}$ at $\theta = \dfrac{\pi}{3}$.

Solution We have $x = a\sec^3\theta$ and $y = a\tan^3\theta$.

Differentiating w.r.t. θ, we get

$$\frac{dx}{d\theta} = 3a\sec^2\theta\frac{d}{d\theta}(\sec\theta) = 3a\sec^3\theta\tan\theta$$

and $$\frac{dy}{d\theta} = 3a\tan^2\theta\frac{d}{d\theta}(\tan\theta) = 3a\tan^2\theta\sec^2\theta.$$

Thus $$\frac{dy}{dx} = \frac{\frac{dy}{d\theta}}{\frac{dx}{d\theta}} = \frac{3a\tan^2\theta\sec^2\theta}{3a\sec^3\theta\tan\theta} = \frac{\tan\theta}{\sec\theta} = \sin\theta.$$

Hence, $\left(\frac{dy}{dx}\right)_{at\,\theta=\frac{\pi}{3}} = \sin\frac{\pi}{3} = \frac{\sqrt{3}}{2}$.

Example 13 If $x^y = e^{x-y}$, prove that $\frac{dy}{dx} = \frac{\log x}{(1+\log x)^2}$.

Solution We have $x^y = e^{x-y}$. Taking logarithm on both sides, we get

$$y \log x = x - y$$

$$\Rightarrow \quad y\,(1 + \log x) = x$$

i.e. $$y = \frac{x}{1+\log x}$$

Differentiating both sides w.r.t. x, we get

$$\frac{dy}{dx} = \frac{(1+\log x).1 - x\left(\frac{1}{x}\right)}{(1+\log x)^2} = \frac{\log x}{(1+\log x)^2}.$$

Example 14 If $y = \tan x + \sec x$, prove that $\frac{d^2y}{dx^2} = \frac{\cos x}{(1-\sin x)^2}$.

Solution We have $y = \tan x + \sec x$. Differentiating w.r.t. x, we get

$$\frac{dy}{dx} = \sec^2 x + \sec x \tan x$$

$$= \frac{1}{\cos^2 x} + \frac{\sin x}{\cos^2 x} = \frac{1+\sin x}{\cos^2 x} = \frac{1+\sin x}{(1+\sin x)(1-\sin x)}.$$

thus $$\frac{dy}{dx} = \frac{1}{1-\sin x}.$$

Now, differentiating again w.r.t. x, we get

$$\frac{d^2y}{dx^2} = \frac{-(-\cos x)}{(1-\sin x)^2} = \frac{\cos x}{(1-\sin x)^2}$$

Example 15 If $f(x) = |\cos x|$, find $f'\left(\frac{3\pi}{4}\right)$.

Solution When $\frac{\pi}{2} < x < \pi$, $\cos x < 0$ so that $|\cos x| = -\cos x$, i.e., $f(x) = -\cos x$

$\Rightarrow f'(x) = \sin x$.

Hence, $f'\left(\frac{3\pi}{4}\right) = \sin\left(\frac{3\pi}{4}\right) = \frac{1}{\sqrt{2}}$

Example 16 If $f(x) = |\cos x - \sin x|$, find $f'\left(\frac{\pi}{6}\right)$.

Solution When $0 < x < \frac{\pi}{4}$, $\cos x > \sin x$, so that $\cos x - \sin x > 0$, i.e.,

$f(x) = \cos x - \sin x$

$\Rightarrow f'(x) = -\sin x - \cos x$

Hence $f'\left(\frac{\pi}{6}\right) = -\sin\frac{\pi}{6} - \cos\frac{\pi}{6} = -\frac{1}{2}(1+\sqrt{3})$.

Example 17 Verify Rolle's theorem for the function, $f(x) = \sin 2x$ in $\left[0, \frac{\pi}{2}\right]$.

Solution Consider $f(x) = \sin 2x$ in $\left[0, \frac{\pi}{2}\right]$. Note that:

(i) The function f is continuous in $\left[0, \frac{\pi}{2}\right]$, as f is a sine function, which is always continuous.

(ii) $f'(x) = 2\cos 2x$, exists in $\left(0, \frac{\pi}{2}\right)$, hence f is derivable in $\left(0, \frac{\pi}{2}\right)$.

(iii) $f(0) = \sin 0 = 0$ and $f\left(\frac{\pi}{2}\right) = \sin\pi = 0 \Rightarrow f(0) = f\left(\frac{\pi}{2}\right)$.

Conditions of Rolle's theorem are satisfied. Hence there exists at least one $c \in \left(0, \frac{\pi}{2}\right)$ such that $f'(c) = 0$. Thus

$$2\cos 2c = 0 \quad \Rightarrow \quad 2c = \frac{\pi}{2} \quad \Rightarrow \quad c = \frac{\pi}{4}.$$

Example 18 Verify mean value theorem for the function $f(x) = (x-3)(x-6)(x-9)$ in [3, 5].

Solution (i) Function f is continuous in [3, 5] as product of polynomial functions is a polynomial, which is continuous.

