FINAL JEE–MAIN EXAMINATION – AUGUST, 2021

(Held On Friday 27th August, 2021) **TIME : 3 : 00 PM to 6 : 00 PM**

MATHEMATICS — TEST PAPER WITH SOLUTION

SECTION-A

1. The angle between the straight lines, whose direction cosines are given by the equations $2l + 2m - n = 0$ and $mn + nl + lm = 0$, is :

(1) $\frac{\pi}{2}$ (2) $\pi - \cos^{-1}\left(\frac{4}{9}\right)$

(3) $\cos^{-1}\left(\frac{8}{9}\right)$ (4) $\frac{\pi}{3}$

Official Ans. by NTA (1)

Sol. $n = 2(\ell + m)$

$\ell m + n(\ell + m) = 0$

$\ell m + 2(\ell + m)^2 = 0$

$2\ell^2 + 2m^2 + 5m\ell = 0$

$2\left(\frac{\ell}{m}\right)^2 + 2 + 5\left(\frac{\ell}{m}\right) = 0.$

$2t^2 + 5t + 2 = 0$

$(t + 2)(2t + 1) = 0$

$\Rightarrow t = -2; -\frac{1}{2}$

(i) $\frac{\ell}{m} = -2$

$\frac{n}{m} = -2$

$(-2m, m, -2m)$

$(-2, 1, -2)$

(ii) $\frac{\ell}{m} = -\frac{1}{2}$

$n = -2\ell$

$(\ell, -2\ell, -2\ell)$

$(1, -2, -2)$

$\cos\theta = \frac{-2-2+4}{\sqrt{9}\sqrt{9}} = 0 \Rightarrow 0 = \frac{\pi}{2}$

2. Let $A = \begin{pmatrix} [x+1] & [x+2] & [x+3] \\ [x] & [x+3] & [x+3] \\ [x] & [x+2] & [x+4] \end{pmatrix}$, where $[t]$ denotes the greatest integer less than or equal to t. If det(A) = 192, then the set of values of x is the interval:

(1) [68, 69) (2) [62, 63)

(3) [65, 66) (4) [60, 61)

Official Ans. by NTA (2)

Sol. $\begin{vmatrix} [x+1] & [x+2] & [x+3] \\ [x] & [x+3] & [x+3] \\ [x] & [x+2] & [x+4] \end{vmatrix} = 192$

$R_1 \to R_1 - R_3$ & $R_2 \to R_2 - R_3$

$\begin{bmatrix} 1 & 0 & -1 \\ 0 & 1 & -1 \\ [x] & [x]+2 & [x]+4 \end{bmatrix} = 192$

$2[x] + 6 + [x] = 192 \Rightarrow [x] = 62$

3. Let M and m respectively be the maximum and minimum values of the function $f(x) = \tan^{-1}(\sin x + \cos x)$ in $\left[0, \frac{\pi}{2}\right]$, Then the value of $\tan(M - m)$ is equal to:

(1) $2+\sqrt{3}$ (2) $2-\sqrt{3}$

(3) $3+2\sqrt{2}$ (4) $3-2\sqrt{2}$

Official Ans. by NTA (4)

Sol. Let $g(x) = \sin x + \cos x = \sqrt{2}\sin\left(x + \frac{\pi}{4}\right)$

$g(x) \in [1, \sqrt{2}]$ for $x \in [0, \pi/2]$

$f(x) = \tan^{-1}(\sin x + \cos x) \in \left[\frac{\pi}{4}, \tan^{-1}\sqrt{2}\right]$

$\tan\left(\tan^{-1}\sqrt{2} - \frac{\pi}{4}\right) = \frac{\sqrt{2}-1}{1+\sqrt{2}} \times \frac{\sqrt{2}-1}{\sqrt{2}-1} = 3 - 2\sqrt{2}$

4. Each of the persons A and B independently tosses three fair coins. The probability that both of them get the same number of heads is :

(1) $\frac{1}{8}$ (2) $\frac{5}{8}$ (3) $\frac{5}{16}$ (4) 1

Official Ans. by NTA (3)

Sol. C – I '0' Head

T T T $\left(\frac{1}{2}\right)^3\left(\frac{1}{2}\right)^3=\frac{1}{64}$

C – II '1' head

H T T $\left(\frac{3}{8}\right)\left(\frac{3}{8}\right)=\frac{9}{64}$

C – III '2' Head

H H T $\left(\frac{3}{8}\right)\left(\frac{3}{8}\right)=\frac{9}{64}$

C–IV '3' Heads

H H H $\left(\frac{1}{8}\right)\left(\frac{1}{8}\right)=\frac{1}{64}$

Total probability = $\frac{5}{16}$.

5. A differential equation representing the family of parabolas with axis parallel to y-axis and whose length of latus rectum is the distance of the point (2, –3) form the line 3x + 4y = 5, is given by :

(1) $10\frac{d^2y}{dx^2}=11$ (2) $11\frac{d^2x}{dy^2}=10$

(3) $10\frac{d^2x}{dy^2}=11$ (4) $11\frac{d^2y}{dx^2}=10$

Official Ans. by NTA (4)

Sol. α. R = $\frac{|3(2)+4(-3)-5|}{5}=\frac{11}{5}$

$(x-h)^2=\frac{11}{5}(y-k)$

differentiate w.r.t 'x' : –

$2(x-h)=\frac{11}{5}\frac{dy}{dx}$

again differentiate

$2=\frac{11}{5}\frac{d^2y}{dx^2}$

$\frac{11d^2y}{dx^2}=10$.

