FINAL JEE–MAIN EXAMINATION – MARCH, 2021

(Held On Tuesday 16th March, 2021) TIME : 9 : 00 AM to 12 : 00 NOON

MATHEMATICS | **TEST PAPER WITH SOLUTION**

SECTION-A

1. The number of elements in the set $\{x \in \mathbb{R} : (|x| - 3)\,|x + 4| = 6\}$ is equal to

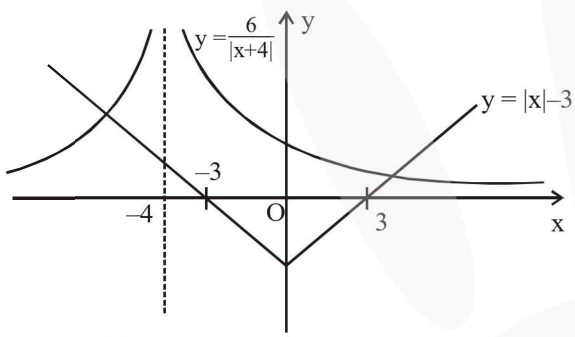
(1) 3 (2) 2 (3) 4 (4) 1

Official Ans. by NTA (2)

Sol. $x \neq -4$

$(|x| - 3)(|x + 4|) = 6$

$\Rightarrow \quad |x| - 3 = \dfrac{6}{|x+4|}$



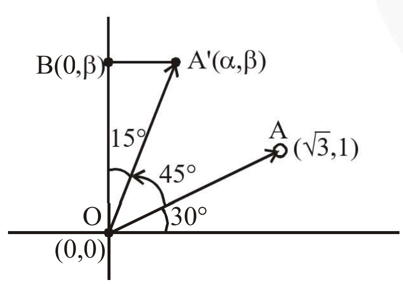
No. of solutions = 2

2. Let a vector $\alpha\hat{i} + \beta\hat{j}$ be obtained by rotating the vector $\sqrt{3}\hat{i} + \hat{j}$ by an angle 45° about the origin in counterclockwise direction in the first quadrant. Then the area of triangle having vertices (α, β), $(0, \beta)$ and $(0, 0)$ is equal to

(1) $\dfrac{1}{2}$ (2) 1 (3) $\dfrac{1}{\sqrt{2}}$ (4) $2\sqrt{2}$

Official Ans. by NTA (1)

Sol.



Area of $\Delta(OA'B) = \dfrac{1}{2} OA'\cos 15° \times OA'\sin 15°$

$= \dfrac{1}{2}(OA')^2 \dfrac{\sin 30°}{2}$

$= (3+1) \times \dfrac{1}{8} = \dfrac{1}{2}$

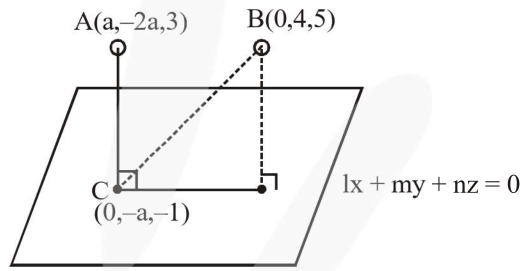
3. If for $a > 0$, the feet of perpendiculars from the points $A(a, -2a, 3)$ and $B(0, 4, 5)$ on the plane $lx + my + nz = 0$ are points $C(0, -a, -1)$ and D respectively, then the length of line segment CD is equal to :

(1) $\sqrt{31}$ (2) $\sqrt{41}$

(3) $\sqrt{55}$ (4) $\sqrt{66}$

Official Ans. by NTA (4)

Sol.



C lies on plane $\Rightarrow -ma - n = 0 \Rightarrow \dfrac{m}{n} = -\dfrac{1}{a}$ ....(1)

$\overrightarrow{CA} \parallel l\hat{i} + m\hat{j} + n\hat{k}$

$\dfrac{a-0}{l} = \dfrac{-a}{m} = \dfrac{4}{n} \Rightarrow \dfrac{m}{n} = -\dfrac{a}{4}$ ....(2)

From (1) & (2)

$-\dfrac{1}{a} = \dfrac{-a}{4} \Rightarrow a^2 = 4 \Rightarrow a = 2$ (since $a > 0$)

From (2) $\dfrac{m}{n} = \dfrac{-1}{2}$

Let $m = -t \Rightarrow n = 2t$

$\dfrac{2}{l} = \dfrac{-2}{-t} \Rightarrow l = t$

So plane : $t(x - y + 2z) = 0$

$BD = \dfrac{6}{\sqrt{6}} = \sqrt{6}$ $\quad C \cong (0, -2, -1)$

$CD = \sqrt{BC^2 - BD^2}$

$= \sqrt{(0^2 + 6^2 + 6^2) - (\sqrt{6})^2}$

$= \sqrt{66}$

4. Consider three observations a, b and c such that b = a + c. If the standard deviation of a + 2, b + 2, c + 2 is d, then which of the following is true ?