(ii) $f'(x) = 3x^2 - 36x + 99$ exists in (3, 5) and hence derivable in (3, 5).

Thus conditions of mean value theorem are satisfied. Hence, there exists at least one $c \in (3, 5)$ such that

$$f'(c) = \frac{f(5) - f(3)}{5-3}$$

$$\Rightarrow 3c^2 - 36c + 99 = \frac{8-0}{2} = 4$$

$$\Rightarrow c = 6 \pm \sqrt{\frac{13}{3}}.$$

Hence $c = 6 - \sqrt{\frac{13}{3}}$ (since other value is not permissible).

Long Answer (L.A.)

Example 19 If $f(x) = \dfrac{\sqrt{2}\cos x - 1}{\cot x - 1}, x \neq \dfrac{\pi}{4}$

find the value of $f\left(\dfrac{\pi}{4}\right)$ so that $f(x)$ becomes continuous at $x = \dfrac{\pi}{4}$.

Solution Given, $f(x) = \dfrac{\sqrt{2}\cos x - 1}{\cot x - 1}, x \neq \dfrac{\pi}{4}$

Therefore,
$$\lim_{x \to \frac{\pi}{4}} f(x) = \lim_{x \to \frac{\pi}{4}} \frac{\sqrt{2}\cos x - 1}{\cot x - 1}$$

$$= \lim_{x \to \frac{\pi}{4}} \frac{\left(\sqrt{2}\cos x - 1\right)\sin x}{\cos x - \sin x}$$

$$= \lim_{x \to \frac{\pi}{4}} \frac{\left(\sqrt{2}\cos x - 1\right)}{\left(\sqrt{2}\cos x + 1\right)} \cdot \frac{\left(\sqrt{2}\cos x + 1\right)}{(\cos x - \sin x)} \cdot \frac{(\cos x + \sin x)}{(\cos x + \sin x)} . \sin x$$

$$= \lim_{x\to\frac{\pi}{4}} \frac{2\cos^2 x - 1}{\cos^2 x - \sin^2 x} \cdot \frac{\cos x + \sin x}{\sqrt{2}\cos x + 1} . (\sin x)$$

$$= \lim_{x\to\frac{\pi}{4}} \frac{\cos 2x}{\cos 2x} \cdot \left(\frac{\cos x + \sin x}{\sqrt{2}\cos x + 1}\right) . (\sin x)$$

$$= \lim_{x\to\frac{\pi}{4}} \frac{(\cos x + \sin x)}{\sqrt{2}\cos x + 1} \sin x$$

$$= \frac{\frac{1}{\sqrt{2}}\left(\frac{1}{\sqrt{2}} + \frac{1}{\sqrt{2}}\right)}{\sqrt{2}.\frac{1}{\sqrt{2}} + 1} = \frac{1}{2}$$

Thus, $\displaystyle \lim_{x\to\frac{\pi}{4}} f(x) = \frac{1}{2}$

If we define $f\left(\frac{\pi}{4}\right) = \frac{1}{2}$, then $f(x)$ will become continuous at $x = \frac{\pi}{4}$. Hence for f to be continuous at $x = \frac{\pi}{4}$, $f\left(\frac{\pi}{4}\right) = \frac{1}{2}$.

Example 20 Show that the function f given by $f(x) = \begin{cases} \dfrac{e^{\frac{1}{x}} - 1}{e^{\frac{1}{x}} + 1}, & if\ x \neq 0 \\ 0, & if\ x = 0 \end{cases}$

is discontinuous at $x = 0$.

Solution The left hand limit of f at $x = 0$ is given by

$$\lim_{x\to 0^-} f(x) = \lim_{x\to 0^-} \frac{e^{\frac{1}{x}} - 1}{e^{\frac{1}{x}} + 1} = \frac{0-1}{0+1} = -1$$

Similarly, $$\lim_{x\to 0^+} f(x) = \lim_{x\to 0^+} \frac{e^{\frac{1}{x}}-1}{e^{\frac{1}{x}}+1}$$

$$= \lim_{x\to 0^+} \frac{1-\frac{1}{e^{\frac{1}{x}}}}{1+\frac{1}{e^{\frac{1}{x}}}} = \lim_{x\to 0^+} \frac{1-e^{\frac{-1}{x}}}{1+e^{\frac{-1}{x}}} = \frac{1-0}{1+0} = 1$$

Thus $\lim_{x\to 0^-} f(x) \neq \lim_{x\to 0^+} f(x)$, therefore, $\lim_{x\to 0} f(x)$ does not exist. Hence f is discontinuous at $x = 0$.

Example 21 Let $f(x) = \begin{cases} \frac{1-\cos 4x}{x^2}, & if\ x<0 \\ a, & if\ x=0 \\ \frac{\sqrt{x}}{\sqrt{16+\sqrt{x}}-4}, & if\ x>0 \end{cases}$

For what value of a, f is continuous at $x = 0$?

