Class XII : Maths
Chapter 11 : Three Dimensional Geometry

Questions and Solutions | Exercise 11.1 - NCERT Books

Question 1:

If a line makes angles 90°, 135°, 45° with *x*, *y* and *z*-axes respectively, find its direction cosines.

Answer

Let direction cosines of the line be *l*, *m*, and *n*.

$$l = \cos 90° = 0$$

$$m = \cos 135° = -\frac{1}{\sqrt{2}}$$

$$n = \cos 45° = \frac{1}{\sqrt{2}}$$

Therefore, the direction cosines of the line are $0, -\frac{1}{\sqrt{2}}, \text{ and } \frac{1}{\sqrt{2}}$.

Question 2:

Find the direction cosines of a line which makes equal angles with the coordinate axes.

Answer

Let the direction cosines of the line make an angle *α* with each of the coordinate axes.

∴ $l = \cos\alpha$, $m = \cos\alpha$, $n = \cos\alpha$

$$l^2 + m^2 + n^2 = 1$$
$$\Rightarrow \cos^2\alpha + \cos^2\alpha + \cos^2\alpha = 1$$
$$\Rightarrow 3\cos^2\alpha = 1$$
$$\Rightarrow \cos^2\alpha = \frac{1}{3}$$
$$\Rightarrow \cos\alpha = \pm\frac{1}{\sqrt{3}}$$

Thus, the direction cosines of the line, which is equally inclined to the coordinate axes, are $\pm\frac{1}{\sqrt{3}}, \pm\frac{1}{\sqrt{3}},$ and $\pm\frac{1}{\sqrt{3}}.$

Question 3:

If a line has the direction ratios −18, 12, −4, then what are its direction cosines?

Answer

If a line has direction ratios of −18, 12, and −4, then its direction cosines are

$$\frac{-18}{\sqrt{(-18)^2+(12)^2+(-4)^2}}, \frac{12}{\sqrt{(-18)^2+(12)^2+(-4)^2}}, \frac{-4}{\sqrt{(-18)^2+(12)^2+(-4)^2}}$$

i.e., $\frac{-18}{22}, \frac{12}{22}, \frac{-4}{22}$

$\frac{-9}{11}, \frac{6}{11}, \frac{-2}{11}$

Thus, the direction cosines are $-\frac{9}{11}, \frac{6}{11},$ and $\frac{-2}{11}$.

Question 4:

Show that the points (2, 3, 4), (−1, −2, 1), (5, 8, 7) are collinear.

Answer

The given points are A (2, 3, 4), B (− 1, − 2, 1), and C (5, 8, 7).

It is known that the direction ratios of line joining the points, (x_1, y_1, z_1) and (x_2, y_2, z_2), are given by, $x_2 - x_1$, $y_2 - y_1$, and $z_2 - z_1$.

The direction ratios of AB are (−1 − 2), (−2 − 3), and (1 − 4) i.e., −3, −5, and −3.

The direction ratios of BC are (5 − (− 1)), (8 − (− 2)), and (7 − 1) i.e., 6, 10, and 6.

It can be seen that the direction ratios of BC are −2 times that of AB i.e., they are proportional.

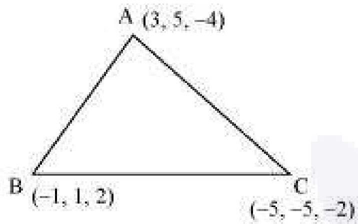
Therefore, AB is parallel to BC. Since point B is common to both AB and BC, points A, B, and C are collinear.

Question 5:

Find the direction cosines of the sides of the triangle whose vertices are (3, 5, − 4), (− 1, 1, 2) and (− 5, − 5, − 2)

Answer

The vertices of ΔABC are A (3, 5, −4), B (−1, 1, 2), and C (−5, −5, −2).



The direction ratios of side AB are (−1 − 3), (1 − 5), and (2 − (−4)) i.e., −4, −4, and 6.

$$\text{Then,} \sqrt{(-4)^2+(-4)^2+(6)^2} = \sqrt{16+16+36}$$
$$= \sqrt{68}$$
$$= 2\sqrt{17}$$

Therefore, the direction cosines of AB are

$$\frac{-4}{\sqrt{(-4)^2+(-4)^2+(6)^2}}, \frac{-4}{\sqrt{(-4)^2+(-4)^2+(6)^2}}, \frac{6}{\sqrt{(-4)^2+(-4)^2+(6)^2}}$$

$$\frac{-4}{2\sqrt{17}}, -\frac{4}{2\sqrt{17}}, \frac{6}{2\sqrt{17}}$$

$$\frac{-2}{\sqrt{17}}, \frac{-2}{\sqrt{17}}, \frac{3}{\sqrt{17}}$$

The direction ratios of BC are (−5 − (−1)), (−5 − 1), and (−2 − 2) i.e., −4, −6, and −4.

Therefore, the direction cosines of BC are

$$\frac{-4}{\sqrt{(-4)^2+(-6)^2+(-4)^2}}, \frac{-6}{\sqrt{(-4)^2+(-6)^2+(-4)^2}}, \frac{-4}{\sqrt{(-4)^2+(-6)^2+(-4)^2}}$$

$$\frac{-4}{2\sqrt{17}}, \frac{-6}{2\sqrt{17}}, \frac{-4}{2\sqrt{17}}$$

The direction ratios of CA are (−5 − 3), (−5 − 5), and (−2 − (−4)) i.e., −8, −10, and 2.