(1) $b^2 = 3(a^2 + c^2) + 9d^2$

(2) $b^2 = a^2 + c^2 + 3d^2$

(3) $b^2 = 3(a^2 + c^2 + d^2)$

(4) $b^2 = 3(a^2 + c^2) - 9d^2$

Official Ans. by NTA (4)

Sol. For a, b, c

mean $= \dfrac{a+b+c}{3} (= \bar{x})$

b = a + c

$\Rightarrow \quad \bar{x} = \dfrac{2b}{3}$ ......(1)

S.D. (a + 2, b + 2, c + 2) = S.D. (a, b, c) = d

$\Rightarrow \quad d^2 = \dfrac{a^2+b^2+c^2}{3} - (\bar{x})^2$

$\Rightarrow \quad d^2 = \dfrac{a^2+b^2+c^2}{3} - \dfrac{4b^2}{9}$

$\Rightarrow \quad 9d^2 = 3(a^2 + b^2 + c^2) - 4b^2$

$\Rightarrow \quad b^2 = 3(a^2 + c^2) - 9d^2$

5. If for $x \in \left(0, \dfrac{\pi}{2}\right)$, $\log_{10}\sin x + \log_{10}\cos x = -1$ and $\log_{10}(\sin x + \cos x) = \dfrac{1}{2}(\log_{10}n - 1)$, n > 0, then the value of n is equal to :

(1) 20 (2) 12 (3) 9 (4) 16

Official Ans. by NTA (2)

Sol. $x \in \left(0, \dfrac{\pi}{2}\right)$

$\log_{10}\sin x + \log_{10}\cos x = -1$

$\Rightarrow \quad \log_{10}\sin x.\cos x = -1$

$\Rightarrow \quad \sin x.\cos x = \dfrac{1}{10}$ ......(1)

$\log_{10}(\sin x + \cos x) = \dfrac{1}{2}(\log_{10} n - 1)$

$\Rightarrow \quad \sin x + \cos x = 10^{\left(\log_{10}\sqrt{n} - \frac{1}{2}\right)} = \sqrt{\dfrac{n}{10}}$

by squaring

$1 + 2\sin x.\cos x = \dfrac{n}{10}$

$\Rightarrow \quad 1 + \dfrac{1}{5} = \dfrac{n}{10} \quad \Rightarrow \quad n = 12$

6. Let $A = \begin{bmatrix} i & -i \\ -i & i \end{bmatrix}$, $i = \sqrt{-1}$. Then, the system of linear equations $A^8\begin{bmatrix} x \\ y \end{bmatrix} = \begin{bmatrix} 8 \\ 64 \end{bmatrix}$ has :

(1) A unique solution

(2) Infinitely many solutions

(3) No solution

(4) Exactly two solutions

Official Ans. by NTA (3)

Sol. $A = \begin{bmatrix} i & -i \\ -i & i \end{bmatrix}$

$A^2 = \begin{bmatrix} -2 & 2 \\ 2 & -2 \end{bmatrix} = 2\begin{bmatrix} -1 & 1 \\ 1 & -1 \end{bmatrix}$

$A^4 = 2^2\begin{bmatrix} 2 & -2 \\ -2 & 2 \end{bmatrix} = 8\begin{bmatrix} 1 & -1 \\ -1 & 1 \end{bmatrix}$

$A^8 = 64\begin{bmatrix} 2 & -2 \\ -2 & 2 \end{bmatrix} = 128\begin{bmatrix} 1 & -1 \\ -1 & 1 \end{bmatrix}$

$A^8\begin{bmatrix} x \\ y \end{bmatrix} = \begin{bmatrix} 8 \\ 64 \end{bmatrix}$

$\Rightarrow \quad 128\begin{bmatrix} 1 & -1 \\ -1 & 1 \end{bmatrix}\begin{bmatrix} x \\ y \end{bmatrix} = \begin{bmatrix} 8 \\ 64 \end{bmatrix}$

$\Rightarrow \quad 128\begin{bmatrix} x-y \\ -x+y \end{bmatrix} = \begin{bmatrix} 8 \\ 64 \end{bmatrix}$

$\Rightarrow \quad x - y = \dfrac{1}{16}$ .....(1)

& $\quad -x + y = \dfrac{1}{2}$ .....(2)

$\Rightarrow$ From (1) & (2) : No solution.

7. If the three normals drawn to the parabola, $y^2 = 2x$ pass through the point (a, 0) $a \neq 0$, then 'a' must be greater than :

(1) $\dfrac{1}{2}$ (2) $-\dfrac{1}{2}$ (3) –1 (4) 1

Official Ans. by NTA (4)

Sol. For standard parabola

For more than 3 normals (on axis)

$x > \dfrac{L}{2}$ (where L is length of L.R.)

For $y^2 = 2x$

L.R. = 2

for (a, 0)

$a > \frac{L.R.}{2} \Rightarrow a > 1$

8. Let the position vectors of two points P and Q be $3\hat{i}-\hat{j}+2\hat{k}$ and $\hat{i}+2\hat{j}-4\hat{k}$, respectively. Let R and S be two points such that the direction ratios of lines PR and QS are (4, –1, 2) and (–2, 1, –2), respectively. Let lines PR and QS intersect at T. If the vector $\overrightarrow{TA}$ is perpendicular to both $\overrightarrow{PR}$ and $\overrightarrow{QS}$ and the length of vector $\overrightarrow{TA}$ is $\sqrt{5}$ units, then the modulus of a position vector of A is :

(1) $\sqrt{482}$ (2) $\sqrt{171}$

(3) $\sqrt{5}$ (4) $\sqrt{227}$

Official Ans. by NTA (2)

Sol. P(3, –1, 2)

