

FINAL JEE-MAIN EXAMINATION – SEPTEMBER, 2020

(Held On Friday 04th SEPTEMBER, 2020) TIME : 9 AM to 12 PM

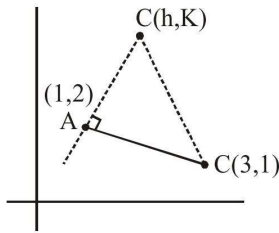
MATHEMATICS	TEST PAPER WITH SOLUTION
<p>1. If $A = \begin{bmatrix} \cos\theta & i\sin\theta \\ i\sin\theta & \cos\theta \end{bmatrix}$, $\left(\theta = \frac{\pi}{24}\right)$ and $A^5 = \begin{bmatrix} a & b \\ c & d \end{bmatrix}$, where $i = \sqrt{-1}$, then which one of the following is not true?</p> <p>(1) $0 \leq a^2 + b^2 \leq 1$ (2) $a^2 - d^2 = 0$</p> <p>(3) $a^2 - b^2 = \frac{1}{2}$ (4) $a^2 - c^2 = 1$</p> <p>Official Ans. by NTA (3)</p> <p>Sol. $A^2 = \begin{pmatrix} \cos 2\theta & i\sin 2\theta \\ i\sin 2\theta & \cos 2\theta \end{pmatrix}$</p> <p>Similarly, $A^5 = \begin{pmatrix} \cos 5\theta & i\sin 5\theta \\ i\sin 5\theta & \cos 5\theta \end{pmatrix} = \begin{pmatrix} a & b \\ c & d \end{pmatrix}$</p> <p>(1) $a^2 + b^2 = \cos^2 5\theta - \sin^2 5\theta = \cos 10\theta = \cos 75^\circ$</p> <p>(2) $a^2 - d^2 = \cos^2 5\theta - \cos^2 5\theta = 0$</p> <p>(3) $a^2 - b^2 = \cos^2 5\theta + \sin^2 5\theta = 1$</p> <p>(4) $a^2 - c^2 = \cos^2 5\theta + \sin^2 5\theta = 1$</p> <p>2. Let $[t]$ denote the greatest integer $\leq t$. Then the equation in x, $[x]^2 + 2[x + 2] - 7 = 0$ has :</p> <p>(1) no integral solution</p> <p>(2) exactly four integral solutions</p> <p>(3) exactly two solutions</p> <p>(4) infinitely many solutions</p> <p>Official Ans. by NTA (4)</p> <p>Sol. $[x]^2 + 2[x + 2] - 7 = 0$ $\Rightarrow [x]^2 + 2[x] + 4 - 7 = 0$ $\Rightarrow [x] = 1, -3$ $\Rightarrow x \in [1, 2) \cup [-3, -2)$</p>	<p>3. Let α and β be the roots of $x^2 - 3x + p = 0$ and γ and δ be the roots of $x^2 - 6x + q = 0$. If $\alpha, \beta, \gamma, \delta$ form a geometric progression. Then ratio $(2q + p) : (2q - p)$ is :</p> <p>(1) 3 : 1 (2) 33 : 31</p> <p>(3) 9 : 7 (4) 5 : 3</p> <p>Official Ans. by NTA (3)</p> <p>Sol. $x^2 - 3x + p = 0 \begin{matrix} \alpha \\ \beta \end{matrix}$</p> <p>$\alpha, \beta, \gamma, \delta$ in G.P. $\alpha + \alpha r = 3 \dots(1)$</p> <p>$x^2 - 6x + q = 0 \begin{matrix} \gamma \\ \delta \end{matrix}$</p> <p>$\alpha r^2 + \alpha r^3 = 6 \dots(2)$</p> <p>(2) \div (1) $r^2 = 2$</p> <p>So, $\frac{2q+p}{2q-p} = \frac{2r^5+r}{2r^5-r} = \frac{2r^4+1}{2r^4-1} = \frac{9}{7}$</p> <p>4. Let $\frac{x^2}{a^2} + \frac{y^2}{b^2} = 1$ ($a > b$) be a given ellipse, length of whose latus rectum is 10. If its eccentricity is the maximum value of the function, $\phi(t) = \frac{5}{12} + t - t^2$, then $a^2 + b^2$ is equal to :</p> <p>(1) 126 (2) 135</p> <p>(3) 145 (4) 116</p> <p>Official Ans. by NTA (1)</p> <p>Sol. $\frac{x^2}{a^2} + \frac{y^2}{b^2} = 1$ ($a > b$); $\frac{2b^2}{a} = 10 \Rightarrow b^2 = 5a \dots(i)$</p> <p>Now, $\phi(t) = \frac{5}{12} + t - t^2 = \frac{8}{12} - \left(t - \frac{1}{2}\right)^2$</p> <p>$\phi(t)_{\max} = \frac{8}{12} = \frac{2}{3} = e \Rightarrow e^2 = 1 - \frac{b^2}{a^2} = \frac{4}{9} \dots(ii)$</p> <p>$\Rightarrow a^2 = 81$ (from (i) & (ii))</p> <p>So, $a^2 + b^2 = 81 + 45 = 126$</p>

