FINAL JEE–MAIN EXAMINATION – FEBRUARY, 2021

(Held On Thursday 25th February, 2021) TIME : 3 : 00 PM to 6 : 00 PM

MATHEMATICS — TEST PAPER WITH SOLUTION

SECTION-A

1. Let A be a 3 × 3 matrix with det(A) = 4. Let R_i denote the i^{th} row of A. If a matrix B is obtained by performing the operation $R_2 \to 2R_2 + 5R_3$ on 2A, then det(B) is equal to :

(1) 16 (2) 80 (3) 128 (4) 64

Official Ans. by NTA (4)

Sol. $|A| = 4$

$\Rightarrow |2A| = 2^3 \times 4 = 32$

$\because$ B is obtained by $R_2 \to 2R_2 + 5R_3$

$\Rightarrow |B| = 2 \times 32 = 64$

option (4)

2. The integral $\int \frac{e^{3\log_e 2x} + 5e^{2\log_e 2x}}{e^{4\log_e x} + 5e^{3\log_e x} - 7e^{2\log_e x}} dx$, $x > 0$, is equal to :

(where c is a constant of integration)

(1) $\log_e |x^2 + 5x - 7| + c$

(2) $4\log_e |x^2 + 5x - 7| + c$

(3) $\frac{1}{4}\log_e |x^2 + 5x - 7| + c$

(4) $\log_e \sqrt{x^2 + 5x - 7} + c$

Official Ans. by NTA (2)

Sol. $$\int \frac{e^{3\log_e 2x} + 5e^{2\log_e 2x}}{e^{4\log_e x} + 5e^{3\log_e x} - 7e^{2\log_e x}} dx, x > 0$$

$$= \int \frac{(2x)^3 + 5(2x)^2}{x^4 + 5x^3 - 7x^2} dx = \int \frac{4x^2(2x+5)}{x^2(x^2+5x-7)} dx$$

$$= 4\int \frac{d(x^2+5x-7)}{(x^2+5x-7)} = 4\log_e |x^2 + 5x - 7| + c$$

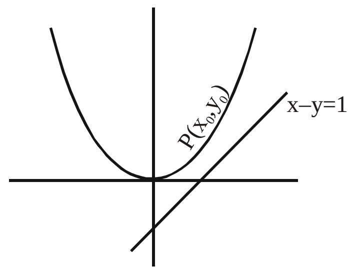
option (2)

3. The shortest distance between the line $x - y = 1$ and the curve $x^2 = 2y$ is :

(1) $\frac{1}{2}$ (2) $\frac{1}{2\sqrt{2}}$ (3) $\frac{1}{\sqrt{2}}$ (4) 0

Official Ans. by NTA (2)

Sol.



Shortest distance between curves is always along common normal.

$$\left.\frac{dy}{dx}\right|_P = \text{slope of line} = 1$$

$\Rightarrow x_0 = 1$ $\therefore y_0 = \frac{1}{2}$

$$\Rightarrow P\left(1, \frac{1}{2}\right)$$

$$\therefore \text{Shortest distance} = \left|\frac{1 - \frac{1}{2} - 1}{\sqrt{1^2 + 1^2}}\right| = \frac{1}{2\sqrt{2}}$$

option (2)

4. If $\alpha, \beta \in R$ are such that $1 - 2i$ (here $i^2 = -1$) is a root of $z^2 + \alpha z + \beta = 0$, then $(\alpha - \beta)$ is equal to :

(1) –3 (2) –7 (3) 7 (4) 3

Official Ans. by NTA (2)

Sol. $\because \alpha, \beta \in R \Rightarrow$ other root is $1 + 2i$

$\alpha = -(\text{sum of roots}) = -(1 - 2i + 1 + 2i) = -2$

$\beta = \text{product of roots} = (1 - 2i)(1 + 2i) = 5$

$\therefore \alpha - \beta = -7$

option (2)

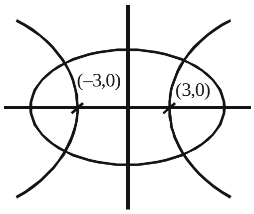
5. A hyperbola passes through the foci of the ellipse $\frac{x^2}{25}+\frac{y^2}{16}=1$ and its transverse and conjugate axes coincide with major and minor axes of the ellipse, respectively. If the product of their eccentricities in one, then the equation of the hyperbola is :

(1) $\frac{x^2}{9}-\frac{y^2}{25}=1$ (2) $\frac{x^2}{9}-\frac{y^2}{16}=1$

(3) $x^2 - y^2 = 9$ (4) $\frac{x^2}{9}-\frac{y^2}{4}=1$

Official Ans. by NTA (2)

Sol.