Solution Here $f(0) = a$ Left hand limit of f at 0 is

$$\lim_{x\to 0^-} f(x) = \lim_{x\to 0^-} \frac{1-\cos 4x}{x^2} = \lim_{x\to 0^-} \frac{2\sin^2 2x}{x^2}$$

$$= \lim_{2x\to 0^-} 8\left(\frac{\sin 2x}{2x}\right)^2 = 8\,(1)^2 = 8.$$

and right hand limit of f at 0 is

$$\lim_{x\to 0^+} f(x) = \lim_{x\to 0^+} \frac{\sqrt{x}}{\sqrt{16+\sqrt{x}}-4}$$

$$= \lim_{x\to 0^+} \frac{\sqrt{x}\,(\sqrt{16+\sqrt{x}}+4)}{(\sqrt{16+\sqrt{x}}+4)(\sqrt{16+\sqrt{x}}-4)}$$

$$= \lim_{x\to 0^+} \frac{\sqrt{x}\,(\sqrt{16+\sqrt{x}}+4)}{16+\sqrt{x}-16} = \lim_{x\to 0^+}\left(\sqrt{16+\sqrt{x}}+4\right) = 8$$

Thus, $\lim_{x\to 0^+} f(x) = \lim_{x\to 0^-} f(x) = 8$. Hence f is continuous at $x = 0$ only if $a = 8$.

Example 22 Examine the differentiability of the function f defined by

$$f(x) = \begin{cases} 2x+3, \text{ if } -3 \le x < -2 \\ x+1 \;\;, \text{ if } -2 \le x < 0 \\ x+2 \;\;, \text{ if } 0 \le x \le 1 \end{cases}$$

Solution The only doubtful points for differentiability of $f(x)$ are $x = -2$ and $x = 0$.
Differentiability at $x = -2$.

Now $\text{L} f'(-2) = \lim_{h\to 0^-} \frac{f(-2+h) - f(-2)}{h}$

$$= \lim_{h\to 0^-} \frac{2(-2+h)+3-(-2+1)}{h} = \lim_{h\to 0^-} \frac{2h}{h} = \lim_{h\to 0^-} 2 = 2.$$

and $\text{R} f'(-2) = \lim_{h\to 0^+} \frac{f(-2+h) - f(-2)}{h}$

$$= \lim_{h\to 0^-} \frac{-2+h+1-(-2+1)}{h}$$

$$= \lim_{h\to 0^-} \frac{h-1-(-1)}{h} = \lim_{h\to 0^+} \frac{h}{h} = 1$$

Thus $\text{R} f'(-2) \neq \text{L} f'(-2)$. Therefore f is not differentiable at $x = -2$.
Similarly, for differentiability at $x = 0$, we have

$$\text{L}\,(f'(0) = \lim_{h\to 0^-} \frac{f(0+h) - f(0)}{h}$$

$$= \lim_{h\to 0^-} \frac{0+h+1-(0+2)}{h}$$

$$= \lim_{h\to 0^-} \frac{h-1}{h} = \lim_{h\to 0^-} \left(1 - \frac{1}{h}\right)$$

which does not exist. Hence f is not differentiable at $x = 0$.

Example 23 Differentiate $\tan^{-1}\left(\frac{\sqrt{1-x^2}}{x}\right)$ with respect to $\cos^{-1}\left(2x\sqrt{1-x^2}\right)$, where $x \in \left(\frac{1}{\sqrt{2}}, 1\right)$.

Solution Let $u = \tan^{-1}\left(\frac{\sqrt{1-x^2}}{x}\right)$ and $v = \cos^{-1}\left(2x\sqrt{1-x^2}\right)$.

We want to find $\frac{du}{dv} = \frac{\frac{du}{dx}}{\frac{dv}{dx}}$

Now $u = \tan^{-1}\left(\frac{\sqrt{1-x^2}}{x}\right)$. Put $x = \sin\theta$. $\left(\frac{\pi}{4} < \theta < \frac{\pi}{2}\right)$.

Then $u = \tan^{-1}\left(\frac{\sqrt{1-\sin^2\theta}}{\sin\theta}\right) = \tan^{-1}(\cot\theta)$

$$= \tan^{-1}\left\{\tan\left(\frac{\pi}{2} - \theta\right)\right\} = \frac{\pi}{2} - \theta = \frac{\pi}{2} - \sin^{-1}x$$

Hence $\frac{du}{dx} = \frac{-1}{\sqrt{1-x^2}}$.

Now $v = \cos^{-1}(2x\sqrt{1-x^2})$

$$= \frac{\pi}{2} - \sin^{-1}(2x\sqrt{1-x^2})$$

$$= \frac{\pi}{2} - \sin^{-1}(2\sin\theta\sqrt{1-\sin^2\theta}) = \frac{\pi}{2} - \sin^{-1}(\sin 2\theta)$$

$$= \frac{\pi}{2} - \sin^{-1}\{\sin(\pi - 2\theta)\} \quad \left[\text{since } \frac{\pi}{2} < 2\theta < \pi\right]$$

$$= \frac{\pi}{2} - (\pi - 2\theta) = \frac{-\pi}{2} + 2\theta$$

$\Rightarrow$ $$v = \frac{-\pi}{2} + 2\sin^{-1}x$$

$\Rightarrow$ $$\frac{dv}{dx} = \frac{2}{\sqrt{1-x^2}}.$$

Hence $$\frac{du}{dv} = \frac{\frac{du}{dx}}{\frac{dv}{dx}} = \frac{\frac{-1}{\sqrt{1-x^2}}}{\frac{2}{\sqrt{1-x^2}}} = \frac{-1}{2}.$$

Objective Type Questions

Choose the correct answer from the given four options in each of the Examples 24 to 35.