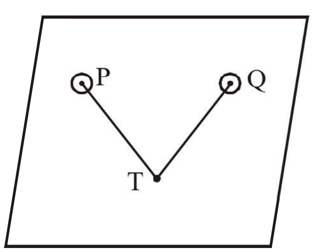
Q(1, 2, –4)

$\overrightarrow{PR} \parallel 4\hat{i}-\hat{j}+2\hat{k}$

$\overrightarrow{QS} \parallel -2\hat{i}+\hat{j}-2\hat{k}$

dr's of normal to the plane containing P, T & Q will be proportional to :

$$\begin{vmatrix} \hat{i} & \hat{j} & \hat{k} \\ 4 & -1 & 2 \\ -2 & 1 & -2 \end{vmatrix}$$



$\therefore \quad \frac{\ell}{0} = \frac{m}{4} = \frac{n}{2}$

For point, T : $\overrightarrow{PT} = \frac{x-3}{4} = \frac{y+1}{-1} = \frac{z-2}{2} = \lambda$

$\overrightarrow{QT} = \frac{x-1}{-2} = \frac{y-1}{1} = \frac{z+4}{-2} = \mu$

T : $(4\lambda + 3, -\lambda - 1, 2\lambda + 2)$

$\cong (2\mu+1, \mu+2, -2\mu-4)$

$4\lambda + 3 = -2\mu + 1 \quad \Rightarrow \quad 2\lambda + \mu = -1$

$\lambda + \mu = -3 \quad \Rightarrow \quad \lambda = 2$

& $\mu = -5 \quad \lambda + \mu = -3 \quad \Rightarrow \quad \lambda = 2$

So point T : (11, –3, 6)

$$\overrightarrow{OA} = \left(11\hat{i}-3\hat{j}+6\hat{k}\right) \pm \left(\frac{2\hat{j}+\hat{k}}{\sqrt{5}}\right)\sqrt{5}$$

$$\overrightarrow{OA} = \left(11\hat{i}-3\hat{j}+6\hat{k}\right) \pm \left(2\hat{j}+\hat{k}\right)$$

$$\overrightarrow{OA} = 11\hat{i}-\hat{j}+7\hat{k}$$

or

$$9\hat{i}-5\hat{j}+5\hat{k}$$

$$|\overrightarrow{OA}| = \sqrt{121+1+49} = \sqrt{171}$$

or

$$\sqrt{81+25+25} = \sqrt{131}$$

9. Let the functions $f : \mathbb{R} \to \mathbb{R}$ and $g : \mathbb{R} \to \mathbb{R}$ be defined as :

$$f(x) = \begin{cases} x+2, & x<0 \\ x^2, & x \geq 0 \end{cases} \text{ and } g(x) = \begin{cases} x^3, & x<1 \\ 3x-2, & x \geq 1 \end{cases}$$

Then, the number of points in $\mathbb{R}$ where $(fog)(x)$ is NOT differentiable is equal to :

(1) 3 (2) 1

(3) 0 (4) 2

Official Ans. by NTA (2)

Sol. $$f(g(x)) = \begin{cases} g(x)+2, & g(x)<0 \\ (g(x))^2, & g(x) \geq 0 \end{cases}$$

$$= \begin{cases} x^3+2, & x<0 \\ x^6, & x \in [0,1) \\ (3x-2)^2, & x \in [1,\infty) \end{cases}$$

$$(fog(x))' = \begin{cases} 3x^2, & x<0 \\ 6x^5, & x \in (0,1) \\ 2(3x-2)\times 3, & x \in (1,\infty) \end{cases}$$

At 'O'

L.H.L. ≠ R.H.L. (Discontinuous)

At '1'

L.H.D. = 6 = R.H.D.

$\Rightarrow \quad fog(x)$ is differentiable for $x \in \mathbb{R} - \{0\}$

10. Which of the following Boolean expression is a tautology ?

(1) $(p \wedge q) \vee (p \vee q)$

(2) $(p \wedge q) \vee (p \rightarrow q)$

(3) $(p \wedge q) \wedge (p \rightarrow q)$

(4) $(p \wedge q) \rightarrow (p \rightarrow q)$

Official Ans. by NTA (4)

Sol.

p	q	$p \wedge q$	$p \rightarrow q$	$(p \wedge q) \rightarrow (p \rightarrow q)$
T	T	T	T	T
T	F	F	F	T
F	T	F	T	T
F	F	F	T	T

$(p \wedge q) \rightarrow (p \rightarrow q)$ is tautology

11. Let a complex number z, $|z| \neq 1$,

satisfy $\log_{\frac{1}{\sqrt{2}}}\left(\frac{|z|+11}{(|z|-1)^2}\right) \leq 2$. Then, the largest value of $|z|$ is equal to ______ .