For ellipse $e_1 = \sqrt{1-\frac{b^2}{a^2}} = \frac{3}{5}$

for hyperbola $e_2 = \frac{5}{3}$

Let hyperbola be

$$\frac{x^2}{a^2}-\frac{y^2}{b^2}=1$$

$\because$ it passes through (3,0) $\Rightarrow \frac{9}{a_2}=1$

$\Rightarrow a^2 = 9$

$\Rightarrow b^2 = a^2(e^2 - 1)$

$=9\left(\frac{25}{9}-1\right)=16$

$\therefore$ Hyperbola is

$\frac{x^2}{9}-\frac{y^2}{16}=1$ ... option 2.

6. If $0<x, y<\pi$ and $\cos x + \cos y - \cos(x+y) = \frac{3}{2}$, then $\sin x + \cos y$ is equal to :

(1) $\frac{1}{2}$ (2) $\frac{1+\sqrt{3}}{2}$

(3) $\frac{\sqrt{3}}{2}$ (4) $\frac{1-\sqrt{3}}{2}$

Official Ans. by NTA (2)

Sol. $\cos x + \cos y - \cos(x + y) = \frac{3}{2}$

$$\cos^2\left(\frac{x+y}{2}\right)-\cos\left(\frac{x+y}{2}\right).\cos\left(\frac{x-y}{2}\right)+\frac{1}{4}.\cos^2\left(\frac{x-y}{2}\right)+\frac{1}{4}\sin^2\left(\frac{x-y}{2}\right)=0$$

$$\Rightarrow\left(\cos\left(\frac{x+y}{2}\right)-\frac{1}{2}\cos\left(\frac{x-y}{2}\right)\right)^2+\frac{1}{4}\sin^2\left(\frac{x-y}{2}\right)=0$$

$\Rightarrow \sin\left(\frac{x-y}{2}\right)=0$ and

$$\cos\left(\frac{x+y}{2}\right)=\frac{1}{2}\cos\left(\frac{x-y}{2}\right)$$

$\Rightarrow x = y$ and $\cos x = \frac{1}{2} = \cos y$

$\therefore \sin x = \frac{\sqrt{3}}{2}$

$\Rightarrow \sin x + \cos y = \frac{1+\sqrt{3}}{2}$

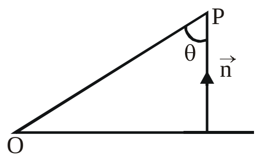
option (2)

7. A plane passes through the points A(1, 2, 3), B(2, 3, 1) and C(2, 4, 2). If O is the origin and P is (2, –1, 1), then the projection of $\overrightarrow{OP}$ on this plane is of length :

(1) $\sqrt{\frac{2}{7}}$ (2) $\sqrt{\frac{2}{3}}$ (3) $\sqrt{\frac{2}{11}}$ (4) $\sqrt{\frac{2}{5}}$

Official Ans. by NTA (3)

Sol.



Normal to plane $\vec{n} = \begin{vmatrix} \hat{i} & \hat{j} & \hat{k} \\ 1 & 1 & -2 \\ 0 & 1 & 1 \end{vmatrix}$

$= 3\hat{i}-\hat{j}+\hat{k}$

$\overrightarrow{OP} = 2\hat{i}-\hat{j}+\hat{k}$

$\cos\theta = \frac{6+1+1}{\sqrt{6}\sqrt{11}} = \frac{8}{\sqrt{66}} \Rightarrow \sin\theta = \sqrt{\frac{2}{66}}$

$\therefore$ Projection of $\overrightarrow{OP}$ on plane $= |\overrightarrow{OP}|\sin\theta$

$= \sqrt{\frac{2}{11}}$

option (3)

8. In a group of 400 people, 160 are smokers and non-vegetarian; 100 are smokers and vegetarian and the remaining 140 are non-smokers and vegetarian. Their chances of getting a particular chest disorder are 35%, 20% and 10% respectively. A person is chosen from the group at random and is found to be suffering from the chest disorder. The probability that the selected person is a smoker and non-vegetarian is :

(1) $\frac{7}{45}$ (2) $\frac{14}{45}$ (3) $\frac{28}{45}$ (4) $\frac{8}{45}$

Official Ans. by NTA (3)

Sol. Consider following events

A : Person chosen is a smoker and non vegetarian.