Therefore, the direction cosines of AC are

$$\frac{-8}{\sqrt{(-8)^2+(10)^2+(2)^2}}, \frac{-5}{\sqrt{(-8)^2+(10)^2+(2)^2}}, \frac{2}{\sqrt{(-8)^2+(10)^2+(2)^2}}$$

$$\frac{-8}{2\sqrt{42}}, \frac{-10}{2\sqrt{42}}, \frac{2}{2\sqrt{42}}$$

Class XII : Maths
Chapter 11 : Three Dimensional Geometry

Questions and Solutions | Exercise 11.2 - NCERT Books

Question 1:

Show that the three lines with direction cosines

$\frac{12}{13}, \frac{-3}{13}, \frac{-4}{13}; \frac{4}{13}, \frac{12}{13}, \frac{3}{13}; \frac{3}{13}, \frac{-4}{13}, \frac{12}{13}$ are mutually perpendicular.

Answer

Two lines with direction cosines, l_1, m_1, n_1 and l_2, m_2, n_2, are perpendicular to each other, if $l_1l_2 + m_1m_2 + n_1n_2 = 0$

(i) For the lines with direction cosines, $\frac{12}{13}, \frac{-3}{13}, \frac{-4}{13}$ and $\frac{4}{13}, \frac{12}{13}, \frac{3}{13}$, we obtain

$$l_1l_2 + m_1m_2 + n_1n_2 = \frac{12}{13} \times \frac{4}{13} + \left(\frac{-3}{13}\right) \times \frac{12}{13} + \left(\frac{-4}{13}\right) \times \frac{3}{13}$$
$$= \frac{48}{169} - \frac{36}{169} - \frac{12}{169}$$
$$= 0$$

Therefore, the lines are perpendicular.

(ii) For the lines with direction cosines, $\frac{4}{13}, \frac{12}{13}, \frac{3}{13}$ and $\frac{3}{13}, \frac{-4}{13}, \frac{12}{13}$, we obtain

$$l_1l_2 + m_1m_2 + n_1n_2 = \frac{4}{13} \times \frac{3}{13} + \frac{12}{13} \times \left(\frac{-4}{13}\right) + \frac{3}{13} \times \frac{12}{13}$$
$$= \frac{12}{169} - \frac{48}{169} + \frac{36}{169}$$
$$= 0$$

Therefore, the lines are perpendicular.

(iii) For the lines with direction cosines, $\frac{3}{13}, \frac{-4}{13}, \frac{12}{13}$ and $\frac{12}{13}, \frac{-3}{13}, \frac{-4}{13}$, we obtain

$$l_1l_2 + m_1m_2 + n_1n_2 = \left(\frac{3}{13}\right)\times\left(\frac{12}{13}\right)+\left(\frac{-4}{13}\right)\times\left(\frac{-3}{13}\right)+\left(\frac{12}{13}\right)\times\left(\frac{-4}{13}\right)$$
$$= \frac{36}{169}+\frac{12}{169}-\frac{48}{169}$$
$$= 0$$

Therefore, the lines are perpendicular.

Thus, all the lines are mutually perpendicular.

Question 2:

Show that the line through the points (1, −1, 2) (3, 4, −2) is perpendicular to the line through the points (0, 3, 2) and (3, 5, 6).

Answer

Let AB be the line joining the points, (1, −1, 2) and (3, 4, − 2), and CD be the line joining the points, (0, 3, 2) and (3, 5, 6).

The direction ratios, a_1, b_1, c_1, of AB are (3 − 1), (4 − (−1)), and (−2 − 2) i.e., 2, 5, and −4.

The direction ratios, a_2, b_2, c_2, of CD are (3 − 0), (5 − 3), and (6 −2) i.e., 3, 2, and 4.

AB and CD will be perpendicular to each other, if $a_1a_2 + b_1b_2 + c_1c_2 = 0$

$a_1a_2 + b_1b_2 + c_1c_2 = 2 \times 3 + 5 \times 2 + (-4) \times 4$

$= 6 + 10 - 16$

$= 0$

Therefore, AB and CD are perpendicular to each other.

Question 3:

Show that the line through the points (4, 7, 8) (2, 3, 4) is parallel to the line through the points (−1, −2, 1), (1, 2, 5).

Answer

Let AB be the line through the points, (4, 7, 8) and (2, 3, 4), and CD be the line through the points, (−1, −2, 1) and (1, 2, 5).

The directions ratios, a_1, b_1, c_1, of AB are (2 − 4), (3 − 7), and (4 − 8) i.e., −2, −4, and −4.

The direction ratios, a_2, b_2, c_2, of CD are (1 − (−1)), (2 − (−2)), and (5 − 1) i.e., 2, 4, and 4.

AB will be parallel to CD, if $\frac{a_1}{a_2}=\frac{b_1}{b_2}=\frac{c_1}{c_2}$

$\frac{a_1}{a_2}=\frac{-2}{2}=-1$

$\frac{b_1}{b_2}=\frac{-4}{4}=-1$

$\frac{c_1}{c_2}=\frac{-4}{4}=-1$

$\therefore\ \frac{a_1}{a_2}=\frac{b_1}{b_2}=\frac{c_1}{c_2}$

Thus, AB is parallel to CD.

Question 4:

Find the equation of the line which passes through the point (1, 2, 3) and is parallel to the vector $3\hat{i}+2\hat{j}-2\hat{k}$.

Answer

It is given that the line passes through the point A (1, 2, 3). Therefore, the position vector through A is $\vec{a}=\hat{i}+2\hat{j}+3\hat{k}$

$\vec{b}=3\hat{i}+2\hat{j}-2\hat{k}$

It is known that the line which passes through point A and parallel to $\vec{b}$ is given by $\vec{r}=\vec{a}+\lambda\vec{b}$, where λ is a constant.

$\Rightarrow \vec{r}=\hat{i}+2\hat{j}+3\hat{k}+\lambda\left(3\hat{i}+2\hat{j}-2\hat{k}\right)$

This is the required equation of the line.