(1) 8 (2) 7 (3) 6 (4) 5

Official Ans. by NTA (2)

Sol. $\log_{\frac{1}{\sqrt{2}}}\left(\frac{|z|+11}{(|z|-1)^2}\right) \leq 2$

$\frac{|z|+11}{(|z|-1)^2} \geq \frac{1}{2}$

$2|z| + 22 \geq (|z| - 1)^2$

$2|z| + 22 \geq |z|^2 + 1 - 2|z|$

$|z|^2 - 4|z| - 21 \leq 0$

$\Rightarrow$ $|z| \leq 7$

$\therefore$ Largest value of $|z|$ is 7

12. If n is the number of irrational terms in the expansion of $(3^{1/4} + 5^{1/8})^{60}$, then$(n - 1)$ is divisible by :

(1) 26 (2) 30 (3) 8 (4) 7

Official Ans. by NTA (1)

Sol. $(3^{1/4} + 5^{1/8})^{60}$

${}^{60}C_r(3^{1/4})^{60-r}.(5^{1/8})^r$

${}^{60}C_r(3)^{\frac{60-r}{4}}.5^{\frac{r}{8}}$

For rational terms.

$\frac{r}{8} = k$; $\quad 0 \leq r \leq 60$

$0 \leq 8k \leq 60$

$0 \leq k \leq \frac{60}{8}$

$0 \leq k \leq 7.5$

$k = 0, 1, 2, 3, 4, 5, 6, 7$

$\frac{60-8k}{4}$ is always divisible by 4 for all value of k.

Total rational terms = 8

Total terms = 61

irrational terms = 53

$n - 1 = 53 - 1 = 52$

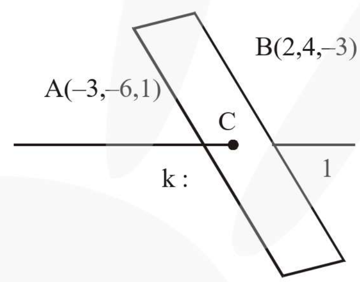
52 is divisible by 26.

13. Let P be a plane $lx + my + nz = 0$ containing the line, $\frac{1-x}{1} = \frac{y+4}{2} = \frac{z+2}{3}$. If plane P divides the line segment AB joining points A(–3, –6, 1) and B(2, 4, –3) in ratio k : 1 then the value of k is equal to :

(1) 1.5 (2) 3 (3) 2 (4) 4

Official Ans. by NTA (3)

Sol.



Point C is

$\left(\frac{2k-3}{k+1}, \frac{4k-6}{k+1}, \frac{-3k+1}{k+1}\right)$

$\frac{x-1}{-1} = \frac{y+4}{2} = \frac{z+2}{3}$

Plane $lx + my + nz = 0$

$l(-1) + m(2) + n(3) = 0$

$-l + 2m + 3n = 0$ ....(1)

It also satisfy point (1, –4, –2)

$l - 4m - 2n = 0$ ....(2)

Solving (1) and (2)

$2m + 3n = 4m + 2n$

$n = 2m$

$l - 4m - 4m = 0$

$l = 8m$

$\frac{l}{8} = \frac{m}{1} = \frac{n}{2}$

$l : m : n = 8 : 1 : 2$

Plane is $8x + y + 2z = 0$

It will satisfy point C

$8\left(\frac{2k-3}{k+1}\right) + \left(\frac{4k-6}{k+1}\right) + 2\left(\frac{-3k+1}{k+1}\right) = 0$

$16k - 24 + 4k - 6 - 6k + 2 = 0$

$14k = 28 \quad \therefore \quad k = 2$

14. The range of $a \in \mathbb{R}$ for which the function

$$f(x) = (4a-3)(x+\log_e 5)+2(a-7)\cot\left(\frac{x}{2}\right)\sin^2\left(\frac{x}{2}\right),$$

$x \neq 2n\pi, n \in \mathbb{N}$, has critical points, is :

(1) (–3, 1) (2) $\left[-\frac{4}{3}, 2\right]$

(3) [1, ∞) (4) (–∞, –1]

Official Ans. by NTA (2)

Sol. $f(x) = (4a-3)(x+\log_e 5)+(a-7)\sin x$

$f(x) = (4a-3)(1)+(a-7)\cos x = 0$

$$\Rightarrow \quad \cos x = \frac{3-4a}{a-7}$$

$$-1 \le \frac{3-4a}{a-7} < 1$$

$$\frac{3-4a}{a-7}+1 \ge 0 \qquad \frac{3-4a}{a-7} < 1$$

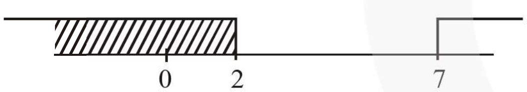
$$\frac{3-4a+a-7}{a-7} \ge 0 \qquad \frac{3-4a}{a-7}-1<0$$

$$\frac{-3a-4}{a-7} \ge 0 \qquad \frac{3-4a-a+7}{a-7}<0$$

$$\frac{3a+4}{a-7} \le 0 \qquad \frac{-5a+10}{a-7}<0$$

$$\frac{5a-10}{a-7}>0$$

$$\frac{5(a-2)}{a-7}>0$$



$$\alpha \in \left[-\frac{4}{3}, 2\right)$$

Check end point $\left[-\frac{4}{3}, 2\right)$

15. A pack of cards has one card missing. Two cards are drawn randomly and are found to be spades. The probability that the missing card is not a spade, is :

(1) $\frac{3}{4}$ (2) $\frac{52}{867}$

(3) $\frac{39}{50}$ (4) $\frac{22}{425}$

Official Ans. by NTA (3)