B : Person chosen is a smoker and vegetarian.

C : Person chosen is a non-smoker and vegetarian.

E : Person chosen has a chest disorder

Given

$$P(A)=\frac{160}{400}\quad P(B)=\frac{100}{400}\quad P(C)=\frac{140}{400}$$

$$P\left(\frac{E}{A}\right)=\frac{35}{100}\quad P\left(\frac{E}{B}\right)=\frac{20}{100}\quad P\left(\frac{E}{C}\right)=\frac{10}{100}$$

To find

$$P\left(\frac{A}{E}\right)=\frac{P(A)P\left(\frac{E}{A}\right)}{P(A).P\left(\frac{E}{A}\right)+P(B).P\left(\frac{E}{B}\right)+P(C).P\left(\frac{E}{C}\right)}$$

$$=\frac{\frac{160}{400}\times\frac{35}{100}}{\frac{160}{400}\times\frac{35}{100}+\frac{100}{400}\times\frac{20}{100}+\frac{140}{400}\times\frac{10}{100}}$$

$$=\frac{28}{45}$$ option (3)

9. $\operatorname{cosec}\left[2\cot^{-1}(5)+\cos^{-1}\left(\frac{4}{5}\right)\right]$ is equal to :

(1) $\frac{56}{33}$ (2) $\frac{65}{56}$ (3) $\frac{65}{33}$ (4) $\frac{75}{56}$

Official Ans. by NTA (2)

Sol. $\operatorname{cosec}\left[2\tan^{-1}\left(\frac{1}{5}\right)+\tan^{-1}\left(\frac{3}{4}\right)\right]$

$\operatorname{cosec}\left[\tan^{-1}\left(\frac{5}{12}\right)+\tan^{-1}\left(\frac{3}{4}\right)\right]$

$=\operatorname{cosec}\left[\tan^{-1}\left(\frac{56}{33}\right)\right]=\frac{65}{56}$ option (2)

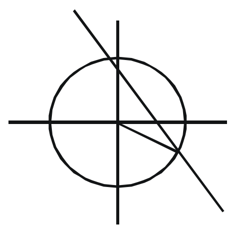
10. If the curve $x^2 + 2y^2 = 2$ intersects the line $x + y = 1$ at two points P and Q, then the angle subtended by the line segment PQ at the origin is :

(1) $\frac{\pi}{2}+\tan^{-1}\left(\frac{1}{3}\right)$ (2) $\frac{\pi}{2}-\tan^{-1}\left(\frac{1}{3}\right)$

(3) $\frac{\pi}{2}-\tan^{-1}\left(\frac{1}{4}\right)$ (4) $\frac{\pi}{2}+\tan^{-1}\left(\frac{1}{4}\right)$

Official Ans. by NTA (4)

Sol.



Homogenising

$x^2+2y^2-2(x+y)^2=0$

$\Rightarrow -x^2-4xy=0 \Rightarrow x^2+4xy=0$

Lines are $x = 0$ and $y=-\frac{x}{4}$

$\therefore$ Angle between lines $=\frac{\pi}{2}+\tan^{-1}\frac{1}{4}$

option (4)

11. The contrapositive of the statement "If you will work, you will earn money" is :

(1) You will earn money, if you will not work

(2) If you will earn money, you will work

(3) If you will not earn money, you will not work

(4) To earn money, you need to work

Official Ans. by NTA (3)

Sol. Constrapositive of $p \rightarrow q$ is $\sim q \rightarrow \sim p$

$\Rightarrow$ If you will not earn money, you will not work.

option (3)

12. A function f(x) is given by $f(x)=\frac{5^x}{5^x+5}$, then the sum of the series

$f\left(\frac{1}{20}\right)+f\left(\frac{2}{20}\right)+f\left(\frac{3}{20}\right)+......+f\left(\frac{39}{20}\right)$ is equal to :

