FINAL JEE–MAIN EXAMINATION – SEPTEMBER, 2020

(Held On Friday 04th SEPTEMBER, 2020) TIME : 3 PM to 6 PM

MATHEMATICS — TEST PAPER WITH SOLUTION

1. The function $f(x) = \begin{cases} \frac{\pi}{4} + \tan^{-1} x, & |x| \le 1 \\ \frac{1}{2}(|x|-1), & |x| > 1 \end{cases}$ is :

(1) continuous on R–{1} and differentiable on R – {–1, 1}.

(2) both continuous and differentiable on R – {–1}.

(3) continuous on R – {–1} and differentiable on R – {–1, 1}.

(4) both continuous and differentiable on R –{1}

Official Ans. by NTA (1)

Sol. $f(x) = \begin{cases} \frac{\pi}{4} + \tan^{-1} x, & x \in (-\infty, -1] \cup [1, \infty) \\ -\frac{(x+1)}{2}, & x \in (-1, 0] \\ \frac{x-1}{2}, & x \in (0, 1) \end{cases}$

for continuity at x = –1

L.H.L. = $\frac{\pi}{4} - \frac{\pi}{4} = 0$

R.H.L. = 0

so, continuous at x = –1

for continuity at x = 1

L.H.L. = 0

R.H.L. = $\frac{\pi}{4} + \frac{\pi}{4} = \frac{\pi}{2}$

so, not continuous at x = 1

For differentiability at x = –1

L.H.D. = $\frac{1}{1+1} = \frac{1}{2}$

R.H.D. = $-\frac{1}{2}$

so, non differentiable at x = –1

2. Let $\bigcup_{i=1}^{50} X_i = \bigcup_{i=1}^{n} Y_i = T$, where each X_i contains 10 elements and each Y_i contains 5 elements. If each element of the set T is an element of exactly 20 of sets X_i's and exactly 6 of sets Y_i's, then n is equal to :

(1) 45 (2) 15

(3) 50 (4) 30

Official Ans. by NTA (4)

Sol. $n(X_i) = 10$. $\bigcup_{i=1}^{50} X_i = T, \Rightarrow$ n (T) = 500

each element of T belongs to exactly 20 elements of $X_i \Rightarrow \frac{500}{20} = 25$ distinct elements

so $\frac{5n}{6} = 25 \Rightarrow n = 30$

3. Let $\lambda \neq 0$ be in R. If α and β are the roots of the equation, $x^2 - x + 2\lambda = 0$ and α and γ are the roots of the equation, $3x^2 - 10x + 27\lambda = 0$, then $\frac{\beta\gamma}{\lambda}$ is equal to :

(1) 36 (2) 27

(3) 9 (4) 18

Official Ans. by NTA (4)

Sol. $\alpha + \beta = 1, \ \alpha\beta = 2\lambda$

$\alpha + \beta = \frac{10}{3}, \quad \alpha\gamma = \frac{27\lambda}{3} = 9\lambda$

$\gamma - \beta = \frac{7}{3},$

$\frac{\gamma}{\beta} = \frac{9}{2} \Rightarrow \gamma = \frac{9}{2}\beta = \frac{9}{2} \times \frac{2}{3} \Rightarrow \gamma = 3$

$\frac{9}{2}\beta - \beta = \frac{7}{3}$

$\frac{9}{2}\beta = \frac{7}{3} \Rightarrow \beta = \frac{2}{3}$

$\alpha = 1 - \frac{2}{3} = \frac{1}{3}$

$2\lambda = \frac{2}{9} \Rightarrow \lambda = \frac{1}{9}$

$\frac{\beta\gamma}{\lambda} = \frac{\frac{2}{3} \times 3}{\frac{1}{9}} = 18$

4. The solution of the differential equation $\frac{dy}{dx} - \frac{y+3x}{\log_e(y+3x)} + 3 = 0$ is :-

(where C is a constant of integration.)