Sol. E_1 : Event denotes spade is missing

$$P(E_1)=\frac{1}{4};\ P\left(\bar{E}_1\right)=\frac{3}{4}$$

A : Event drawn two cards are spade

$$P(A)=\frac{\frac{1}{4}\times\left(\frac{^{12}C_2}{^{51}C_2}\right)+\frac{3}{4}\times\left(\frac{^{13}C_2}{^{51}C_2}\right)+\frac{3}{4}\times\left(\frac{^{13}C_2}{^{51}C_2}\right)}{\frac{1}{4}\times\left(\frac{^{12}C_2}{^{51}C_2}\right)+\frac{3}{4}\times\left(\frac{^{13}C_2}{^{51}C_2}\right)}$$

$$=\frac{39}{50}$$

16. Let [x] denote greatest integer less than or equal to x. If for $n \in \mathbb{N}$, $(1-x+x^3)^n = \sum_{j=0}^{3n} a_j x^j$,

then $\sum_{j=0}^{\left[\frac{3n}{2}\right]} a_{2j} + 4\sum_{j=0}^{\left[\frac{3n-1}{2}\right]} a_{2j}+1$ is equal to :

(1) 2 (2) 2^{n-1}

(3) 1 (4) n

Official Ans. by NTA (3)

Sol. $(1-x+x^3)^n = \sum_{j=0}^{3n} a_j x^j$

$(1-x+x^3)^n = a_0 + a_1x + a_2x^2 \,.......\, + a_{3n}x^{3n}$

$\sum_{j=0}^{\left[\frac{3n}{2}\right]} a_{2j}$ = Sum of $a_0 + a_2 + a_4$ ....

$\sum_{j=0}^{\left[\frac{3n-1}{2}\right]} a_{2j}+1$ = Sum of $a_1 + a_3 + a_5$ ....

put x = 1

$1 = a_0 + a_1 + a_2 + a_3 \,.........\, + a_{3n}$ ....(A)

Put x = –1

$1 = a_0 - a_1 + a_2 - a_3 \,.........\, + (-1)^{3n}a_{3n}$ ....(B)

Solving (A) and (B)

$a_0 + a_2 + a_4 \,.....\, = 1$

$a_1 + a_3 + a_5 \,.......\, = 0$

$$\sum_{j=0}^{\left[\frac{3n}{2}\right]} a_{2j} + 4\sum_{j=0}^{\left[\frac{3n-1}{2}\right]} a_{2j+1} = 1$$

17. If y = y(x) is the solution of the differential equation, $\frac{dy}{dx}+2y\tan x=\sin x,\ y\left(\frac{\pi}{3}\right)=0$, then the maximum value of the function y(x) over $\mathbb{R}$ is equal to :

(1) 8 (2) $\frac{1}{2}$ (3) $-\frac{15}{4}$ (4) $\frac{1}{8}$

Official Ans. by NTA (4)

Sol. $\frac{dy}{dx}+2y\tan x=\sin x$

I.F. = $e^{\int 2\tan x dx}=e^{2\ell n \sec x}$

I.F. = $\sec^2 x$

$y.(\sec^2 x)=\int \sin x.\sec^2 x dx$

$y.(\sec^2 x)=\int \sec x \tan x dx$

$y.(\sec^2 x)=\sec x + C$

$x=\frac{\pi}{3};\ y=0$

$\Rightarrow$ C = –2

$\Rightarrow\ y=\frac{\sec x-2}{\sec^2 x}=\cos x-2\cos^2 x$

$y = t - 2t^2 \Rightarrow \frac{dy}{dt}=1-4t=0 \Rightarrow t=\frac{1}{4}$

$\therefore$ max $=\frac{1}{4}-\frac{1}{8}=\frac{2-1}{8}=\frac{1}{8}$

18. The locus of the midpoints of the chord of the circle, $x^2 + y^2 = 25$ which is tangent to the hyperbola, $\frac{x^2}{9}-\frac{y^2}{16}=1$ is :

(1) $(x^2 + y^2)^2 - 16x^2 + 9y^2 = 0$

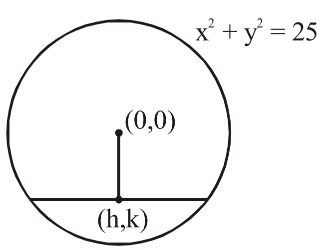
(2) $(x^2 + y^2)^2 - 9x^2 + 144y^2 = 0$

(3) $(x^2 + y^2)^2 - 9x^2 - 16y^2 = 0$

(4) $(x^2 + y^2)^2 - 9x^2 + 16y^2 = 0$

Official Ans. by NTA (4)

Sol.



Equation of chord

$y - k = -\frac{h}{k}(x-h)$

$ky - k^2 = -hx + h^2$

$hx + ky = h^2 + k^2$

$y=-\frac{hx}{k}+\frac{h^2+k^2}{k}$

tangent to $\frac{x^2}{9}-\frac{y^2}{16}=1$

$c^2 = a^2m^2 - b^2$

$\left(\frac{h^2+k^2}{k}\right)^2=9\left(-\frac{h}{k}\right)^2-16$

$(x^2 + y^2)^2 = 9x^2 - 16y^2$

19. The number of roots of the equation,

$(81)^{\sin^2 x}+(81)^{\cos^2 x}=30$

in the interval $[0, \pi]$ is equal to :

(1) 3 (2) 4 (3) 8 (4) 2

Official Ans. by NTA (2)