Example 24 The function $f(x) = \begin{cases} \frac{\sin x}{x} + \cos x, & \text{if } x \neq 0 \\ k & \text{, if } x = 0 \end{cases}$

is continuous at $x = 0$, then the value of k is

(A) 3 (B) 2

(C) 1 (D) 1.5

Solution (B) is the Correct answer.

Example 25 The function $f(x) = [x]$, where $[x]$ denotes the greatest integer function, is continuous at

(A) 4 (B) – 2

(C) 1 (D) 1.5

Solution (D) is the correct answer. The greatest integer function$[x]$ is discontinuous at all integral values of x. Thus D is the correct answer.

Example 26 The number of points at which the function $f(x) = \frac{1}{x-[x]}$ is not continuous is

(A) 1 (B) 2

(C) 3 (D) none of these

Solution (D) is the correct answer. As $x - [x] = 0$, when x is an integer so $f(x)$ is discontinuous for all $x \in \mathbf{Z}$.

Example 27 The function given by $f(x) = \tan x$ is discontinuous on the set

(A) $\{n\pi : n \in \mathbf{Z}\}$ (B) $\{2n\pi : n \in \mathbf{Z}\}$

(C) $\left\{(2n+1)\frac{\pi}{2} : n \in \mathbf{Z}\right\}$ (D) $\left\{\frac{n\pi}{2} : n \in \mathbf{Z}\right\}$

Solution C is the correct answer.

Example 28 Let $f(x) = |\cos x|$. Then,

(A) f is everywhere differentiable.

(B) f is everywhere continuous but not differentiable at $n = n\pi$, $n \in \mathbf{Z}$.

(C) f is everywhere continuous but not differentiable at $x = (2n + 1)\frac{\pi}{2}$, $n \in \mathbf{Z}$.

(D) none of these.

Solution C is the correct answer.

Example 29 The function $f(x) = |x| + |x - 1|$ is

(A) continuous at $x = 0$ as well as at $x = 1$.

(B) continuous at $x = 1$ but not at $x = 0$.

(C) discontinuous at $x = 0$ as well as at $x = 1$.

(D) continuous at $x = 0$ but not at $x = 1$.

Solution Correct answer is A.

Example 30 The value of k which makes the function defined by

$$f(x) = \begin{cases} \sin\frac{1}{x}, & \text{if } x \neq 0 \\ k, & \text{if } x = 0 \end{cases}, \text{ continuous at } x = 0 \text{ is}$$

(A) 8 (B) 1

(C) –1 (D) none of these

Solution (D) is the correct answer. Indeed $\lim_{x \to 0} \sin\frac{1}{x}$ does not exist.

Example 31 The set of points where the functions f given by $f(x) = |x - 3| \cos x$ is differentiable is

(A) **R** (B) **R** – {3}
(C) (0, ∞) (D) none of these

Solution B is the correct answer.

Example 32 Differential coefficient of sec ($\tan^{-1}x$) w.r.t. x is

(A) $\dfrac{x}{\sqrt{1+x^2}}$ (B) $\dfrac{x}{1+x^2}$

(C) $x\sqrt{1+x^2}$ (D) $\dfrac{1}{\sqrt{1+x^2}}$

Solution (A) is the correct answer.

Example 33 If $u = \sin^{-1}\left(\dfrac{2x}{1+x^2}\right)$ and $v = \tan^{-1}\left(\dfrac{2x}{1-x^2}\right)$, then $\dfrac{du}{dv}$ is

(A) $\dfrac{1}{2}$ (B) x (C) $\dfrac{1-x^2}{1+x^2}$ (D) 1

Solution (D) is the correct answer.

Example 34 The value of c in Rolle's Theorem for the function $f(x) = e^x \sin x$, $x \in [0, \pi]$ is

(A) $\dfrac{\pi}{6}$ (B) $\dfrac{\pi}{4}$ (C) $\dfrac{\pi}{2}$ (D) $\dfrac{3\pi}{4}$

Solution (D) is the correct answer.

Example 35 The value of c in Mean value theorem for the function $f(x) = x(x-2)$, $x \in [1, 2]$ is

(A) $\dfrac{3}{2}$ (B) $\dfrac{2}{3}$ (C) $\dfrac{1}{2}$ (D) $\dfrac{3}{2}$

Solution (A) is the correct answer.