(1) $\frac{19}{2}$ (2) $\frac{49}{2}$ (3) $\frac{29}{2}$ (4) $\frac{39}{2}$

Official Ans. by NTA (4)

Sol. $f(x)=\frac{5^x}{5^x+5}$ $\quad f(2-x)=\frac{5}{5^x+5}$

$f(x) + f(2 - x) = 1$

$\Rightarrow f\left(\frac{1}{20}\right)+f\left(\frac{2}{20}\right)+...+f\left(\frac{39}{20}\right)$

$=\left(f\left(\frac{1}{20}\right)+f\left(\frac{39}{20}\right)\right)+...+\left(f\left(\frac{19}{20}\right)+f\left(\frac{21}{20}\right)+f\left(\frac{20}{20}\right)\right)$

$=19+\frac{1}{2}=\frac{39}{2}$

13. If for the matrix, $A=\begin{bmatrix}1 & -\alpha\\ \alpha & \beta\end{bmatrix}$, $AA^T = I_2$, then the value of $\alpha^4 + \beta^4$ is :

(1) 4 (2) 2 (3) 3 (4) 1

Official Ans. by NTA (4)

Sol. $A=\begin{bmatrix}1 & -\alpha\\ \alpha & \beta\end{bmatrix}$ $\quad AA^T = I_2$

$\Rightarrow \begin{bmatrix}1 & -\alpha\\ \alpha & \beta\end{bmatrix}\begin{bmatrix}1 & \alpha\\ -\alpha & \beta\end{bmatrix}=\begin{bmatrix}1 & 0\\ 0 & 1\end{bmatrix}$

$\Rightarrow \begin{bmatrix}1+\alpha^2 & \alpha-\alpha\beta\\ \alpha-\alpha\beta & \alpha^2+\beta^2\end{bmatrix}=\begin{bmatrix}1 & 0\\ 0 & 1\end{bmatrix}$

$\Rightarrow \alpha^2 = 0$ & $\beta^2 = 1$

$\therefore \alpha^4 + \beta^4 = 1$

14. The minimum value of $f(x)=a^{a^x}+a^{1-a^x}$, where $a, x \in R$ and $a > 0$, is equal to :

(1) 2a (2) $2\sqrt{a}$

(3) $a+\frac{1}{a}$ (4) a + 1

Official Ans. by NTA (2)

Sol. A.M. $\geq$ G.M.

$f(x)=a^{a^x}+a^{1-a^x}=a^{a^x}+\frac{a}{a^{a^x}}\geq 2\sqrt{a}$

15. If $I_n=\int_{\frac{\pi}{4}}^{\frac{\pi}{2}}\cot^n x\,dx$, then :

(1) $\frac{1}{I_2+I_4},\frac{1}{I_3+I_5},\frac{1}{I_4+I_6}$ are in G.P.

(2) $I_2 + I_4$, $I_3 + I_5$, $I_4 + I_6$ are in A.P.

(3) $I_2 + I_4$, $(I_3 + I_5)^2$, $I_4 + I_6$ are in G.P.

(4) $\frac{1}{I_2+I_4},\frac{1}{I_3+I_5},\frac{1}{I_4+I_6}$ are in A.P.

Official Ans. by NTA (4)

Sol. $I_n=\int_{\pi/4}^{\pi/2}\cot^n x\,dx=\int_{\pi/4}^{\pi/2}\cot^{n-2}x(\text{cosec}^2x-1)\,dx$

$=-\left[\frac{\cot^{n-1}x}{n-1}\right]_{\pi/4}^{\pi/2}-I_{n-2}$

$=\frac{1}{n-1}-I_{n-2}$

$\Rightarrow I_n + I_{n-2} = \frac{1}{n-1}$

$\Rightarrow I_2 + I_4 = \frac{1}{3}$

$I_3 + I_5 = \frac{1}{4}$

$I_4 + I_6 = \frac{1}{5}$

$\therefore \frac{1}{I_2+I_4},\frac{1}{I_3+I_5},\frac{1}{I_4+I_6}$ are in A.P.