5. A triangle ABC lying in the first quadrant has two vertices as A(1, 2) and B(3, 1). If $\angle BAC = 90^\circ$, and $\text{ar}(\Delta ABC) = 5\sqrt{5}$ sq. units, then the abscissa of the vertex C is :

- (1) $2 + \sqrt{5}$ (2) $1 + \sqrt{5}$
 (3) $1 + 2\sqrt{5}$ (4) $2\sqrt{5} - 1$

Official Ans. by NTA (3)

Sol.



$$\left(\frac{K-2}{h-1}\right)\left(\frac{1-2}{3-1}\right) = -1 \Rightarrow K = 2h \quad \dots(1)$$

$$\sqrt{5} |h-1| = 10$$

$$\therefore [\Delta ABC] = 5\sqrt{5}$$

$$\Rightarrow \frac{1}{2}(\sqrt{5})\sqrt{(h-1)^2 + (K-2)^2} = 5\sqrt{5} \quad \dots(2)$$

$$\Rightarrow h = 2\sqrt{5} + 1 \quad (h > 0)$$

6. Let $f(x) = |x - 2|$ and $g(x) = f(f(x))$, $x \in [0, 4]$.

Then $\int_0^3 (g(x) - f(x)) dx$ is equal to :

- (1) $\frac{3}{2}$ (2) 0
 (3) $\frac{1}{2}$ (4) 1

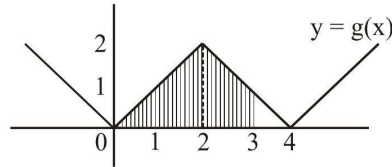
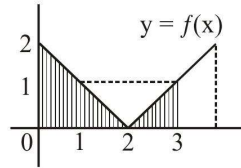
Official Ans. by NTA (4)

Sol.

$$\int_0^3 g(x) - f(x) dx = \int_0^3 |x-2| - 2 |dx| - \int_0^3 |x-2| dx$$

$$= \left(\frac{1}{2} \times 2 \times 2 + 1 + \frac{1}{2} \times 1 \times 1\right) - \left(\frac{1}{2} \times 2 \times 2 + \frac{1}{2} \times 1 \times 1\right)$$

$$= \left(2 + 1 + \frac{1}{2}\right) - \left(2 + \frac{1}{2}\right) = 1$$



7. Given the following two statements :

- (S₁) : $(q \vee p) \rightarrow (p \leftrightarrow \sim q)$ is a tautology.
 (S₂) : $\sim q \wedge (\sim p \leftrightarrow q)$ is a fallacy.

Then :

- (1) only (S₁) is correct.
 (2) both (S₁) and (S₂) are correct.
 (3) both (S₁) and (S₂) are not correct.
 (4) only (S₂) is correct.