(1) $x-2\log_e(y+3x)=C$

(2) $x-\log_e(y+3x)=C$

(3) $x-\frac{1}{2}(\log_e(y+3x))^2 = C$

(4) $y + 3x-\frac{1}{2}(\log_e x)^2 = C$

Official Ans. by NTA (3)

Sol. $\ell n(y + 3x) = z$ (let)

$\frac{1}{y+3x}\cdot\left(\frac{dy}{dx}+3\right) = \frac{dz}{dx}$ ..(1)

$\frac{dy}{dx}+3 = \frac{y+3x}{\ell n(y+3x)}$ (given)

$\frac{dz}{dx} = \frac{1}{z}$

$\Rightarrow z\,dz = dx \Rightarrow \frac{z^2}{2} = x + C$

$\Rightarrow \frac{1}{2}\ell n^2(y+3x) = x+C$

$\Rightarrow x - \frac{1}{2}(\ell n(y + 3x))^2 = C$

5. Let $a_1, a_2..., a_n$ be a given A.P. whose common difference is an integer and $S_n = a_1 + a_2 + ..+ a_n$. If $a_1 = 1$, $a_n = 300$ and $15 \le n \le 50$, then the ordered pair (S_{n-4}, a_{n-4}) is equal to :

(1) (2480, 249) (2) (2490, 249)

(3) (2490, 248) (4) (2480, 248)

Official Ans. by NTA (3)

Sol. $a_n = a_1 + (n - 1)d$

$\Rightarrow 300 = 1 + (n - 1) d$

$\Rightarrow (n - 1)d = 299 = 13 \times 23$

since, $n \in [15, 50]$

$\therefore$ n = 24 and d = 13

$a_{n-4} = a_{20} = 1 + 19 \times 13 = 248$

$\Rightarrow a_{n-4} = 248$

$S_{n-4} = \frac{20}{2}\{1+248\} = 2490$

6. The distance of the point (1, – 2, 3) from the plane x–y+z = 5 measured parallel to the line $\frac{x}{2} = \frac{y}{3} = \frac{z}{-6}$ is :

(1) 7 (2) 1 (3) $\frac{1}{7}$ (4) $\frac{7}{5}$

Official Ans. by NTA (2)

Sol. equation of line parallel to $\frac{x}{2} = \frac{y}{3} = \frac{z}{-6}$ passes through (1, –2, 3) is

$\frac{x-1}{2} = \frac{y+2}{3} = \frac{z-3}{-6} = r$

x = 2r + 1

y = 3r –2,

z = –6r + 3

So 2r + 1 – 3r + 2– 6r + 3 = 5

$\Rightarrow$ –7r + 1 = 0

$r = \frac{1}{7}$

$x = \frac{9}{7}$, $y = \frac{-11}{7}$,, $z = \frac{15}{7}$

Distance is $= \sqrt{\left(\frac{9}{7}-1\right)^2 + \left(2-\frac{11}{7}\right)^2 + \left(3-\frac{15}{7}\right)^2}$

$= \sqrt{\left(\frac{2}{7}\right)^2 + \left(\frac{3}{7}\right)^2 + \left(\frac{6}{7}\right)^2}$

$= \frac{1}{7}\sqrt{4+9+36}$

$= \frac{1}{7}\sqrt{49} = 1$

7. Let $f : (0, \infty) \to (0, \infty)$ be a differentiable function such that $f(1) = e$ and

$$\lim_{t \to x} \frac{t^2 f^2(x) - x^2 f^2(t)}{t - x} = 0$$

If $f(x) = 1$, then x is equal to :

(1) 2e (2) $\frac{1}{2e}$ (3) e (4) $\frac{1}{e}$

Official Ans. by NTA (4)

Sol. $L = \lim_{t \to x} \frac{t^2 f^2(x) - x^2 f^2(t)}{t - x}$

using L.H. rule

$L = \lim_{t \to x} \frac{2t f^2(x) - x^2 \cdot 2f'(t).f(t)}{1}$

$\Rightarrow L = 2xf(x)\,(f(x) - x f'(x)) = 0$ (given)

$\Rightarrow f(x) = xf'(x) \Rightarrow \int \frac{f'(x)dx}{f(x)} = \int \frac{dx}{x}$

$\Rightarrow \ell n\,|f(x)| = \ell n\,|x| + C$

$\because f(1) = e,\ x > 0,\ f(x) > 0$

$\Rightarrow f(x) = ex$, if $f(x) = 1 \Rightarrow x = \frac{1}{e}$

8. If the system of equations

$x + y + z = 2$

$2x + 4y - z = 6$

$3x + 2y + \lambda z = \mu$

has infinitely many solutions, then :