16. $\lim_{n\to\infty}\left[\frac{1}{n}+\frac{n}{(n+1)^2}+\frac{n}{(n+2)^2}+........+\frac{n}{(2n-1)^2}\right]$

is equal to :

(1) $\frac{1}{2}$ (2) 1

(3) $\frac{1}{3}$ (4) $\frac{1}{4}$

Official Ans. by NTA (1)

Sol. $\lim_{n\to\infty}\left[\frac{1}{n}+\frac{n}{(n+1)^2}+\frac{n}{(n+2)^2}+...+\frac{n}{(2n-1)^2}\right]$

$$=\lim_{n\to\infty}\sum_{r=0}^{n-1}\frac{n}{(n+r)^2}=\lim_{n\to\infty}\sum_{r=0}^{n-1}\frac{n}{n^2+2nr+r^2}$$

$$=\lim_{n\to\infty}\frac{1}{n}\sum_{r=0}^{n-1}\frac{1}{(r/n)^2+2(r/n)+1}$$

$$=\int_0^1\frac{dx}{(x+1)^2}=\left[\frac{-1}{(x+1)}\right]_0^1=\frac{1}{2}$$

17. Let A be a set of all 4-digit natural numbers whose exactly one digit is 7. Then the probability that a randomly chosen element of A leaves remainder 2 when divided by 5 is :

(1) $\frac{2}{9}$ (2) $\frac{122}{297}$

(3) $\frac{97}{297}$ (4) $\frac{1}{5}$

Official Ans. by NTA (3)

Sol. n(s) = n(when 7 appears on thousands place)

+ n(7 does not appear on thousands place)

$= 9 \times 9 \times 9 + 8 \times 9 \times 9 \times 3$

$= 33 \times 9 \times 9$

n(E) = n(last digit 7 & 7 appears once)

+ n(last digit 2 when 7 appears once)

$= 8 \times 9 \times 9 + (9 \times 9 + 8 \times 9 \times 2)$

$$\therefore\ P(E)=\frac{8\times9\times9+9\times25}{33\times9\times9}=\frac{97}{297}$$

18. Let α and β be the roots of $x^2 - 6x - 2 = 0$. If $a_n = \alpha^n - \beta^n$ for $n \geq 1$, then the value of $\frac{a_{10}-2a_8}{3a_9}$ is :

(1) 2 (2) 1 (3) 4 (4) 3

Official Ans. by NTA (1)

Sol. $\alpha^2 - 6\alpha - 2 = 0$

$\alpha^{10} - 6\alpha^9 - 2\alpha^8 = 0$

Similarly $\beta^{10} - 6\beta^9 - 2\beta^8 = 0$

$(\alpha^{10} - \beta^{10}) - 6(\alpha^9 - \beta^9) - 2(\alpha^8 - \beta^8) = 0$

$\Rightarrow a_{10} - 6a_9 - 2a_8 = 0$

$$\Rightarrow \frac{a_{10}-2a_8}{3a_9}=2$$

19. Let x denote the total number of one-one functions from a set A with 3 elements to a set B with 5 elements and y denote the total number of one-one functions from the set A to the set A × B. Then :

(1) $y = 273x$ (2) $2y = 91x$

(3) $y = 91x$ (4) $2y = 273x$

Official Ans. by NTA (2)

Sol. $x = {}^5C_3 \times 3! = 60$

$y = {}^{15}C_3 \times 3! = 15 \times 14 \times 13 = 30 \times 91$

$\therefore 2y = 91x$

20. The following system of linear equations

$2x + 3y + 2z = 9$

$3x + 2y + 2z = 9$

$x - y + 4z = 8$

(1) has a solution (α, β, γ) satisfying $\alpha + \beta^2 + \gamma^3 = 12$

(2) has infinitely many solutions

(3) does not have any solution

(4) has a unique solution

Official Ans. by NTA (4)

Sol. $2x + 3y + 2z = 9$ ...(1)

$3x + 2y + 2z = 9$ ...(2)

$x - y + 4z = 8$ ...(3)

(1) – (2) $\Rightarrow -x + y = 0 \Rightarrow x - y = 0$

from (3) $4z = 8 \Rightarrow z = 2$

from (1) $2x + 3y = 5$

$\Rightarrow x = y = 1$

$\therefore$ system has unique solution

SECTION-B

1. The total number of two digit numbers 'n', such that $3^n + 7^n$ is a multiple of 10, is ______.

Official Ans. by NTA (45)