Official Ans. by NTA (3)

Sol. Let TV(r) denotes truth value of a statement r.

Now, if $\text{TV}(p) = \text{TV}(q) = T$

$$\Rightarrow \text{TV}(S_1) = F$$

Also, if $\text{TV}(p) = T$ & $\text{TV}(q) = F$

$$\Rightarrow \text{TV}(S_2) = T$$

8. Let P(3, 3) be a point on the hyperbola,

$$\frac{x^2}{a^2} - \frac{y^2}{b^2} = 1. \text{ If the normal to it at P intersects}$$

the x-axis at (9, 0) and e is its eccentricity, then the ordered pair (a^2, e^2) is equal to :

- (1) $\left(\frac{9}{2}, 3\right)$ (2) $\left(\frac{9}{2}, 2\right)$
 (3) $\left(\frac{3}{2}, 2\right)$ (4) (9, 3)

Official Ans. by NTA (1)

Sol. Since, (3, 3) lies on $\frac{x^2}{a^2} - \frac{y^2}{b^2} = 1$

$$\frac{9}{a^2} - \frac{9}{b^2} = 1 \quad \dots(1)$$

Now, normal at (3, 3) is $y - 3 = -\frac{a^2}{b^2}(x - 3)$,

which passes through (9, 0) $\Rightarrow b^2 = 2a^2 \quad \dots(2)$

So, $e^2 = 1 + \frac{b^2}{a^2} = 3$

Also, $a^2 = \frac{9}{2}$ (from (i) & (ii))

Thus, $(a^2, e^2) = \left(\frac{9}{2}, 3\right)$

9. Let $f(x) = \int \frac{\sqrt{x}}{(1+x)^2} dx$ ($x \geq 0$). Then $f(3) - f(1)$

is equal to :

(1) $-\frac{\pi}{6} + \frac{1}{2} + \frac{\sqrt{3}}{4}$ (2) $\frac{\pi}{6} + \frac{1}{2} - \frac{\sqrt{3}}{4}$

(3) $-\frac{\pi}{12} + \frac{1}{2} + \frac{\sqrt{3}}{4}$ (4) $\frac{\pi}{12} + \frac{1}{2} - \frac{\sqrt{3}}{4}$

Official Ans. by NTA (4)

Sol. $f(x) = \int_1^3 \frac{\sqrt{x}}{(1+x)^2} dx = \int_1^{\sqrt{3}} \frac{t \cdot 2t dt}{(1+t^2)^2}$ (put $\sqrt{x} = t$)

$$= \left(-\frac{t}{1+t^2}\right)_1^{\sqrt{3}} + (\tan^{-1} t)_1^{\sqrt{3}} \quad [\text{Applying by parts}]$$

$$= -\left(\frac{\sqrt{3}}{4} - \frac{1}{2}\right) + \frac{\pi}{3} - \frac{\pi}{4}$$

$$= \frac{1}{2} - \frac{\sqrt{3}}{4} + \frac{\pi}{12}$$

10. A survey shows that 63% of the people in a city read newspaper A whereas 76% read newspaper B. If x% of the people read both the newspapers, then a possible value of x can be:

- (1) 65 (2) 37
 (3) 29 (4) 55

Official Ans. by NTA (4)

Sol. $n(B) \leq n(A \cup B) \leq n(U)$
 $\Rightarrow 76 \leq 76 + 63 - x \leq 100$
 $\Rightarrow -63 \leq -x \leq -39$
 $\Rightarrow 63 \geq x \geq 39$

11. Let $u = \frac{2z+i}{z-ki}$, $z = x + iy$ and $k > 0$. If the curve represented by $\text{Re}(u) + \text{Im}(u) = 1$ intersects the y-axis at the points P and Q where $PQ = 5$, then the value of k is :

- (1) 3/2 (2) 4
 (3) 2 (4) 1/2

Official Ans. by NTA (3)