Question 5:

Find the equation of the line in vector and in Cartesian form that passes through the point with position vector $2\hat{i}-\hat{j}+4\hat{k}$ and is in the direction $\hat{i}+2\hat{j}-\hat{k}$.

Answer

It is given that the line passes through the point with position vector

$\vec{a} = 2\hat{i} - \hat{j} + 4\hat{k}$ ...(1)

$\vec{b} = \hat{i} + 2\hat{j} - \hat{k}$ ...(2)

It is known that a line through a point with position vector $\vec{a}$ and parallel to $\vec{b}$ is given by the equation, $\vec{r} = \vec{a} + \lambda\vec{b}$

$\Rightarrow \vec{r} = 2\hat{i} - \hat{j} + 4\hat{k} + \lambda\left(\hat{i} + 2\hat{j} - \hat{k}\right)$

This is the required equation of the line in vector form.

$\vec{r} = x\hat{i} - y\hat{j} + z\hat{k}$

$\Rightarrow x\hat{i} - y\hat{j} + z\hat{k} = (\lambda + 2)\hat{i} + (2\lambda - 1)\hat{j} + (-\lambda + 4)\hat{k}$

Eliminating λ, we obtain the Cartesian form equation as

$$\frac{x-2}{1} = \frac{y+1}{2} = \frac{z-4}{-1}$$

This is the required equation of the given line in Cartesian form.

Question 6:

Find the Cartesian equation of the line which passes through the point (−2, 4, −5) and parallel to the line given by $\frac{x+3}{3} = \frac{y-4}{5} = \frac{z+8}{6}$

Answer

It is given that the line passes through the point (−2, 4, −5) and is parallel to

$$\frac{x+3}{3} = \frac{y-4}{5} = \frac{z+8}{6}$$

The direction ratios of the line, $\frac{x+3}{3} = \frac{y-4}{5} = \frac{z+8}{6}$, are 3, 5, and 6.

The required line is parallel to $\frac{x+3}{3} = \frac{y-4}{5} = \frac{z+8}{6}$

Therefore, its direction ratios are $3k$, $5k$, and $6k$, where $k \neq 0$

It is known that the equation of the line through the point (x_1, y_1, z_1) and with direction ratios, a, b, c, is given by $\frac{x-x_1}{a} = \frac{y-y_1}{b} = \frac{z-z_1}{c}$

Therefore the equation of the required line is

$$\frac{x+2}{3k}=\frac{y-4}{5k}=\frac{z+5}{6k}$$

$$\Rightarrow \frac{x+2}{3}=\frac{y-4}{5}=\frac{z+5}{6}=k$$

Question 7:

The Cartesian equation of a line is $\frac{x-5}{3}=\frac{y+4}{7}=\frac{z-6}{2}$. Write its vector form.

Answer

The Cartesian equation of the line is

$$\frac{x-5}{3}=\frac{y+4}{7}=\frac{z-6}{2} \quad \ldots(1)$$

The given line passes through the point (5, −4, 6). The position vector of this point is

$\vec{a}=5\hat{i}-4\hat{j}+6\hat{k}$

Also, the direction ratios of the given line are 3, 7, and 2.

This means that the line is in the direction of vector, $\vec{b}=3\hat{i}+7\hat{j}+2\hat{k}$

It is known that the line through position vector $\vec{a}$ and in the direction of the vector $\vec{b}$ is given by the equation, $\vec{r}=\vec{a}+\lambda\vec{b}, \lambda \in R$

$$\Rightarrow \vec{r}=\left(5\hat{i}-4\hat{j}+6\hat{k}\right)+\lambda\left(3\hat{i}+7\hat{j}+2\hat{k}\right)$$

This is the required equation of the given line in vector form.

Question 8**:**

Find the angle between the following pairs of lines:

(i) $\vec{r} = 2\hat{i} - 5\hat{j} + \hat{k} + \lambda\left(3\hat{i} - 2\hat{j} + 6\hat{k}\right)$ and

$\vec{r} = 7\hat{i} - 6\hat{k} + \mu\left(\hat{i} + 2\hat{j} + 2\hat{k}\right)$

(ii) $\vec{r} = 3\hat{i} + \hat{j} - 2\hat{k} + \lambda\left(\hat{i} - \hat{j} - 2\hat{k}\right)$ and

$\vec{r} = 2\hat{i} - \hat{j} - 56\hat{k} + \mu\left(3\hat{i} - 5\hat{j} - 4\hat{k}\right)$

Answer

(i) Let Q be the angle between the given lines.

The angle between the given pairs of lines is given by, $\cos Q = \left|\frac{\vec{b}_1 \cdot \vec{b}_2}{\left|\vec{b}_1\right|\left|\vec{b}_2\right|}\right|$

The given lines are parallel to the vectors, $\vec{b}_1 = 3\hat{i} + 2\hat{j} + 6\hat{k}$ and $\vec{b}_2 = \hat{i} + 2\hat{j} + 2\hat{k}$, respectively.