Sol. for $3^n + 7^n$ to be divisible by 10

n can be any odd number

$\therefore$ Number of odd two digit numbers = 45

2. A function f is defined on [–3, 3] as

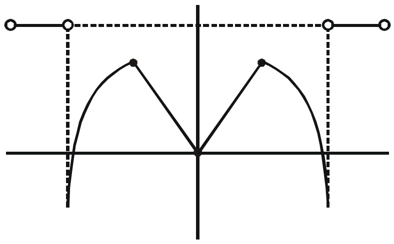
$$f(x) = \begin{cases} \min\{|x|, 2 - x^2\}, & -2 \le x \le 2 \\ [|x|] & , 2 < |x| \le 3 \end{cases}$$

where [x] denotes the greatest integer $\le$ x. The number of points, where f is not differentiable in (–3, 3) is ______.

Official Ans. by NTA (5)

Sol. $$f(x) = \begin{cases} \min\{|x|, 2 - x^2\} & , \quad -2 \le x \le 2 \\ [|x|] & , \quad 2 < |x| \le 3 \end{cases}$$

$\Rightarrow x \in [-3, -2) \cup (2, 3]$



Number of points of non-differentiability in (–3, 3) = 5

3. Let $\vec{a} = \hat{i} + \alpha\hat{j} + 3\hat{k}$ and $\vec{b} = 3\hat{i} - \alpha\hat{j} + \hat{k}$. If the area of the parallelogram whose adjacent sides are represented by the vectors $\vec{a}$ and $\vec{b}$ is $8\sqrt{3}$ square units, then $\vec{a} \cdot \vec{b}$ is equal to ______:

Official Ans. by NTA (2)

Sol. $\vec{a} = \hat{i} + \alpha\hat{j} + 3\hat{k}$

$\vec{b} = 3\hat{i} - \alpha\hat{j} + \hat{k}$

area of parallelogram = $|\vec{a} \times \vec{b}| = 8\sqrt{3}$.

$$\vec{a} \times \vec{b} = \begin{vmatrix} \hat{i} & \hat{j} & \hat{k} \\ 1 & \alpha & 3 \\ 3 & -\alpha & 1 \end{vmatrix} = \hat{i}(4\alpha) - \hat{j}(-8) + \hat{k}(-4\alpha)$$

$\therefore |\vec{a} \times \vec{b}| = \sqrt{64 + 32\alpha^2} = 8\sqrt{3}$

$\Rightarrow 2 + \alpha^2 = 6 \Rightarrow \alpha^2 = 4$

$\therefore \vec{a} \cdot \vec{b} = 3 - \alpha^2 + 3 = 2$

4. If the remainder when x is divided by 4 is 3, then the remainder when $(2020 + x)^{2022}$ is divided by 8 is ______.

Official Ans. by NTA (1)

Sol. $x = 4k + 3$

$\therefore (2020 + x)^{2022} = (2020 + 4k + 3)^{2022}$

$= (4(505 + k) + 3)^{2022}$

$= (4\lambda + 3)^{2022} = (16\lambda^2 + 24\lambda + 9)^{1011}$

$= (8(2\lambda^2 + 3\lambda + 1) + 1)^{1011}$

$= (8p + 1)^{1011}$

$\therefore$ Remainder when divided by 8 = 1

5. If the curves $x = y^4$ and $xy = k$ cut at right angles, then $(4k)^6$ is equal to ______.

Official Ans. by NTA (4)

Sol. $x = y^4 xy = k$

for intersection $\quad y^5 = k \ldots(1)$

Also $\quad x = y^4$

$$\Rightarrow 1 = 4y^3 \frac{dy}{dx} \Rightarrow \frac{dy}{dx} = \frac{1}{4y^3}$$

for $xy = k \Rightarrow x = \dfrac{k}{y}$

$$\Rightarrow 1 = -\frac{k}{y^2} \cdot \frac{dy}{dx}$$

$$\Rightarrow \frac{dy}{dx} = \frac{-y^2}{k}$$

$\because$ Curve cut orthogonally

$$\Rightarrow \frac{1}{4y^3} \times \left(\frac{-y^2}{k}\right) = -1$$

$$\Rightarrow y = \frac{1}{4k}$$

$\therefore$ from (1) $y^5 = k$

$$\Rightarrow \frac{1}{(4k)^5} = k$$

$\Rightarrow 4 = (4k)^6$