Example 36 Match the following

COLUMN-I	COLUMN-II
(A) If a function $f(x)=\begin{cases}\dfrac{\sin 3x}{x}, & if\ x \neq 0\\ \dfrac{k}{2}, & \text{if } x = 0\end{cases}$ is continuous at $x = 0$, then k is equal to	(a) $\lvert x\rvert$

(B)	Every continuous function is differentiable	(b) True
(C)	An example of a function which is continuous everywhere but not differentiable at exactly one point	(c) 6
(D)	The identity function i.e. $f(x) = x \ \forall x \in R$ is a continuous function	(d) False

Solution A $\to$ *c*, B $\to$ *d*, C $\to$ *a*, D $\to$ *b*

Fill in the blanks in each of the Examples 37 to 41.

Example 37 The number of points at which the function $f(x) = \frac{1}{\log |x|}$ is discontinuous is ________.

Solution The given function is discontinuous at $x = 0, \pm 1$ and hence the number of points of discontinuity is 3.

Example 38 If $f(x) = \begin{cases} ax + 1 & \text{if } x \geq 1 \\ x + 2 & \text{if } x < 1 \end{cases}$ is continuous, then *a* should be equal to _______.

Solution $a = 2$

Example 39 The derivative of $\log_{10} x$ w.r.t. *x* is ________.

Solution $(\log_{10} e)\frac{1}{x}$.

Example 40 If $y = \sec^{-1}\left(\frac{\sqrt{x}+1}{\sqrt{x}-1}\right) + \sin^{-1}\left(\frac{\sqrt{x}-1}{\sqrt{x}+1}\right)$, then $\frac{dy}{dx}$ is equal to ______.

Solution 0.

Example 41 The deriative of sin *x* w.r.t. cos *x* is ________.

Solution $-\cot x$

State whether the statements are True or False in each of the Exercises 42 to 46.

Example 42 For continuity, at $x = a$, each of $\lim_{x \to a^+} f(x)$ and $\lim_{x \to a^-} f(x)$ is equal to $f(a)$.

Solution True.

Example 43 $y = |x - 1|$ is a continuous function.

Solution True.

Example 44 A continuous function can have some points where limit does not exist.

Solution False.

Example 45 $|\sin x|$ is a differentiable function for every value of *x*.

Solution False.

Example 46 cos $|x|$ is differentiable everywhere.

Solution True.

5.3 EXERCISE

Short Answer (S.A.)

1. Examine the continuity of the function

$f(x) = x^3 + 2x^2 - 1$ at $x = 1$

Find which of the functions in Exercises 2 to 10 is continuous or discontinuous at the indicated points:

2. $f(x)=\begin{cases} 3x+5, \text{ if } x\geq 2 \\ x^2, \text{ if } x<2 \end{cases}$

at $x=2$

3. $f(x)=\begin{cases} \dfrac{1-\cos 2x}{x^2}, & if\ x\neq 0 \\ 5, & if\ x=0 \end{cases}$

at $x=0$

4. $f(x)=\begin{cases} \dfrac{2x^2-3x-2}{x-2}, & if\ x\neq 2 \\ 5, & if\ x=2 \end{cases}$

at $x=2$

5. $f(x)=\begin{cases} \dfrac{|x-4|}{2(x-4)}, & if\ x\neq 4 \\ 0, & if\ x=4 \end{cases}$

at $x = 4$

6. $f(x)=\begin{cases} |x|\cos\dfrac{1}{x}, & if\ x\neq 0 \\ 0, & if\ x=0 \end{cases}$

at $x = 0$

7. $f(x)=\begin{cases} |x-a|\sin\dfrac{1}{x-a}, & if\ x\neq 0 \\ 0, & if\ x=a \end{cases}$

at $x = a$

8. $f(x)=\begin{cases} \dfrac{e^{\frac{1}{x}}}{1+e^{\frac{1}{x}}}, & if\ x\neq 0 \\ 0, & if\ x=0 \end{cases}$

at $x = 0$

9. $f(x)=\begin{cases} \dfrac{x^2}{2}, & if\ 0\leq x\leq 1 \\ 2x^2-3x+\dfrac{3}{2}, & if\ 1<x\leq 2 \end{cases}$

at $x = 1$

10. $f(x)=|x|+|x-1|$ at $x = 1$

Find the value of *k* in each of the Exercises 11 to 14 so that the function *f* is continuous at the indicated point:

11. $f(x)=\begin{cases}3x-8, & \text{if } x\le 5\\ 2k, & \text{if } x>5\end{cases}$ at $x=5$

12. $f(x)=\begin{cases}\dfrac{2^{x+2}-16}{4^x-16}, & \text{if } x\ne 2\\ k, & \text{if } x=2\end{cases}$ at $x=2$

13. $f(x)=\begin{cases}\dfrac{\sqrt{1+kx}-\sqrt{1-kx}}{x}, & \text{if } -1\le x<0\\ \dfrac{2x+1}{x-1}, & \text{if } 0\le x\le 1\end{cases}$ at $x = 0$

14. $f(x)=\begin{cases}\dfrac{1-\cos kx}{x\sin x}, & \text{if } x\ne 0\\ \dfrac{1}{2}, & \text{if } x=0\end{cases}$ at $x = 0$

15. Prove that the function *f* defined by

$$f(x)=\begin{cases}\dfrac{x}{|x|+2x^2}, & x\ne 0\\ k, & x=0\end{cases}$$

remains discontinuous at $x = 0$, regardless the choice of *k*.