(1) $\lambda - 2\mu = -5$ (2) $2\lambda - \mu = 5$

(3) $2\lambda + \mu = 14$ (4) $\lambda + 2\mu = 14$

Official Ans. by NTA (3)

Sol. For infinite solutions

$\Delta = \Delta_x = \Delta_y = \Delta_z = 0$

Now $\Delta = 0 \Rightarrow \begin{vmatrix} 1 & 1 & 1 \\ 2 & 4 & -1 \\ 3 & 2 & \lambda \end{vmatrix} = 0$

$\Rightarrow \lambda = \frac{9}{2}$

$\Delta_{x=0} \Rightarrow \begin{vmatrix} 2 & 1 & 1 \\ 6 & 4 & -1 \\ \mu & 2 & -\frac{9}{2} \end{vmatrix} = 0$

$\Rightarrow \mu = 5$

For $\lambda = \frac{9}{2}$ & $\mu = 5$, $\Delta_y = \Delta_z = 0$

Now check option $2\lambda + \mu = 14$

9. The minimum value of $2^{\sin x} + 2^{\cos x}$ is :-

(1) $2^{1-\frac{1}{\sqrt{2}}}$ (2) $2^{-1+\sqrt{2}}$

(3) $2^{1-\sqrt{2}}$ (4) $2^{-1+\frac{1}{\sqrt{2}}}$

Official Ans. by NTA (1)

Sol. Usnign AM ≥ GM

$\Rightarrow \frac{2^{\sin x} + 2^{\cos x}}{2} \geq \sqrt{2^{\sin x} \cdot 2^{\cos x}}$

$\Rightarrow 2^{\sin x} + 2^{\cos x} \geq 2^{1+\left(\frac{\sin x + \cos x}{2}\right)}$

$\Rightarrow \min\left(2^{\sin x} + 2^{\cos x}\right) = 2^{1-\frac{1}{\sqrt{2}}}$

10. $\int_{\pi/6}^{\pi/3} \tan^3 x \cdot \sin^2 3x (2\sec^2 x \cdot \sin^2 3x + 3\tan x \cdot \sin 6x)dx$

is equal to :

(1) $\frac{9}{2}$ (2) $-\frac{1}{9}$ (3) $-\frac{1}{18}$ (4) $\frac{7}{18}$

Official Ans. by NTA (3)

Sol. $I = \int_{\pi/6}^{\pi/3} ((2\tan^3 x \cdot \sec^2 x \cdot \sin^4 3x) + (3\tan^4 x \cdot \sin^3 3x \cdot \cos 3x))dx$

$\Rightarrow I = \frac{1}{2} \int_{\frac{\pi}{6}}^{\frac{\pi}{3}} d((\sin 3x)^4 (\tan x)^4)$

$\Rightarrow I = ((\sin 3x)^4 (\tan x)^4)_{\pi/6}^{\pi/3}$

$\Rightarrow I = -\frac{1}{18}$

11. The circle passing through the intersection of the circles, $x^2 + y^2 - 6x = 0$ and $x^2 + y^2 - 4y = 0$, having its centre on the line, $2x - 3y + 12 = 0$, also passes through the point :

(1) (1, –3) (2) (–1, 3)

(3) (–3, 1) (4) (–3, 6)

Official Ans. by NTA (4)

Sol. Let S be the circle pasing through point of intersection of S_1 & S_2

$\therefore\ S = S_1 + \lambda S_2 = 0$

$\Rightarrow S : (x^2 + y^2 - 6x) + \lambda\,(x^2 + y^2 - 4y) = 0$

$\Rightarrow S : x^2 + y^2 - \left(\frac{6}{1+\lambda}\right)x - \left(\frac{4\lambda}{1+\lambda}\right) y = 0$ ...(1)

Centre $\left(\frac{3}{1+\lambda}, \frac{2\lambda}{1+\lambda}\right)$ lies on

$2x - 3y + 12 = 0 \Rightarrow \lambda = -3$

put in (1) $\Rightarrow S : x^2 + y^2 + 3x - 6y = 0$

Now check options point $(-3, 6)$

lies on S.