$\therefore \left|\vec{b}_1\right| = \sqrt{3^2 + 2^2 + 6^2} = 7$

$\left|\vec{b}_2\right| = \sqrt{(1)^2 + (2)^2 + (2)^2} = 3$

$$\begin{aligned}\vec{b}_1 \cdot \vec{b}_2 &= \left(3\hat{i} + 2\hat{j} + 6\hat{k}\right) \cdot \left(\hat{i} + 2\hat{j} + 2\hat{k}\right) \\ &= 3 \times 1 + 2 \times 2 + 6 \times 2 \\ &= 3 + 4 + 12 \\ &= 19\end{aligned}$$

$$\Rightarrow \cos Q=\frac{19}{7\times 3}$$
$$\Rightarrow Q=\cos^{-1}\left(\frac{19}{21}\right)$$

(ii) The given lines are parallel to the vectors, $\vec{b}_1=\hat{i}-\hat{j}-2\hat{k}$ and $\vec{b}_2=3\hat{i}-5\hat{j}-4\hat{k}$, respectively.

$$\therefore \left|\vec{b}_1\right|=\sqrt{(1)^2+(-1)^2+(-2)^2}=\sqrt{6}$$
$$\left|\vec{b}_2\right|=\sqrt{(3)^2+(-5)^2+(-4)^2}=\sqrt{50}=5\sqrt{2}$$
$$\vec{b}_1\cdot\vec{b}_2=\left(\hat{i}-\hat{j}-2\hat{k}\right)\cdot\left(3\hat{i}-5\hat{j}-4\hat{k}\right)$$
$$=1\cdot 3-1(-5)-2(-4)$$
$$=3+5+8$$
$$=16$$
$$\cos Q=\left|\frac{\vec{b}_1\cdot\vec{b}_2}{\left|\vec{b}_1\right|\left|\vec{b}_2\right|}\right|$$
$$\Rightarrow \cos Q=\frac{16}{\sqrt{6}\cdot 5\sqrt{2}}=\frac{16}{\sqrt{2}\cdot\sqrt{3}\cdot 5\sqrt{2}}=\frac{16}{10\sqrt{3}}$$
$$\Rightarrow \cos Q=\frac{8}{5\sqrt{3}}$$
$$\Rightarrow Q=\cos^{-1}\left(\frac{8}{5\sqrt{3}}\right)$$

Question 9:

Find the angle between the following pairs of lines:

(i) $\frac{x-2}{2}=\frac{y-1}{5}=\frac{z+3}{-3}$ and $\frac{x+2}{-1}=\frac{y-4}{8}=\frac{z-5}{4}$

(ii) $\frac{x}{2}=\frac{y}{2}=\frac{z}{1}$ and $\frac{x-5}{4}=\frac{y-2}{1}=\frac{z-3}{8}$

i. Answer

ii. Let $\vec{b}_1$ and $\vec{b}_2$ be the vectors parallel to the pair of lines,

$\frac{x-2}{2}=\frac{y-1}{5}=\frac{z+3}{-3}$ and $\frac{x+2}{-1}=\frac{y-4}{8}=\frac{z-5}{4}$, respectively.

$\therefore \vec{b}_1 = 2\hat{i}+5\hat{j}-3\hat{k}$ and $\vec{b}_2 = -\hat{i}+8\hat{j}+4\hat{k}$

$$\left|\vec{b}_1\right| = \sqrt{(2)^2+(5)^2+(-3)^2} = \sqrt{38}$$

$$\left|\vec{b}_2\right| = \sqrt{(-1)^2+(8)^2+(4)^2} = \sqrt{81} = 9$$

$$\vec{b}_1\cdot\vec{b}_2 = \left(2\hat{i}+5\hat{j}-3\hat{k}\right)\cdot\left(-\hat{i}+8\hat{j}+4\hat{k}\right)$$

$$= 2(-1)+5\times 8+(-3)\cdot 4$$

$$= -2+40-12$$

$$= 26$$

The angle, Q, between the given pair of lines is given by the relation,

$$\cos Q = \left|\frac{\vec{b}_1\cdot\vec{b}_2}{\left|\vec{b}_1\right|\left|\vec{b}_2\right|}\right|$$

$$\Rightarrow \cos Q = \frac{26}{9\sqrt{38}}$$

$$\Rightarrow Q = \cos^{-1}\left(\frac{26}{9\sqrt{38}}\right)$$

(ii) Let $\vec{b}_1, \vec{b}_2$ be the vectors parallel to the given pair of lines, $\frac{x}{2}=\frac{y}{2}=\frac{z}{1}$ and $\frac{x-5}{4}=\frac{y-5}{1}=\frac{z-3}{8}$, respectively.

$\vec{b}_1 = 2\hat{i} + 2\hat{j} + \hat{k}$

$\vec{b}_2 = 4\hat{i} + \hat{j} + 8\hat{k}$

$\therefore \left|\vec{b}_1\right| = \sqrt{(2)^2 + (2)^2 + (1)^2} = \sqrt{9} = 3$

$\left|\vec{b}_2\right| = \sqrt{4^2 + 1^2 + 8^2} = \sqrt{81} = 9$

$$\begin{aligned}\vec{b}_1 \cdot \vec{b}_2 &= \left(2\hat{i} + 2\hat{j} + \hat{k}\right)\cdot\left(4\hat{i} + \hat{j} + 8\hat{k}\right)\\ &= 2\times4 + 2\times1 + 1\times8\\ &= 8 + 2 + 8\\ &= 18\end{aligned}$$

If Q is the angle between the given pair of lines, then $\cos Q = \left|\frac{\vec{b}_1 \cdot \vec{b}_2}{\left|\vec{b}_1\right|\left|\vec{b}_2\right|}\right|$

$\Rightarrow \cos Q = \frac{18}{3\times9} = \frac{2}{3}$

$\Rightarrow Q = \cos^{-1}\left(\frac{2}{3}\right)$

Question 10:

Find the values of p so the line $\frac{1-x}{3} = \frac{7y-14}{2p} = \frac{z-3}{2}$ and $\frac{7-7x}{3p} = \frac{y-5}{1} = \frac{6-z}{5}$ are at right angles.