Sol. $(81)^{\sin^2 x}+(81)^{\cos^2 x}=30$

$(81)^{\sin^2 x}+\frac{(81)^1}{(18)^{\sin^2 x}}=30$

$(81)^{\sin^2 x}=t$

$t+\frac{81}{t}=30$

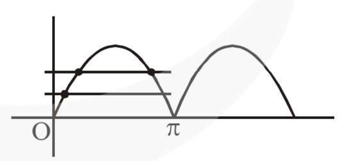
$t^2 - 30t + 81 = 0$

$(t - 3)(t - 27) = 0$

$(81)^{\sin^2 x}=3^1$ or $(81)^{\sin^2 x}=3^3$

$3^{4\sin^2 x}=3^1$ or $3^{4\sin^2 x}=3^3$

$\sin^2 x=\frac{1}{4}$ or $\sin^2 x=\frac{3}{4}$



Total sol. = 4

20. Let $S_k=\sum_{r=1}^{k}\tan^{-1}\left(\frac{6^r}{2^{2r+1}+3^{2r+1}}\right)$. Then $\lim_{k\to\infty} S_k$ is equal to :

(1) $\tan^{-1}\left(\frac{3}{2}\right)$ (2) $\frac{\pi}{2}$

(3) $\cot^{-1}\left(\frac{3}{2}\right)$ (4) $\tan^{-1}(3)$

Official Ans. by NTA (3)

Sol. $S_k = \sum_{r=1}^{k} \tan^{-1}\left(\frac{6^r}{2^{2r+1}+3^{2r+1}}\right)$

Divide by 3^{2r}

$$\sum_{r=1}^{k} \tan^{-1}\left(\frac{\left(\frac{2}{3}\right)^r}{\left(\frac{2}{3}\right)^{2r}.2+3}\right)$$

$$\sum_{r=1}^{k} \tan^{-1}\left(\frac{\left(\frac{2}{3}\right)^r}{3\left(\left(\frac{2}{3}\right)^{2r+1}+1\right)}\right)$$

Let $\left(\frac{2}{3}\right)^r = t$

$$\sum_{r=1}^{k} \tan^{-1}\left(\frac{\frac{t}{3}}{1+\frac{2}{3}t^2}\right)$$

$$\sum_{r=1}^{k} \tan^{-1}\left(\frac{t-\frac{2t}{3}}{1+t.\frac{2t}{3}}\right)$$

$$\sum_{r=1}^{k}\left(\tan^{-1}(t)-\tan^{-1}\left(\frac{2t}{3}\right)\right)$$

$$\sum_{r=1}^{k}\left(\tan^{-1}\left(\frac{2}{3}\right)^r-\tan^{-1}\left(\frac{2}{3}\right)^{r+1}\right)$$

$$S_k = \tan^{-1}\left(\frac{2}{3}\right)-\tan^{-1}\left(\frac{2}{3}\right)^{k+1}$$

$$S_\infty = \lim_{k\to\infty}\left(\tan^{-1}\left(\frac{2}{3}\right)-\tan^{-1}\left(\frac{2}{3}\right)^{k+1}\right)$$

$$= \tan^{-1}\left(\frac{2}{3}\right)-\tan^{-1}(0)$$

$$\therefore \quad S_\infty = \tan^{-1}\left(\frac{2}{3}\right) = \cot^{-1}\left(\frac{3}{2}\right)$$

SECTION-B

1. Consider an arithmetic series and a geometric series having four initial terms from the set {11, 8, 21, 16, 26, 32, 4}. If the last terms of these series are the maximum possible four digit numbers, then the number of common terms in these two series is equal to ______ .

Official Ans. by NTA (3)

Sol. **GP :** 4, 8, 16, 32, 64, 128, 256, 512, 1024, 2048, 4096, 8192

AP : 11, 16, 21, 26, 31, 36

Common terms : 16, 256, 4096 only

2. Let $f : (0, 2) \to \mathbb{R}$ be defined as

$$f(x) = \log_2\left(1+\tan\left(\frac{\pi x}{4}\right)\right).$$

Then, $\lim_{n\to\infty}\frac{2}{n}\left(f\left(\frac{1}{n}\right)+f\left(\frac{2}{n}\right)+....+f(1)\right)$ is equal to ________ .

Official Ans. by NTA (1)

Sol. $E = 2\lim_{n\to\infty}\sum_{r=1}^{n}\frac{1}{n}f\left(\frac{r}{n}\right)$

$$E = \frac{2}{\ell n2}\int_0^1 \ell n\left(1+\tan\frac{\pi x}{4}\right)dx \quad .....(i)$$

replacing $x \to 1 - x$

$$E = \frac{2}{\ell n2}\int_0^1 \ell n\left(1+\tan\frac{\pi}{4}(1-x)\right)dx$$

$$E = \frac{2}{\ell n2}\int_0^1 \ell n\left(1+\tan\left(\frac{\pi}{4}-\frac{\pi}{4}x\right)\right)dx$$

$$E = \frac{2}{\ell n2}\int_0^1 \ell n\left(1+\frac{1+\tan\frac{\pi}{4}x}{1+\tan\frac{\pi}{4}x}\right)dx$$

$$E = \frac{2}{\ell n2}\int_0^1 \ell n\left(\frac{2}{1+\tan\frac{\pi x}{4}}\right)dx$$

$$E = \frac{2}{\ell n2}\int_0^1\left(\ell n2-\ell n\left(1+\tan\frac{\pi x}{4}\right)\right)dx \quad .....(ii)$$