12. The angle of elevation of a cloud C from a point P, 200 m above a still lake is 30º. If the angle of depression of the image of C in the lake from the point P is 60º, then PC (in m) is equal to :

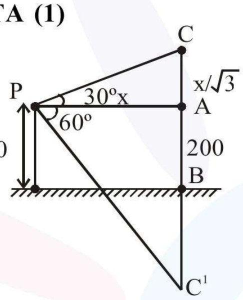
(1) 400 (2) $400\sqrt{3}$

(3) 100 (4) $200\sqrt{3}$

Official Ans. by NTA (1)

Sol. Let PA = x

For ΔAPC



$AC = \frac{PA}{\sqrt{3}} = \frac{x}{\sqrt{3}}$

$AC^1 = AB + BC^1$

$AC^1 = AB + BC$

$AC^1 = 400 + \frac{x}{\sqrt{3}}$

From $\Delta C^1PA : AC^1 = \sqrt{3}\,PA$

$\Rightarrow \left(400 + \frac{x}{\sqrt{3}}\right) = \sqrt{3}x \Rightarrow x = (200)(\sqrt{3})$

from Δ APC : $PC = \frac{2x}{\sqrt{3}} \Rightarrow PC = 400$

13. If a and b are real numbers such that $(2 + \alpha)^4 = a + b\alpha$, where $\alpha = \frac{-1+i\sqrt{3}}{2}$, then a + b is equal to :

(1) 57 (2) 33 (3) 24 (4) 9

Official Ans. by NTA (4)

Sol. $\alpha = \omega$ $(\omega^3 = 1)$

$\Rightarrow (2 + \omega)^4 = a + b\omega$

$\Rightarrow 2^4 + 4.2^3\,\omega + 6.2^2\omega^3 + 4.2\,.\,\omega^3 + \omega^4 = a + b\omega$

$\Rightarrow 16 + 32\,\omega + 24\,\omega^2 + 8 + \omega = a + b\omega$

$\Rightarrow 24 + 24\,\omega^2 + 33\omega = a + b\omega$

$\Rightarrow -24\omega + 33\omega = a + b\omega$

$\Rightarrow a = 0,\ b = 9$

14. In a game two players A and B take turns in throwing a pair of fair dice starting with player A and total of scores on the two dice, in each throw is noted. A wins the game if he throws a total of 6 before B throws a total of 7 and B wins the game if he throws a total of 7 before A throws a total of six The game stops as soon as either of the players wins. The probability of A winning the game is :

(1) $\frac{31}{61}$ (2) $\frac{5}{6}$

(3) $\frac{5}{31}$ (4) $\frac{30}{61}$

Official Ans. by NTA (4)

Sol. $P(6) = \frac{1}{6}$, $P(7) = \frac{5}{36}$

P(A) = W + FFW + FFFFW + ...

$= \frac{1}{6} + \frac{5}{6} \times \frac{31}{36} \times \frac{1}{6} + \left(\frac{5}{6}\right)^2 \left(\frac{31}{36}\right)^2 \frac{1}{6} + ...$

$= \frac{\frac{1}{6}}{1 - \frac{155}{216}} = \frac{36}{61}$

15. Let x = 4 be a directrix to an ellipse whose centre is at the origin and its eccentricity is $\frac{1}{2}$. If P $(1, \beta)$, $\beta > 0$ is a point on this ellipse, then the equation of the normal to it at P is :-

(1) $7x - 4y = 1$ (2) $4x - 2y = 1$

(3) $4x - 3y = 2$ (4) $8x - 2y = 5$

Official Ans. by NTA (2)

Sol. Ellipse : $\frac{x^2}{a^2} + \frac{y^2}{b^2} = 1$

directrix : $x = \frac{a}{e} = 4$ & $e = \frac{1}{2}$

$\Rightarrow a = 2$ & $b^2 = a^2(1 - e^2) = 3$

$\Rightarrow$ Ellipse is $\frac{x^2}{4} + \frac{y^2}{3} = 1$

P is $\left(1, \frac{3}{2}\right)$

Normal is : $\frac{4x}{1}-\frac{3y}{3/2}=4-3$

$\Rightarrow 4x - 2y = 1$

16. Contrapositive of the statement:

'If a function f is differentiable at a, then it is also continuous at a', is :-

(1) If a function f is continuous at a, then it is not differentiable at a.