Sol. $u = \frac{2z+i}{z-ki}$
 $= \frac{2x^2 + (2y+1)(y-k)}{x^2 + (y-k)^2} + i \frac{(x(2y+1) - 2x(y-k))}{x^2 + (y-k)^2}$

Since $\text{Re}(u) + \text{Im}(u) = 1$
 $\Rightarrow 2x^2 + (2y+1)(y-k) + x(2y+1) - 2x(y-k) = x^2 + (y-k)^2$

$$\left. \begin{matrix} P(0, y_1) \\ Q(0, y_2) \end{matrix} \right\} \Rightarrow y^2 + y - k - k^2 = 0 \left\{ \begin{matrix} y_1 + y_2 = -1 \\ y_1 y_2 = -k - k^2 \end{matrix} \right.$$

$\therefore PQ = 5$
 $\Rightarrow |y_1 - y_2| = 5 \Rightarrow k^2 + k - 6 = 0$
 $\Rightarrow k = -3, 2$
 So, $k = 2$ ($k > 0$)

16. If $1+(1-2^2.1)+(1-4^2.3)+(1-6^2.5)+\dots+(1-20^2.19) = \alpha - 220\beta$, then an ordered pair (α, β) is equal to :

- (1) (10, 97) (2) (11, 103)
 (3) (10, 103) (4) (11, 97)

Official Ans. by NTA (2)

Sol. $1 + (1 - 2^2.1) + (1 - 4^2.3) + \dots + (1 - 20^2.19) = \alpha - 220\beta$
 $= 11 - (2^2.1 + 4^2.3 + \dots + 20^2.19)$
 $= 11 - 2^2 \cdot \sum_{r=1}^{10} r^2(2r-1) = 11 - 4 \left(\frac{110^2}{2} - 35 \times 11 \right)$
 $= 11 - 220(103)$
 $\Rightarrow \alpha = 11, \beta = 103$

17. Let $y = y(x)$ be the solution of the differential equation, $xy' - y = x^2(x \cos x + \sin x)$, $x > 0$.

If $y(\pi) = \pi$, then $y''\left(\frac{\pi}{2}\right) + y\left(\frac{\pi}{2}\right)$ is equal to :

- (1) $2 + \frac{\pi}{2}$ (2) $1 + \frac{\pi}{2}$
 (3) $1 + \frac{\pi}{2} + \frac{\pi^2}{4}$ (4) $2 + \frac{\pi}{2} + \frac{\pi^2}{4}$

Official Ans. by NTA (1)

Sol. $x \frac{dy}{dx} - y = x^2(x \cos x + \sin x)$, $x > 0$

$$\frac{dy}{dx} - \frac{y}{x} = x(x \cos x + \sin x) \Rightarrow \frac{dy}{dx} + Py = Q$$

so, I.F. = $e^{\int -\frac{1}{x} dx} = \frac{1}{|x|} = \frac{1}{x}$ ($x > 0$)

Thus, $\frac{y}{x} = \int \frac{1}{x}(x(x \cos x + \sin x)) dx$

$$\Rightarrow \frac{y}{x} = x \sin x + C$$

$\therefore y(\pi) = \pi \Rightarrow C = 1$

so, $y = x^2 \sin x + x \Rightarrow (y)_{\pi/2} = \frac{\pi^2}{4} + \frac{\pi}{2}$

Also, $\frac{dy}{dx} = x^2 \cos x + 2x \sin x + 1$

$$\Rightarrow \frac{d^2y}{dx^2} = -x^2 \sin x + 4x \cos x + 2 \sin x$$

$$\Rightarrow \frac{d^2y}{dx^2} \Big|_{\frac{\pi}{2}} = -\frac{\pi^2}{4} + 2$$

Thus, $y\left(\frac{\pi}{2}\right) + \frac{d^2y}{dx^2}\left(\frac{\pi}{2}\right) = \frac{\pi}{2} + 2$