Answer

The given equations can be written in the standard form as

$\frac{x-1}{-3} = \frac{y-2}{\frac{2p}{7}} = \frac{z-3}{2}$ and $\frac{x-1}{\frac{-3p}{7}} = \frac{y-5}{1} = \frac{z-6}{-5}$

The direction ratios of the lines are −3, $\frac{2p}{7}$, 2 and $\frac{-3p}{7}, 1, -5$ respectively.

Two lines with direction ratios, a_1, b_1, c_1 and a_2, b_2, c_2, are perpendicular to each other, if $a_1a_2 + b_1 b_2 + c_1c_2 = 0$

$$\therefore (-3)\cdot\left(\frac{-3p}{7}\right)+\left(\frac{2p}{7}\right)\cdot(1)+2\cdot(-5)=0$$

$$\Rightarrow \frac{9p}{7}+\frac{2p}{7}=10$$

$$\Rightarrow 11p=70$$

$$\Rightarrow p=\frac{70}{11}$$

Thus, the value of *p* is $\frac{70}{11}$.

Question 13:

Show that the lines $\frac{x-5}{7}=\frac{y+2}{-5}=\frac{z}{1}$ and $\frac{x}{1}=\frac{y}{2}=\frac{z}{3}$ are perpendicular to each other.

Answer

The equations of the given lines are $\frac{x-5}{7}=\frac{y+2}{-5}=\frac{z}{1}$ and $\frac{x}{1}=\frac{y}{2}=\frac{z}{3}$

The direction ratios of the given lines are 7, −5, 1 and 1, 2, 3 respectively.

Two lines with direction ratios, a_1, b_1, c_1 and a_2, b_2, c_2, are perpendicular to each other, if $a_1a_2 + b_1 b_2 + c_1c_2 = 0$

$\therefore$ 7 × 1 + (−5) × 2 + 1 × 3

= 7 − 10 + 3

= 0

Therefore, the given lines are perpendicular to each other.

Question 14:

Find the shortest distance between the lines

$$\vec{r}=\left(\hat{i}+2\hat{j}+\hat{k}\right)+\lambda\left(\hat{i}-\hat{j}+\hat{k}\right) \text{and}$$

$$\vec{r}=2\hat{i}-\hat{j}-\hat{k}+\mu\left(2\hat{i}+\hat{j}+2\hat{k}\right)$$

Answer

The equations of the given lines are

$$\vec{r}=\left(\hat{i}+2\hat{j}+\hat{k}\right)+\lambda\left(\hat{i}-\hat{j}+\hat{k}\right)$$

$$\vec{r}=2\hat{i}-\hat{j}-\hat{k}+\mu\left(2\hat{i}+\hat{j}+2\hat{k}\right)$$

It is known that the shortest distance between the lines, $\vec{r}=\vec{a}_1+\lambda\vec{b}_1$ and $\vec{r}=\vec{a}_2+\mu\vec{b}_2$, is given by,

$$d=\left|\frac{\left(\vec{b}_1\times\vec{b}_2\right)\cdot\left(\vec{a}_2-\vec{a}_2\right)}{\left|\vec{b}_1\times\vec{b}_2\right|}\right| \quad \ldots(1)$$

Comparing the given equations, we obtain

$$\vec{a}_1=\hat{i}+2\hat{j}+\hat{k}$$

$$\vec{b}_1=\hat{i}-\hat{j}+\hat{k}$$

$$\vec{a}_2=2\hat{i}-\hat{j}-\hat{k}$$

$$\vec{b}_2=2\hat{i}+\hat{j}+2\hat{k}$$

$$\vec{a}_2-\vec{a}_1=\left(2\hat{i}-\hat{j}-\hat{k}\right)-\left(\hat{i}+2\hat{j}+\hat{k}\right)=\hat{i}-3\hat{j}-2\hat{k}$$

$$\vec{b}_1\times\vec{b}_2=\begin{vmatrix}\hat{i} & \hat{j} & \hat{k}\\ 1 & -1 & 1\\ 2 & 1 & 2\end{vmatrix}$$

$$\vec{b}_1\times\vec{b}_2=(-2-1)\hat{i}-(2-2)\hat{j}+(1+2)\hat{k}=-3\hat{i}+3\hat{k}$$

$$\Rightarrow\left|\vec{b}_1\times\vec{b}_2\right|=\sqrt{(-3)^2+(3)^2}=\sqrt{9+9}=\sqrt{18}=3\sqrt{2}$$

Substituting all the values in equation (1), we obtain

$$d=\left|\frac{\left(-3\hat{i}+3\hat{k}\right)\cdot\left(\hat{i}-3\hat{j}-2\hat{k}\right)}{3\sqrt{2}}\right|$$

$$\Rightarrow d=\left|\frac{-3.1+3(-2)}{3\sqrt{2}}\right|$$

$$\Rightarrow d=\left|\frac{-9}{3\sqrt{2}}\right|$$

$$\Rightarrow d=\frac{3}{\sqrt{2}}=\frac{3\times\sqrt{2}}{\sqrt{2}\times\sqrt{2}}=\frac{3\sqrt{2}}{2}$$

Therefore, the shortest distance between the two lines is $\frac{3\sqrt{2}}{2}$ units.