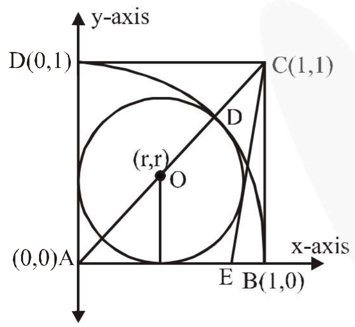
equation (i) + (ii)

E = 1

3. Let ABCD be a square of side of unit length. Let a circle C_1 centered at A with unit radius is drawn. Another circle C_2 which touches C_1 and the lines AD and AB are tangent to it, is also drawn. Let a tangent line from the point C to the circle C_2 meet the side AB at E. If the length of EB is $\alpha+\sqrt{3}\beta$, where α, β are integers, then $\alpha + \beta$ is equal to_______.

Official Ans. by NTA (1)

Sol.



Here AO + OD = 1 or $(\sqrt{2}+1)r = 1$

$\Rightarrow \quad r = \sqrt{2-1}$

equation of circle $(x-r)^2 + (y-r)^2 = r^2$

Equation of CE

$y - 1 = m(x-1)$

$mx - y + 1 - M = 0$

It is tangent to circle

$$\therefore \quad \left|\frac{mr-r+1-m}{\sqrt{m^2+1}}\right| = r$$

$$\left|\frac{(m-1)r+1-m}{\sqrt{m^2+1}}\right| = r$$

$$\frac{(m-1)^2(r-1)^2}{m^2+1} = r^2$$

Put $r=\sqrt{2}-1$

On solving $m = 2-\sqrt{3},\ 2+\sqrt{3}$

Taking greater slope of CE as $2+\sqrt{3}$

$y - 1 = (2+\sqrt{3})(x-1)$

Put $y = 0$

$-1 = (2+\sqrt{3})(x-1)$

$$\frac{-1}{2+\sqrt{3}} \times \left(\frac{2-\sqrt{3}}{2-\sqrt{3}}\right) = x-1$$

$x - 1 = \sqrt{3} - 1$

$EB = 1 - x = 1 - (\sqrt{3} - 1)$

$EB = 2 - \sqrt{3}$

4. If $\lim\limits_{x\to 0}\dfrac{ae^x - b\cos x + ce^{-x}}{x\sin x} = 2$, then a + b + c is equal to _______.

Official Ans. by NTA (4)

Sol. $\lim\limits_{x\to 0}\dfrac{ae^x - b\cos x + ce^{-x}}{x\sin x} = 2$

$$\Rightarrow \lim_{x\to 0}\frac{a\left(1+x+\frac{x^2}{2!}...\right)-b\left(1-\frac{x^2}{2!}+...\right)+c\left(1-x+\frac{x^2}{2!}\right)}{\left(\frac{x\sin x}{x}\right)x} = 2$$

$a - b + c = 0 \qquad .....(1)$

$a - c = 0 \qquad .....(2)$

$\& \quad \dfrac{a+b+c}{2} = 2$

$\Rightarrow \boxed{a+b+c=4}$

5. The total number of 3 × 3 matrices A having enteries from the set (0, 1, 2, 3) such that the sum of all the diagonal entries of AA^T is 9, is equal to _____.

Official Ans. by NTA (766)

Sol. Let $A = \begin{bmatrix} a & b & c \\ d & e & f \\ g & h & i \end{bmatrix}$

diagonal elements of

AA^T, $a^2 + b^2 + c^2$, $d^2 + e^2 + f^2$, $g^2 + b^2 + c^2$

Sum $= a^2 + b^2 + c^2 + d^2 + e^2 + f^2 + g^2 + h^2 + i^2 = 9$

a, b, c, d, e, f, g, h, i $\in$ {0, 1, 2, 3}

	Case	No. of Matrices
(1)	All – 1s	$\frac{9!}{9!}=1$
(2)	One → 3 remaining-0	$\frac{9!}{1!\times 8!}=9$
(3)	One-2 five-1s three-0s	$\frac{9!}{1!\times 5!\times 3!}=8\times 63$
(4)	two – 2's one-1 six-0's	$\frac{9!}{2!\times 6!}=63\times 4$

Total no. of ways = 1 + 9 + 8 × 63 + 63 × 4

= $\boxed{766}$

6. Let

$$P=\begin{bmatrix}-30 & 20 & 56\\ 90 & 140 & 112\\ 120 & 60 & 14\end{bmatrix} \text{ and } A=\begin{bmatrix}2 & 7 & \omega^2\\ -1 & -\omega & 1\\ 0 & -\omega & -\omega+1\end{bmatrix}$$

where $\omega=\frac{-1+i\sqrt{3}}{2}$, and I_3 be the identity matrix of order 3. If the determinant of the matrix $(P^{-1}AP - I_3)^2$ is $\alpha\omega^2$, then the value of α is equal to ______ .