18. The value of $\sum_{r=0}^{20} {}^{50-r}C_6$ is equal to :

- (1) ${}^{51}C_7 + {}^{30}C_7$ (2) ${}^{51}C_7 - {}^{30}C_7$
 (3) ${}^{50}C_7 - {}^{30}C_7$ (4) ${}^{50}C_6 - {}^{30}C_6$

Official Ans. by NTA (2)

Sol. $\sum_{r=0}^{20} {}^{50-r}C_6 = {}^{50}C_6 + {}^{49}C_6 + {}^{48}C_6 + \dots + {}^{30}C_6$
 $= {}^{50}C_6 + {}^{49}C_6 + \dots + {}^{31}C_6 + ({}^{30}C_6 + {}^{30}C_7) - {}^{30}C_7$
 $= {}^{50}C_6 + {}^{49}C_6 + \dots + ({}^{31}C_6 + {}^{31}C_7) - {}^{30}C_7$
 $= {}^{50}C_6 + {}^{50}C_7 - {}^{30}C_7$
 $= {}^{51}C_7 - {}^{30}C_7$

$$\boxed{{}^nC_r + {}^nC_{r-1} = {}^{n+1}C_r}$$

19. Let f be a twice differentiable function on $(1, 6)$. If $f(2) = 8$, $f'(2) = 5$, $f'(x) \geq 1$ and $f''(x) \geq 4$, for all $x \in (1, 6)$, then :

- (1) $f(5) \leq 10$ (2) $f'(5) + f''(5) \leq 20$
 (3) $f(5) + f'(5) \geq 28$ (4) $f(5) + f'(5) \leq 26$

Official Ans. by NTA (3)

Sol. $f(2) = 8$, $f'(2) = 5$, $f'(x) \geq 1$, $f''(x) \geq 4$, $\forall x \in (1, 6)$

$$f''(x) = \frac{f'(5) - f'(2)}{5 - 2} \geq 4 \Rightarrow f'(5) \geq 17 \quad \dots(1)$$

$$f'(x) = \frac{f(5) - f(2)}{5 - 2} \geq 1 \Rightarrow f(5) \geq 11 \quad \dots(2)$$

$$\overline{f'(5) + f(5)} \geq 28$$

20. If $(a + \sqrt{2} b \cos x)(a - \sqrt{2} b \cos y) = a^2 - b^2$,

where $a > b > 0$, then $\frac{dx}{dy}$ at $(\frac{\pi}{4}, \frac{\pi}{4})$ is :

- (1) $\frac{a-b}{a+b}$ (2) $\frac{a+b}{a-b}$
 (3) $\frac{2a+b}{2a-b}$ (4) $\frac{a-2b}{a+2b}$

Official Ans. by NTA (2)

Sol. $(a + \sqrt{2} b \cos x)(a - \sqrt{2} b \cos y) = a^2 - b^2$

$$\Rightarrow a^2 - \sqrt{2} ab \cos y + \sqrt{2} ab \cos x - 2b^2 \cos x \cos y = a^2 - b^2$$

Differentiating both sides :

$$0 - \sqrt{2} ab \left(-\sin y \frac{dy}{dx} \right) + \sqrt{2} ab (-\sin x) - 2b^2 \left[\cos x \left(-\sin y \frac{dy}{dx} \right) + \cos y (-\sin x) \right] = 0$$

At $(\frac{\pi}{4}, \frac{\pi}{4})$:

$$ab \frac{dy}{dx} - ab - 2b^2 \left(-\frac{1}{2} \frac{dy}{dx} - \frac{1}{2} \right) = 0$$

$$\Rightarrow \frac{dx}{dy} = \frac{ab + b^2}{ab - b^2} = \frac{a+b}{a-b} ; \quad a, b > 0$$

21. If the system of equations

$$x - 2y + 3z = 9$$

$$2x + y + z = b$$

$$x - 7y + az = 24,$$

has infinitely many solutions, then $a - b$ is equal to _____ .