Question 11:

Find the shortest distance between the lines $\frac{x+1}{7}=\frac{y+1}{-6}=\frac{z+1}{1}$ and $\frac{x-3}{1}=\frac{y-5}{-2}=\frac{z-7}{1}$

Answer

The given lines are $\frac{x+1}{7}=\frac{y+1}{-6}=\frac{z+1}{1}$ and $\frac{x-3}{1}=\frac{y-5}{-2}=\frac{z-7}{1}$

It is known that the shortest distance between the two lines,

$\frac{x-x_1}{a_1}=\frac{y-y_1}{b_1}=\frac{z-z_1}{c_1}$ and $\frac{x-x_2}{a_2}=\frac{y-y_2}{b_2}=\frac{z-z_2}{c_2}$, is given by,

$$d=\frac{\begin{vmatrix}x_2-x_1 & y_2-y_1 & z_2-z_1\\ a_1 & b_1 & c_1\\ a_2 & b_2 & c_2\end{vmatrix}}{\sqrt{\left(b_1c_2-b_2c_1\right)^2+\left(c_1a_2-c_2a_1\right)^2+\left(a_1b_2-a_2b_1\right)^2}}\quad\ldots(1)$$

Comparing the given equations, we obtain

$x_1 = -1,\ y_1 = -1,\ z_1 = -1$

$a_1 = 7,\ b_1 = -6,\ c_1 = 1$

$x_2 = 3,\ y_2 = 5,\ z_2 = 7$

$a_2 = 1,\ b_2 = -2,\ c_2 = 1$

$$\text{Then, }\begin{vmatrix} x_2 - x_1 & y_2 - y_1 & z_2 - z_1 \\ a_1 & b_1 & c_1 \\ a_2 & b_2 & c_2 \end{vmatrix} = \begin{vmatrix} 4 & 6 & 8 \\ 7 & -6 & 1 \\ 1 & -2 & 1 \end{vmatrix}$$

$$= 4(-6+2) - 6(7-1) + 8(-14+6)$$

$$= -16 - 36 - 64$$

$$= -116$$

$$\Rightarrow \sqrt{(b_1c_2 - b_2c_1)^2 + (c_1a_2 - c_2a_1)^2 + (a_1b_2 - a_2b_1)^2} = \sqrt{(-6+2)^2 + (1+7)^2 + (-14+6)^2}$$

$$= \sqrt{16+36+64}$$

$$= \sqrt{116}$$

$$= 2\sqrt{29}$$

Substituting all the values in equation (1), we obtain

$$d = \frac{-116}{2\sqrt{29}} = \frac{-58}{\sqrt{29}} = \frac{-2\times 29}{\sqrt{29}} = -2\sqrt{29}$$

Since distance is always non-negative, the distance between the given lines is $2\sqrt{29}$ units.

Question 14:

Find the shortest distance between the lines whose vector equations are

$$\vec{r} = \left(\hat{i} + 2\hat{j} + 3\hat{k}\right) + \lambda\left(\hat{i} - 3\hat{j} + 2\hat{k}\right)$$

$$\text{and } \vec{r} = 4\hat{i} + 5\hat{j} + 6\hat{k} + \mu\left(2\hat{i} + 3\hat{j} + \hat{k}\right)$$

Answer

The given lines are $\vec{r} = \hat{i} + 2\hat{j} + 3\hat{k} + \lambda\left(\hat{i} - 3\hat{j} + 2\hat{k}\right)$ and $\vec{r} = 4\hat{i} + 5\hat{j} + 6\hat{k} + \mu\left(2\hat{i} + 3\hat{j} + \hat{k}\right)$

It is known that the shortest distance between the lines, $\vec{r} = \vec{a}_1 + \lambda\vec{b}_1$ and $\vec{r} = \vec{a}_2 + \mu\vec{b}_2$, is given by,

$$d = \left| \frac{(\vec{b}_1 \times \vec{b}_2) \cdot (\vec{a}_2 - \vec{a}_2)}{|\vec{b}_1 \times \vec{b}_2|} \right| \qquad ...(1)$$

Comparing the given equations with $\vec{r} = \vec{a}_1 + \lambda \vec{b}_1$ and $\vec{r} = \vec{a}_2 + \mu \vec{b}_2$, we obtain

$\vec{a}_1 = \hat{i} + 2\hat{j} + 3\hat{k}$

$\vec{b}_1 = \hat{i} - 3\hat{j} + 2\hat{k}$

$\vec{a}_2 = 4\hat{i} + 5\hat{j} + 6\hat{k}$

$\vec{b}_2 = 2\hat{i} + 3\hat{j} + \hat{k}$

$\vec{a}_2 - \vec{a}_1 = (4\hat{i} + 5\hat{j} + 6\hat{k}) - (\hat{i} + 2\hat{j} + 3\hat{k}) = 3\hat{i} + 3\hat{j} + 3\hat{k}$

$$\vec{b}_1 \times \vec{b}_2 = \begin{vmatrix} \hat{i} & \hat{j} & \hat{k} \\ 1 & -3 & 2 \\ 2 & 3 & 1 \end{vmatrix} = (-3-6)\hat{i} - (1-4)\hat{j} + (3+6)\hat{k} = -9\hat{i} + 3\hat{j} + 9\hat{k}$$

$\Rightarrow |\vec{b}_1 \times \vec{b}_2| = \sqrt{(-9)^2 + (3)^2 + (9)^2} = \sqrt{81+9+81} = \sqrt{171} = 3\sqrt{19}$

$$\begin{aligned}(\vec{b}_1 \times \vec{b}_2) \cdot (\vec{a}_2 - \vec{a}_1) &= (-9\hat{i} + 3\hat{j} + 9\hat{k}) \cdot (3\hat{i} + 3\hat{j} + 3\hat{k}) \\ &= -9 \times 3 + 3 \times 3 + 9 \times 3 \\ &= 9\end{aligned}$$

Substituting all the values in equation (1), we obtain

$$d = \left| \frac{9}{3\sqrt{19}} \right| = \frac{3}{\sqrt{19}}$$

Therefore, the shortest distance between the two given lines is $\frac{3}{\sqrt{19}}$ units.