Official Ans. by NTA (36)

Sol. Let $M = (P^{-1}AP - I)^2$

$= (P^{-1}AP)^2 - 2P^{-1}AP + I$

$= P^{-1}A^2P - 2P^{-1}AP + I$

$PM = A^2P - 2AP + P$

$= (A^2 - 2A.I + I^2)P$

$\Rightarrow$ Det(PM) = Det($(A - I)^2 \times P$)

$\Rightarrow$ DetP.DetM = Det$(A - I)^2 \times$ Det(P)

$\Rightarrow$ Det M = (Det(A – I))^2

Now $A-I=\begin{bmatrix}1 & 7 & w^2\\ -1 & -w-1 & 1\\ 0 & -w & -w\end{bmatrix}$

Det(A – I) = $(w^2 + w + w) + 7(-w) + w^3 = -6w$

Det((A – I))^2 = $36w^2$

$\Rightarrow$ $\alpha = 36$

7. If the normal to the curve $y(x)=\int_0^x (2t^2-15t+10)\,dt$ at a point (a,b) is parallel to the line x + 3y = –5, a > 1, then the value of |a + 6b| is equal to ________ .

Official Ans. by NTA (406)

Sol. $y(x)=\int_0^x (2t^2-15t+10)\,dt$

$y'(x)\big]_{x=a}=\left[2x^2-15x+10\right]_a=2a^2-15a+10$

Slope of normal $=-\frac{1}{3}$

$\Rightarrow$ $2a^2-15a+10=3 \Rightarrow a=7$

& $a = \frac{1}{2}$ (rejected)

$b = y(7) = \int_0^7 (2t^2-15t+10)dt$

$=\left[\frac{2t^3}{3}-\frac{15t^2}{2}+10t\right]_0^7$

$\Rightarrow$ $6b=4\times 7^3-45\times 49+60\times 7$

|a + 6b| = 406

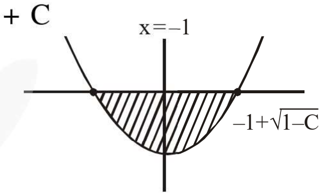
8. Let the curve y = y(x) be the solution of the differential equation, $\frac{dy}{dx}=2(x+1)$. If the numerical value of area bounded by the curve y = y(x) and x-axis is $\frac{4\sqrt{8}}{3}$, then the value of y(1) is equal to ________ .

Official Ans. by NTA (2)

Sol. $\frac{dy}{dx}=2(x+1)$

$\Rightarrow$ $\int dy=\int 2(x+1)dx$

$\Rightarrow$ $y(x) = x^2 + 2x + C$



Area $=\frac{4\sqrt{8}}{3}$

$-1+\sqrt{1-C}$

$\Rightarrow$ $2\int_{-1}^{-1+\sqrt{1-C}}(-(x+1)^2-C+1)dx=\frac{4\sqrt{8}}{3}$

$\Rightarrow$ $2\left[-\frac{(x+1)^3}{3}-Cx+x\right]_{-1}^{-1+\sqrt{1-C}}=\frac{4\sqrt{8}}{3}$

$\Rightarrow$ $-(\sqrt{1-C})^3+3c-3C\sqrt{1-C}$

$-3+3\sqrt{1-C}-3C+3=2\sqrt{8}$

$\Rightarrow$ C = –1

$\Rightarrow$ $f(x) = x^2 + 2x - 1,\ f(1) = 2$

9. Let $f : \mathbb{R} \to \mathbb{R}$ be a continuous function such that $f(x) + f(x + 1) = 2$, for all $x \in \mathbb{R}$. If $I_1 = \int_0^8 f(x)dx$ and $I_2 = \int_{-1}^3 f(x)dx$, then the value of $I_1 + 2I_2$ is equal to ________ .

Official Ans. by NTA (16)

Sol. $f(x) + f(x + 1) = 2$

$\Rightarrow$ $f(x)$ is periodic with period = 2

$$I_1 = \int_0^8 f(x)dx = 4\int_0^2 f(x)dx$$

$$= 4\int_0^1 \left(f(x) + f(1+x)\right)dx = 8$$

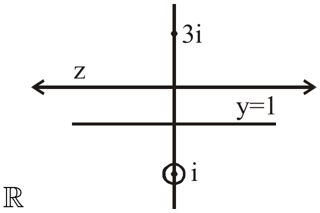
Similarly $I_2 = 2 \times 2 = 4$

$$I_1 + 2I_2 = 16$$

10. Let z and w be two complex numbers such that $w = z\bar{z} - 2z + 2$, $\left|\frac{z+i}{z-3i}\right| = 1$ and $\text{Re}(w)$ has minimum value. Then, the minimum value of $n \in \mathbb{N}$ for which w^n is real, is equal to _____.

Official Ans. by NTA (4)

Sol. $\omega = z\bar{z} - 2z + 2$



$$\left|\frac{z+i}{z-3i}\right| = 1$$

$\Rightarrow$ $|z + i| = |z - 3i|$

$\Rightarrow$ $z = x + i,\ x \in \mathbb{R}$

$\omega = (x + i)(x - i) - 2(x + i) + 2$

$= x^2 + 1 - 2x - 2i + 2$

$\text{Re}(\omega) = x^2 - 2x + 3$

For min $(\text{Re}(\omega))$, $x = 1$

$\Rightarrow$ $\omega = 2 - 2i = 2(1 - i) = 2\sqrt{2}\,e^{-i\frac{\pi}{4}}$

$$\omega^n = (2\sqrt{2})^n e^{-i\frac{n\pi}{4}}$$

For real & minimum value of n,

$n = 4$