16. Find the values of *a* and *b* such that the function *f* defined by

$$f(x)=\begin{cases}\dfrac{x-4}{|x-4|}+a, & \text{if } x<4\\ a+b, & \text{if } x=4\\ \dfrac{x-4}{|x-4|}+b, & \text{if } x>4\end{cases}$$

is a continuous function at $x = 4$.

17. Given the function $f(x)=\dfrac{1}{x+2}$. Find the points of discontinuity of the composite function $y = f(f(x))$.

18. Find all points of discontinuity of the function $f(t)=\dfrac{1}{t^2+t-2}$, where $t=\dfrac{1}{x-1}$.

19. Show that the function $f(x) = |\sin x+\cos x|$ is continuous at $x = \pi$.

Examine the differentiability of f, where f is defined by

20. $f(x) = \begin{cases} x[x], & \text{, if } 0\le x<2 \\ (x-1)x, & \text{if } 2\le x<3 \end{cases}$

at $x = 2$.

21. $f(x) = \begin{cases} x^2 \sin\dfrac{1}{x} & \text{, if } x \neq 0 \\ 0 & \text{, if } x = 0 \end{cases}$

at $x = 0$.

22. $f(x) = \begin{cases} 1+x & \text{, if } x \le 2 \\ 5-x & \text{, if } x > 2 \end{cases}$

at $x = 2$.

23. Show that $f(x) = |x-5|$ is continuous but not differentiable at $x = 5$.

24. A function $f : \mathbf{R} \to \mathbf{R}$ satisfies the equation $f(x + y) = f(x) f(y)$ for all $x, y \in \mathbf{R}$, $f(x) \neq 0$. Suppose that the function is differentiable at $x = 0$ and $f'(0) = 2$. Prove that $f'(x) = 2 f(x)$.

Differentiate each of the following w.r.t. x (Exercises 25 to 43) :

25. $2^{\cos^2 x}$

26. $\dfrac{8^x}{x^8}$

27. $\log\left(x+\sqrt{x^2+a}\right)$

28. $\log\left[\log\left(\log x^5\right)\right]$

29. $\sin\sqrt{x}+\cos^2\sqrt{x}$

30. $\sin^n(ax^2+bx+c)$

31. $\cos\left(\tan\sqrt{x+1}\right)$

32. $\sin x^2 + \sin^2 x + \sin^2(x^2)$

33. $\sin^{-1}\left(\dfrac{1}{\sqrt{x+1}}\right)$

34. $(\sin x)^{\cos x}$

35. $\sin^m x \, . \, \cos^n x$

36. $(x + 1)^2 (x + 2)^3 (x + 3)^4$

37. $\cos^{-1}\left(\frac{\sin x + \cos x}{\sqrt{2}}\right), \frac{-\pi}{4} < x < \frac{\pi}{4}$

38. $\tan^{-1}\left(\sqrt{\frac{1-\cos x}{1+\cos x}}\right), -\frac{\pi}{4} < x < \frac{\pi}{4}$

39. $\tan^{-1}(\sec x + \tan x), -\frac{\pi}{2} < x < \frac{\pi}{2}$

40. $\tan^{-1}\left(\frac{a\cos x - b\sin x}{b\cos x + a\sin x}\right), -\frac{\pi}{2} < x < \frac{\pi}{2}$ and $\frac{a}{b}\tan x > -1$

41. $\sec^{-1}\left(\frac{1}{4x^3 - 3x}\right), 0 < x < \frac{1}{\sqrt{2}}$

42. $\tan^{-1}\left(\frac{3a^2x - x^3}{a^3 - 3ax^2}\right), \frac{-1}{\sqrt{3}} < \frac{x}{a} < \frac{1}{\sqrt{3}}$

43. $\tan^{-1}\left(\frac{\sqrt{1+x^2}+\sqrt{1-x^2}}{\sqrt{1+x^2}-\sqrt{1-x^2}}\right), -1 < x < 1, x \neq 0$

Find $\frac{dy}{dx}$ of each of the functions expressed in parametric form in Exercises from 44 to 48.

44. $x = t + \frac{1}{t}, \; y = t - \frac{1}{t}$

45. $x = e^{\theta}\left(\theta + \frac{1}{\theta}\right), \; y = e^{-\theta}\left(\theta - \frac{1}{\theta}\right)$

46. $x = 3\cos\theta - 2\cos^3\theta, \; y = 3\sin\theta - 2\sin^3\theta.$

47. $\sin x = \frac{2t}{1+t^2}, \; \tan y = \frac{2t}{1-t^2}.$

48. $x = \frac{1+\log t}{t^2}, \; y = \frac{3+2\log t}{t}.$

49. If $x = e^{\cos 2t}$ and $y = e^{\sin 2t}$, prove that $\frac{dy}{dx} = \frac{-y\log x}{x\log y}$.

50. If $x = a\sin 2t\,(1 + \cos 2t)$ and $y = b\cos 2t\,(1 - \cos 2t)$, show that $\left(\frac{dy}{dx}\right)_{\text{at } t=\frac{\pi}{4}} = \frac{b}{a}$.