(2) If a function f is not continuous at a, then it is differentiable at a.

(3) If a function f is not continuous at a, then it is not differentiable at a.

(4) If a function f is continuous at a, then it is differentiable at a.

Official Ans. by NTA (3)

Sol. p = function is differantiable at a

q = function is continuous at a

contrapositive of statement $p \to q$ is

$\sim q \to \sim p$

17. The area (in sq. units) of the largest rectangle ABCD whose vertices A and B lie on the x-axis and vertices C and D lie on the parabola, $y= x^2 -1$ below the x-axis, is :

(1) $\frac{4}{3\sqrt{3}}$ (2) $\frac{1}{3\sqrt{3}}$ (3) $\frac{4}{3}$ (4) $\frac{2}{3\sqrt{3}}$

Official Ans. by NTA (1)

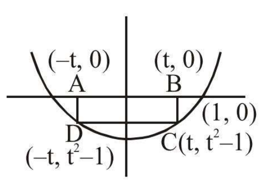
Sol. Area (A) = $2t \cdot (1 - t^2)$

$(0 < t < 1)$

$A = 2t - 2t^3$

$\frac{dA}{dt}=2-6t^2$

$t = \frac{1}{\sqrt{3}}$



$\Rightarrow A_{max} = \frac{2}{\sqrt{3}}\left(1-\frac{1}{3}\right)=\frac{4}{3\sqrt{3}}$

18. If for some positive integer n, the coefficients of three consecutive terms in the binomial expansion of $(1+x)^{n+5}$ are in the ratio 5 : 10 : 14, then the largest coefficient in this expansion is :-

(1) 792 (2) 252 (3) 462 (4) 330

Official Ans. by NTA (3)

Sol. Let n + 5 = N

$^NC_{r-1} : {}^NC_r : {}^NC_{r+1} = 5 : 10 : 14$

$\Rightarrow \frac{^NC_r}{^NC_{r-1}}=\frac{N+1-r}{r}=2$

$\frac{^NC_{r+1}}{^NC_r}=\frac{N-r}{r+1}=\frac{7}{5}$

$\Rightarrow$ r = 4, N = 11

$\Rightarrow (1 + x)^{11}$

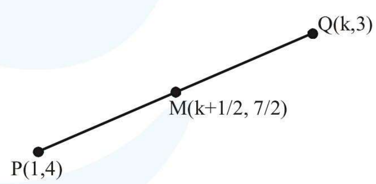
Largest coefficient = $^{11}C_6 = 462$

19. If the perpendicular bisector of the line segment joining the points P (1, 4) and Q (k, 3) has y-intercept equal to –4, then a value of k is :-

(1) $\sqrt{15}$ (2) –2 (3) $\sqrt{14}$ (4) –4

Official Ans. by NTA (4)

Sol.



Slope = $m=\frac{1}{1-k}$

Equation of $\perp^r$ bisector is

$y + 4 = (k -1) (x- 0)$

$\Rightarrow y + 4 = x(k -1)$

$\Rightarrow \frac{7}{2}+4=\frac{k+1}{2}(k-1)$

$\Rightarrow \frac{15}{2}=\frac{k^2-1}{2}\Rightarrow k^2=16\Rightarrow k=4,-4$

20. Suppose the vectors x_1, x_2 and x_3 are the solutions of the system of linear equations, Ax = b when the vector b on the right side is equal to b_1, b_2 and b_3 respectively. If

$$x=\begin{bmatrix}1\\1\\1\end{bmatrix}, x_2=\begin{bmatrix}0\\2\\1\end{bmatrix}, x_3=\begin{bmatrix}0\\0\\1\end{bmatrix}, b_1=\begin{bmatrix}1\\0\\0\end{bmatrix}$$

$b_2=\begin{bmatrix}0\\2\\0\end{bmatrix}$ and $b_3=\begin{bmatrix}0\\0\\2\end{bmatrix}$, then the determinant of A is equal to :-