Question 15:

Find the shortest distance between the lines whose vector equations are

$\vec{r} = (1-t)\hat{i} + (t-2)\hat{j} + (3-2t)\hat{k}$ and

$\vec{r} = (s+1)\hat{i} + (2s-1)\hat{j} - (2s+1)\hat{k}$

Answer

The given lines are

$\vec{r}=(1-t)\hat{i}+(t-2)\hat{j}+(3-2t)\hat{k}$

$\Rightarrow \vec{r}=\left(\hat{i}-2\hat{j}+3\hat{k}\right)+t\left(-\hat{i}+\hat{j}-2\hat{k}\right) \qquad ...(1)$

$\vec{r}=(s+1)\hat{i}+(2s-1)\hat{j}-(2s+1)\hat{k}$

$\Rightarrow \vec{r}=\left(\hat{i}-\hat{j}+\hat{k}\right)+s\left(\hat{i}+2\hat{j}-2\hat{k}\right) \qquad ...(2)$

It is known that the shortest distance between the lines, $\vec{r}=\vec{a}_1+\lambda\vec{b}_1$ and $\vec{r}=\vec{a}_2+\mu\vec{b}_2$, is given by,

$$d=\left|\frac{\left(\vec{b}_1\times\vec{b}_2\right)\cdot\left(\vec{a}_2-\vec{a}_2\right)}{\left|\vec{b}_1\times\vec{b}_2\right|}\right| \qquad ...(3)$$

For the given equations,

$\vec{a}_1=\hat{i}-2\hat{j}+3\hat{k}$

$\vec{b}_1=-\hat{i}+\hat{j}-2\hat{k}$

$\vec{a}_2=\hat{i}-\hat{j}-\hat{k}$

$\vec{b}_2=\hat{i}+2\hat{j}-2\hat{k}$

$\vec{a}_2-\vec{a}_1=\left(\hat{i}-\hat{j}-\hat{k}\right)-\left(\hat{i}-2\hat{j}+3\hat{k}\right)=\hat{j}-4\hat{k}$

$$\vec{b}_1\times\vec{b}_2=\begin{vmatrix}\hat{i} & \hat{j} & \hat{k}\\ -1 & 1 & -2\\ 1 & 2 & -2\end{vmatrix}=(-2+4)\hat{i}-(2+2)\hat{j}+(-2-1)\hat{k}=2\hat{i}-4\hat{j}-3\hat{k}$$

$\Rightarrow \left|\vec{b}_1\times\vec{b}_2\right|=\sqrt{(2)^2+(-4)^2+(-3)^2}=\sqrt{4+16+9}=\sqrt{29}$

$\therefore \left(\vec{b}_1\times\vec{b}_2\right)\cdot\left(\vec{a}_2-\vec{a}_1\right)=\left(2\hat{i}-4\hat{j}-3\hat{k}\right)\cdot\left(\hat{j}-4\hat{k}\right)=-4+12=8$

Substituting all the values in equation (3), we obtain

$$d=\left|\frac{8}{\sqrt{29}}\right|=\frac{8}{\sqrt{29}}$$

Therefore, the shortest distance between the lines is $\frac{8}{\sqrt{29}}$ units.

Class XII : Maths
Chapter 11 : Three Dimensional Geometry

Questions and Solutions | Miscellaneous Exercise 11 - NCERT Books

Question 1:

Find the angle between the lines whose direction ratios are *a*, *b*, *c* and *b* − *c*, *c* − *a*, *a* − *b*.

Answer

The angle *Q* between the lines with direction cosines, *a*, *b*, *c* and *b* − *c*, *c* − *a*, *a* − *b*, is given by,

$$\cos Q=\left|\frac{a(b-c)+b(c-a)+c(a-b)}{\sqrt{a^2+b^2+c^2}+\sqrt{(b-c)^2+(c-a)^2+(a-b)^2}}\right|$$

$$\Rightarrow \cos Q=0$$

$$\Rightarrow Q=\cos^{-1}0$$

$$\Rightarrow Q=90^\circ$$

Thus, the angle between the lines is 90°.

Question 2:

Find the equation of a line parallel to *x*-axis and passing through the origin.

Answer

The line parallel to *x*-axis and passing through the origin is *x*-axis itself.

Let A be a point on *x*-axis. Therefore, the coordinates of A are given by (*a*, 0, 0), where *a* ∈ R.

Direction ratios of OA are (*a* − 0) = *a*, 0, 0

The equation of OA is given by,

$$\frac{x-0}{a}=\frac{y-0}{0}=\frac{z-0}{0}$$

$$\Rightarrow \frac{x}{1}=\frac{y}{0}=\frac{z}{0}=a$$

Thus, the equation of line parallel to *x*-axis and passing through origin is

$$\frac{x}{1}=\frac{y}{0}=\frac{z}{0}$$

Question 3:

If the lines $\frac{x-1}{-3}=\frac{y-2}{2k}=\frac{z-3}{2}$ and $\frac{x-1}{3k}=\frac{y-1}{1}=\frac{z-6}{-5}$ are perpendicular, find the value of k.

Answer

The direction of ratios of the lines, $\frac{x-1}{-3}=\frac{y-2}{2k}=\frac{z-3}{2}$ and $\frac{x-1}{3k}=\frac{y-1}{1}=\frac{z-6}{-5}$, are −3, $2k$, 2 and $3k$, 1, −5 respectively.

It is known that two lines with direction ratios, a_1, b_1, c_1 and a_2, b_2, c_2, are perpendicular, if $a_1a_2 + b_1b_2 + c_1c_2 = 0$

$\therefore -3(3k)+2k\times 1+2(-5)=0$

$\Rightarrow -9k+2k-10=0$

$\Rightarrow 7k=-10$

$\Rightarrow k=\frac{-10}{7}$

Therefore, for $k=-\frac{10}{7}$, the given lines are perpendicular to each other.