51. If $x = 3\sin t - \sin 3t$, $y = 3\cos t - \cos 3t$, find $\frac{dy}{dx}$ at $t = \frac{\pi}{3}$.

52. Differentiate $\frac{x}{\sin x}$ w.r.t. $\sin x$.

53. Differentiate $\tan^{-1}\left(\frac{\sqrt{1+x^2}-1}{x}\right)$ w.r.t. $\tan^{-1} x$ when $x \neq 0$.

Find $\frac{dy}{dx}$ when x and y are connected by the relation given in each of the Exercises 54 to 57.

54. $\sin(xy) + \frac{x}{y} = x^2 - y$

55. $\sec(x + y) = xy$

56. $\tan^{-1}(x^2 + y^2) = a$

57. $(x^2 + y^2)^2 = xy$

58. If $ax^2 + 2hxy + by^2 + 2gx + 2fy + c = 0$, then show that $\frac{dy}{dx}.\frac{dx}{dy} = 1$.

59. If $x = e^{\frac{x}{y}}$, prove that $\frac{dy}{dx} = \frac{x-y}{x\log x}$.

60. If $y^x = e^{y-x}$, prove that $\frac{dy}{dx} = \frac{(1+\log y)^2}{\log y}$.

61. If $y = (\cos x)^{(\cos x)^{(\cos x)\ldots\ldots\infty}}$, show that $\frac{dy}{dx} = \frac{y^2 \tan x}{y\log\cos x - 1}$.

62. If $x\sin(a + y) + \sin a\cos(a + y) = 0$, prove that $\frac{dy}{dx} = \frac{\sin^2(a+y)}{\sin a}$.

63. If $\sqrt{1-x^2} + \sqrt{1-y^2} = a(x - y)$, prove that $\frac{dy}{dx} = \sqrt{\frac{1-y^2}{1-x^2}}$.

64. If $y = \tan^{-1}x$, find $\frac{d^2y}{dx^2}$ in terms of y alone.

Verify the Rolle's theorem for each of the functions in Exercises 65 to 69.

65. $f(x) = x(x-1)^2$ in [0, 1].

66. $f(x) = \sin^4 x + \cos^4 x$ in $\left[0, \frac{\pi}{2}\right]$.

67. $f(x) = \log(x^2 + 2) - \log 3$ in [–1, 1].

68. $f(x) = x(x+3)e^{-x/2}$ in [–3, 0].

69. $f(x) = \sqrt{4 - x^2}$ in [– 2, 2].

70. Discuss the applicability of Rolle's theorem on the function given by

$$f(x) = \begin{cases} x^2 + 1, & \text{if } 0 \le x \le 1 \\ 3 - x, & \text{if } 1 \le x \le 2 \end{cases}.$$

71. Find the points on the curve $y = (\cos x - 1)$ in $[0, 2\pi]$, where the tangent is parallel to x-axis.

72. Using Rolle's theorem, find the point on the curve $y = x(x-4)$, $x \in [0, 4]$, where the tangent is parallel to x-axis.

Verify mean value theorem for each of the functions given Exercises 73 to 76.

73. $f(x) = \frac{1}{4x-1}$ in [1, 4].

74. $f(x) = x^3 - 2x^2 - x + 3$ in [0, 1].

75. $f(x) = \sin x - \sin 2x$ in $[0, \pi]$.

76. $f(x) = \sqrt{25 - x^2}$ in [1, 5].

77. Find a point on the curve $y = (x-3)^2$, where the tangent is parallel to the chord joining the points (3, 0) and (4, 1).

78. Using mean value theorem, prove that there is a point on the curve $y = 2x^2 - 5x + 3$ between the points A(1, 0) and B (2, 1), where tangent is parallel to the chord AB. Also, find that point.

Long Answer (L.A.)

79. Find the values of p and q so that

$$f(x)=\begin{cases} x^2+3x+p, & if \ x \le 1 \\ qx+2, & if \ x > 1 \end{cases}$$

is differentiable at $x = 1$.

80. If $x^m.y^n = (x + y)^{m+n}$, prove that

(i) $\frac{dy}{dx}=\frac{y}{x}$ and (ii) $\frac{d^2y}{dx^2}=0$.

81. If $x = \sin t$ and $y = \sin pt$, prove that $(1-x^2)\frac{d^2y}{dx^2} - x\frac{dy}{dx}+p^2y=0$.

82. Find $\frac{dy}{dx}$, if $y = x^{\tan x} + \sqrt{\frac{x^2+1}{2}}$.

Objective Type Questions

Choose the correct answers from the given four options in each of the Exercises 83 to 96.