(1) $\frac{1}{2}$ (2) 4 (3) $\frac{3}{2}$ (4) 2

Official Ans. by NTA (4)

Sol. $Ax_1 = b_1$
$Ax_2 = b_2$
$Ax_3 = b_3$

$$\Rightarrow |A|\begin{vmatrix}1&0&0\\1&2&0\\1&1&1\end{vmatrix}=\begin{vmatrix}1&0&0\\0&2&0\\0&0&2\end{vmatrix}$$

$$\Rightarrow |A| = \frac{4}{2} = 2$$

21. A test consists of 6 multiple choice questions, each having 4 alternative answers of which only one is correct. The number of ways, in which a candidate answers all six questions such that exactly four of the answers are correct, is ____

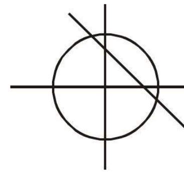
Official Ans. by NTA (135)

Sol. Ways = ${}^6C_4 \cdot 1^4 \cdot 3^2$
= 15 × 9
= 135

22. Let PQ be a diameter of the circle $x^2+y^2=9$. If α and β are the lengths of the perpendiculars from P and Q on the straight line, x + y = 2 respectively, then the maximum value of $\alpha\beta$ is ____

Official Ans. by NTA (7)

Sol.



Let P $(3\cos\theta, 3\sin\theta)$
Q $(-3\cos\theta, -3\sin\theta)$

$$\Rightarrow \alpha\beta = \frac{|(3\cos\theta+3\sin\theta)^2-4|}{2}$$

$$\Rightarrow \alpha\beta = \frac{5+9\sin 2\theta}{2} \le 7$$

23. Let {x} and [x] denote the fractional part of x and the greatest integer ≤ x respectively of a real number x. If $\int_0^n \{x\}dx, \int_0^n [x]dx$ and $10(n^2-n)$, $(n \in N, n > 1)$ are three consecutive terms of a G.P., then n is equal to____

Official Ans. by NTA (21)

Sol. $\int_0^n \{x\}dx = n\int_0^1 \{x\}dx = n\int_0^1 x\,dx = \frac{n}{2}$

$$\int_0^n [x]dx = \int_0^n (x-\{x\})dx = \frac{n^2}{2}-\frac{n}{2}$$

$$\Rightarrow \left(\frac{n^2-n}{2}\right)^2 = \frac{n}{2}\cdot 10\cdot n(n-1) \quad (\text{where } n > 1)$$

$$\Rightarrow \frac{n-1}{4} = 5 \Rightarrow n = 21$$

24. If $\vec{a} = 2\hat{i}+\hat{j}+2\hat{k}$, then the value of $\left|\hat{i}\times(\vec{a}\times\hat{i})\right|^2+\left|\hat{j}\times(\vec{a}\times\hat{j})\right|^2+\left|\hat{k}\times(\vec{a}\times\hat{k})\right|^2$ is equal to ____

Official Ans. by NTA (18)

Sol. $\Sigma|\vec{a}-(\vec{a}\cdot i)i|^2$

$\Rightarrow \Sigma\left(|a|^2+(\vec{a}\cdot i)^2-2(\vec{a}\cdot i)^2\right)$

$\Rightarrow 3|\vec{a}|^2-\Sigma(\vec{a}\cdot i)^2$

$\Rightarrow 2|\vec{a}|^2$

$\Rightarrow 18$

25. If the variance of the following frequency distribution :

Class :	10–20	20–30	30–40
Frequency :	2	x	2

is 50, then x is equal to ____

Official Ans. by NTA (4)

Sol. $\because$ Variance is independent of shifting of origin

$\Rightarrow$ x_i : 15 25 35 or –10 0 10
f_i : 2 x 2 2 x 2

$$\Rightarrow \text{Variance } (\sigma^2) = \frac{\Sigma x_i^2 f_i}{\Sigma f_i} - (\vec{x})^2$$

$$\Rightarrow 50 = \frac{200+0+200}{x+4} - 0 \qquad \{\bar{x}=0\}$$

$\Rightarrow$ 200 + 50x = 200 + 200

$\Rightarrow$ x = 4