Official Ans. by NTA (5)

Sol. $D = \begin{vmatrix} 1 & -2 & 3 \\ 2 & 1 & 1 \\ 1 & -7 & a \end{vmatrix} = 0 \Rightarrow a = 8$

also, $D_1 = \begin{vmatrix} 9 & -2 & 3 \\ b & 1 & 1 \\ 24 & -7 & 8 \end{vmatrix} = 0 \Rightarrow b = 3$

hence, $a - b = 8 - 3 = 5$

22. The probability of a man hitting a target is $\frac{1}{10}$.

The least number of shots required, so that the probability of his hitting the target at least once

is greater than $\frac{1}{4}$, is _____.

Official Ans. by NTA (3)

Sol. We have, $1 - (\text{probability of all shots result in failure}) > \frac{1}{4}$

$$\Rightarrow 1 - \left(\frac{9}{10}\right)^n > \frac{1}{4}$$

$$\Rightarrow 1 - \left(\frac{9}{10}\right)^n > \frac{1}{4}$$

$$\Rightarrow \frac{3}{4} > \left(\frac{9}{10}\right)^n \Rightarrow n \geq 3$$

23. Suppose a differentiable function $f(x)$ satisfies the identity $f(x + y) = f(x) + f(y) + xy^2 + x^2y$,

for all real x and y . If $\lim_{x \rightarrow 0} \frac{f(x)}{x} = 1$, then $f'(3)$

is equal to _____.

Official Ans. by NTA (10)

Sol. Since, $\lim_{x \rightarrow 0} \frac{f(x)}{x}$ exist $\Rightarrow f(0) = 0$

$$\text{Now, } f'(x) = \lim_{h \rightarrow 0} \frac{f(x+h) - f(x)}{h}$$

$$= \lim_{h \rightarrow 0} \frac{f(h) + xh^2 + x^2h}{h} \quad (\text{take } y = h)$$

$$= \lim_{h \rightarrow 0} \frac{f(h)}{h} + \lim_{h \rightarrow 0} (xh) + x^2$$

$$\Rightarrow f'(x) = 1 + 0 + x^2$$

$$\Rightarrow f'(3) = 10$$

24. Let $(2x^2 + 3x + 4)^{10} = \sum_{r=0}^{20} a_r x^r$. Then $\frac{a_7}{a_{13}}$ is equal to _____.

Official Ans. by NTA (8)

Sol. Given $(2x^2 + 3x + 4)^{10} = \sum_{r=0}^{20} a_r x^r$ (1)

replace x by $\frac{2}{x}$ in above identity :-

$$\frac{2^{10} (2x^2 + 3x + 4)^{10}}{x^{20}} = \sum_{r=0}^{20} \frac{a_r 2^r}{x^r}$$

$$\Rightarrow 2^{10} \sum_{r=0}^{20} a_r x^r = \sum_{r=0}^{20} a_r 2^r x^{(20-r)} \text{ (from (i))}$$

now, comparing coefficient of x^7 from both sides

(take $r = 7$ in L.H.S. & $r = 13$ in R.H.S.)

$$2^{10} a_7 = a_{13} 2^{13} \Rightarrow \frac{a_7}{a_{13}} = 2^3 = 8$$

25. If the equation of a plane P, passing through the intersection of the planes, $x + 4y - z + 7 = 0$ and $3x + y + 5z = 8$ is $ax + by + 6z = 15$ for some $a, b \in \mathbb{R}$, then the distance of the point $(3, 2, -1)$ from the plane P is _____.

Official Ans. by NTA (3)

Sol. $D_1 = \begin{vmatrix} -7 & 4 & -1 \\ 8 & 1 & 5 \\ 15 & b & 6 \end{vmatrix} = 0 \Rightarrow b = -3$

$$D = \begin{vmatrix} 1 & 4 & -1 \\ 3 & 1 & 5 \\ a & b & 6 \end{vmatrix} = 0 \Rightarrow 21a - 8b - 66 = 0 \text{ (1)}$$

$$P : 2x - 3y + 6z = 15$$

$$\text{so required distance} = \frac{21}{7} = 3$$