Question 4:

Find the shortest distance between lines $\vec{r} = 6\hat{i} + 2\hat{j} + 2\hat{k} + \lambda\left(\hat{i} - 2\hat{j} + 2\hat{k}\right)$

and $\vec{r} = -4\hat{i} - \hat{k} + \mu\left(3\hat{i} - 2\hat{j} - 2\hat{k}\right)$.

Answer

The given lines are

$$\vec{r} = 6\hat{i} + 2\hat{j} + 2\hat{k} + \lambda\left(\hat{i} - 2\hat{j} + 2\hat{k}\right) \quad \ldots(1)$$

$$\vec{r} = -4\hat{i} - \hat{k} + \mu\left(3\hat{i} - 2\hat{j} - 2\hat{k}\right) \quad \ldots(2)$$

It is known that the shortest distance between two lines, $\vec{r} = \vec{a}_1 + \lambda\vec{b}_1$ and $\vec{r} = \vec{a}_2 + \lambda\vec{b}_2$, is given by

$$d = \left|\frac{\left(\vec{b}_1 \times \vec{b}_2\right) \cdot \left(\vec{a}_2 - \vec{a}_1\right)}{\left|\vec{b}_1 \times \vec{b}_2\right|}\right| \quad \ldots(3)$$

Comparing $\vec{r} = \vec{a}_1 + \lambda\vec{b}_1$ and $\vec{r} = \vec{a}_2 + \lambda\vec{b}_2$ to equations (1) and (2), we obtain

$$\vec{a}_1 = 6\hat{i} + 2\hat{j} + 2\hat{k}$$

$$\vec{b}_1 = \hat{i} - 2\hat{j} + 2\hat{k}$$

$$\vec{a}_2 = -4\hat{i} - \hat{k}$$

$$\vec{b}_2 = 3\hat{i} - 2\hat{j} - 2\hat{k}$$

$$\Rightarrow \vec{a}_2 - \vec{a}_1 = \left(-4\hat{i} - \hat{k}\right) - \left(6\hat{i} + 2\hat{j} + 2\hat{k}\right) = -10\hat{i} - 2\hat{j} - 3\hat{k}$$

$$\Rightarrow \vec{b}_1 \times \vec{b}_2 = \begin{vmatrix} \hat{i} & \hat{j} & \hat{k} \\ 1 & -2 & 2 \\ 3 & -2 & -2 \end{vmatrix} = (4+4)\hat{i} - (-2-6)\hat{j} + (-2+6)\hat{k} = 8\hat{i} + 8\hat{j} + 4\hat{k}$$

$$\therefore \left|\vec{b}_1 \times \vec{b}_2\right| = \sqrt{(8)^2 + (8)^2 + (4)^2} = 12$$

$$\left(\vec{b}_1 \times \vec{b}_2\right) \cdot \left(\vec{a}_2 - \vec{a}_1\right) = \left(8\hat{i} + 8\hat{j} + 4\hat{k}\right) \cdot \left(-10\hat{i} - 2\hat{j} - 3\hat{k}\right) = -80 - 16 - 12 = -108$$

Substituting all the values in equation (1), we obtain

$$d = \left|\frac{-108}{12}\right| = 9$$

Therefore, the shortest distance between the two given lines is 9 units.

Question 5:

Find the vector equation of the line passing through the point (1, 2, − 4) and perpendicular to the two lines: $\frac{x-8}{3} = \frac{y+19}{-16} = \frac{z-10}{7}$ and $\frac{x-15}{3} = \frac{y-29}{8} = \frac{z-5}{-5}$

Answer

Let the required line be parallel to the vector $\vec{b}$ given by, $\vec{b} = b_1\hat{i} + b_2\hat{j} + b_3\hat{k}$

The position vector of the point (1, 2, − 4) is $\vec{a}=\hat{i}+2\hat{j}-4\hat{k}$

The equation of the line passing through (1, 2, −4) and parallel to vector $\vec{b}$ is

$\vec{r}=\vec{a}+\lambda\vec{b}$

$\Rightarrow \vec{r}\left(\hat{i}+2\hat{j}-4\hat{k}\right)+\lambda\left(b_1\hat{i}+b_2\hat{j}+b_3\hat{k}\right) \qquad ...(1)$

The equations of the lines are

$\frac{x-8}{3}=\frac{y+19}{-16}=\frac{z-10}{7} \qquad ...(2)$

$\frac{x-15}{3}=\frac{y-29}{8}=\frac{z-5}{-5} \qquad ...(3)$

Line (1) and line (2) are perpendicular to each other.

$\therefore 3b_1-16b_2+7b_3=0 \qquad ...(4)$

Also, line (1) and line (3) are perpendicular to each other.

$\therefore 3b_1+8b_2-5b_3=0 \qquad ...(5)$

From equations (4) and (5), we obtain

$\frac{b_1}{(-16)(-5)-8\times 7}=\frac{b_2}{7\times 3-3(-5)}=\frac{b_3}{3\times 8-3(-16)}$

$\Rightarrow \frac{b_1}{24}=\frac{b_2}{36}=\frac{b_3}{72}$

$\Rightarrow \frac{b_1}{2}=\frac{b_2}{3}=\frac{b_3}{6}$

∴Direction ratios of $\vec{b}$ are 2, 3, and 6.

$\therefore \vec{b}=2\hat{i}+3\hat{j}+6\hat{k}$

Substituting $\vec{b}=2\hat{i}+3\hat{j}+6\hat{k}$ in equation (1), we obtain

$\vec{r}=\left(\hat{i}+2\hat{j}-4\hat{k}\right)+\lambda\left(2\hat{i}+3\hat{j}+6\hat{k}\right)$

This is the equation of the required line.