83. If $f(x) = 2x$ and $g(x) = \frac{x^2}{2}+1$, then which of the following can be a discontinuous function

(A) $f(x) + g(x)$ (B) $f(x) - g(x)$

(C) $f(x) \,.\, g(x)$ (D) $\frac{g(x)}{f(x)}$

84. The function $f(x) = \frac{4-x^2}{4x-x^3}$ is

(A) discontinuous at only one point

(B) discontinuous at exactly two points

(C) discontinuous at exactly three points

(D) none of these

85. The set of points where the function f given by $f(x) = |2x-1| \sin x$ is differentiable is

(A) $\mathbf{R}$ (B) $\mathbf{R} - \left\{\frac{1}{2}\right\}$

(C) $(0, \infty)$ (D) none of these

86. The function $f(x) = \cot x$ is discontinuous on the set

(A) $\{x = n\pi : n \in \mathbf{Z}\}$ (B) $\{x = 2n\pi : n \in \mathbf{Z}\}$

(C) $\left\{x = (2n+1)\frac{\pi}{2} ; n \in \mathbf{Z}\right\}$ (iv) $\left\{x = \frac{n\pi}{2} ; n \in \mathbf{Z}\right\}$

87. The function $f(x) = e^{|x|}$ is

(A) continuous everywhere but not differentiable at $x = 0$
(B) continuous and differentiable everywhere
(C) not continuous at $x = 0$
(D) none of these.

88. If $f(x) = x^2 \sin\frac{1}{x}$, where $x \neq 0$, then the value of the function f at $x = 0$, so that the function is continuous at $x = 0$, is

(A) 0 (B) – 1
(C) 1 (D) none of these

89. If $f(x) = \begin{cases} mx+1 & ,if\ x \leq \frac{\pi}{2} \\ \sin x + n, & if\ x > \frac{\pi}{2} \end{cases}$, is continuous at $x = \frac{\pi}{2}$, then

(A) $m = 1, n = 0$ (B) $m = \frac{n\pi}{2} + 1$

(C) $n = \frac{m\pi}{2}$ (D) $m = n = \frac{\pi}{2}$

90. Let $f(x) = |\sin x|$. Then

(A) f is everywhere differentiable
(B) f is everywhere continuous but not differentiable at $x = n\pi$, $n \in \mathbf{Z}$.
(C) f is everywhere continuous but not differentiable at $x = (2n + 1)\frac{\pi}{2}$, $n \in \mathbf{Z}$.
(D) none of these

91. If $y = \log\left(\frac{1-x^2}{1+x^2}\right)$, then $\frac{dy}{dx}$ is equal to

(A) $\dfrac{4x^3}{1-x^4}$ (B) $\dfrac{-4x}{1-x^4}$

(C) $\dfrac{1}{4-x^4}$ (D) $\dfrac{-4x^3}{1-x^4}$

92. If $y = \sqrt{\sin x + y}$, then $\dfrac{dy}{dx}$ is equal to

(A) $\dfrac{\cos x}{2y-1}$ (B) $\dfrac{\cos x}{1-2y}$

(C) $\dfrac{\sin x}{1-2y}$ (D) $\dfrac{\sin x}{2y-1}$

93. The derivative of $\cos^{-1}(2x^2 - 1)$ w.r.t. $\cos^{-1}x$ is

(A) 2 (B) $\dfrac{-1}{2\sqrt{1-x^2}}$

(C) $\dfrac{2}{x}$ (D) $1 - x^2$

94. If $x = t^2$, $y = t^3$, then $\dfrac{d^2y}{dx^2}$ is

(A) $\dfrac{3}{2}$ (B) $\dfrac{3}{4t}$

(C) $\dfrac{3}{2t}$ (D) $\dfrac{3}{4}$

95. The value of c in Rolle's theorem for the function $f(x) = x^3 - 3x$ in the interval $[0, \sqrt{3}]$ is

(A) 1 (B) – 1

(C) $\frac{3}{2}$ (D) $\frac{1}{3}$

96. For the function $f(x) = x + \frac{1}{x}, x \in [1, 3]$, the value of c for mean value theorem is

(A) 1 (B) $\sqrt{3}$

(C) 2 (D) none of these

Fill in the blanks in each of the Exercises 97 to 101:

97. An example of a function which is continuous everywhere but fails to be differentiable exactly at two points is __________ .

98. Derivative of x^2 w.r.t. x^3 is _________.

99. If $f(x) = |\cos x|$, then $f'\left(\frac{\pi}{4}\right)$ = _______ .

100. If $f(x) = |\cos x - \sin x|$, then $f'\left(\frac{\pi}{3}\right)$ = _______.

101. For the curve $\sqrt{x} + \sqrt{y} = 1$, $\frac{dy}{dx}$ at $\left(\frac{1}{4}, \frac{1}{4}\right)$ is _________.

State **True** or **False** for the statements in each of the Exercises 102 to 106.

102. Rolle's theorem is applicable for the function $f(x) = |x - 1|$ in [0, 2].

103. If f is continuous on its domain D, then $|f|$ is also continuous on D.

104. The composition of two continuous function is a continuous function.

105. Trigonometric and inverse - trigonometric functions are differentiable in their respective domain.

106. If $f \cdot g$ is continuous at $x = a$, then f and g are separately continuous at $x = a$.