6. If two tangents drawn from a point P to the parabola y^2 = 16(x – 3) are at right angles, then the locus of point P is :

(1) x + 3 = 0 (2) x + 1 = 0

(3) x + 2 = 0 (4) x + 4 = 0

Official Ans. by NTA (2)

Sol. Locus is directrix of parabola

$x-3+4=0 \Rightarrow x+1=0.$

7. The equation of the plane passing through the line of intersection of the planes $\vec{r}.(\hat{i}+\hat{j}+\hat{k})=1$ and $\vec{r}.(2\hat{i}+3\hat{j}-\hat{k})+4=0$ and parallel to the x-axis is:

(1) $\vec{r}.(\hat{j}-3\hat{k})+6=0$ (2) $\vec{r}.(\hat{i}+3\hat{k})+6=0$

(3) $\vec{r}.(\hat{i}-3\hat{k})+6=0$ (4) $\vec{r}.(\hat{j}-3\hat{k})-6=0$

Official Ans. by NTA (1)

Sol. Equation of planes are

$\vec{r}\,.\,(\hat{i}+\hat{j}+\hat{k})-1=0 \Rightarrow x+y+z-1=0$

and $\vec{r}\,.(2\hat{i}+3\hat{j}-\hat{k})+4=0 \Rightarrow 2x+3y-z+4=0$

equation of planes through line of intersection of these planes is :–

$(x+y+z-1)+\lambda(2x+3y-z+4)=0$

$\Rightarrow (1+2\lambda)x+(1+3\lambda)y+(1-\lambda)z-1+4\lambda=0$

But this plane is parallel to x–axis whose direction are (1, 0, 0)

$\therefore$ $(1+2\lambda)1+(1+3\lambda)0+(1-\lambda)0=0$

$\lambda=-\frac{1}{2}$

$\therefore$ Required plane is

$0x+\left(1-\frac{3}{2}\right)y+\left(1+\frac{1}{2}\right)z-1+4\left(\frac{-1}{2}\right)=0$

$\Rightarrow \frac{-y}{2}+\frac{3}{2}z-3=0$

$\Rightarrow y-3z+6=0$

$\Rightarrow \boxed{\vec{r}.(\hat{j}-3\hat{k})+6=0}$ Ans.

8. If the solution curve of the differential equation $(2x - 10y^3)\,dy + y\,dx = 0$, passes through the points (0, 1) and (2, β), then β is a root of the equation:

(1) $y^5 - 2y - 2 = 0$ (2) $2y^5 - 2y - 1 = 0$

(3) $2y^5 - y^2 - 2 = 0$ (4) $y^5 - y^2 - 1 = 0$

Official Ans. by NTA (4)

Sol. $(2x - 10y^3)\,dy + y\,dx = 0$

$$\Rightarrow \frac{dx}{dy} + \left(\frac{2}{y}\right)x = 10y^2$$

$$\text{I. F.} = e^{\int \frac{2}{y}dy} = e^{2\ell n(y)} = y^2$$

Solution of D.E. is

$$\therefore \quad x.\,y = \int \left(10y^2\right) y^2 .dy$$

$$xy^2 = \frac{10y^5}{5} + C \Rightarrow xy^2 = 2y^5 + C$$

It passes through (0, 1) → $0 = 2 + C \Rightarrow C = -2$

$\therefore$ Curve is $\boxed{xy^2 = 2y^5 - 2}$

Now, it passes through (2,β)

$2\beta^2 = 2\beta^5 - 2 \Rightarrow \beta^5 - \beta^2 - 1 = 0$

$\therefore$ β is root of an equation $\boxed{y^5 - y^2 - 1 = 0}$ Ans.

9. Let A(a, 0), B(b, 2b +1) and C(0, b), $b \neq 0$, $|b| \neq 1$, be points such that the area of triangle ABC is 1 sq. unit, then the sum of all possible values of a is :

(1) $\frac{-2b}{b+1}$ (2) $\frac{2b}{b+1}$

(3) $\frac{2b^2}{b+1}$ (4) $\frac{-2b^2}{b+1}$

Official Ans. by NTA (4)

Sol.
$$\left|\frac{1}{2}\begin{vmatrix} a & 0 & 1 \\ b & 2b+1 & 1 \\ 0 & b & 1 \end{vmatrix}\right| = 1$$

$$\Rightarrow \begin{vmatrix} a & 0 & 1 \\ b & 2b+1 & 1 \\ 0 & b & 1 \end{vmatrix} = \pm 2$$

$$\Rightarrow a\,(2b + 1 - b) - 0 + 1\,(b^2 - 0) = \pm 2$$

$$\Rightarrow a = \frac{\pm 2 - b^2}{b+1}$$

$$\therefore \quad a = \frac{2 - b^2}{b+1} \text{ and } a = \frac{-2 - b^2}{b+1}$$

sum of possible values of 'a' is

$$= \frac{-2b^2}{a+1} \text{ Ans.}$$

10. Let [λ] be the greatest integer less than or equal to λ. The set of all values of λ for which the system of linear equations $x + y + z = 4$, $3x + 2y + 5z = 3$, $9x + 4y + (28 + [\lambda])z = [\lambda]$ has a solution is:

(1) **R**

(2) $(-\infty, -9) \cup (-9, \infty)$

(3) [–9, –8)

(4) $(-\infty, -9) \cup [-8, \infty)$

Official Ans. by NTA (1)

Sol. $D = \begin{vmatrix} 1 & 1 & 1 \\ 3 & 2 & 5 \\ 9 & 4 & 28+[\lambda] \end{vmatrix} = -24 - [\lambda] + 15 = -[\lambda] - 9$

if $[\lambda] + 9 \neq 0$ then unique solution

if $[\lambda] + 9 = 0$ then $D_1 = D_2 = D_3 = 0$

so infinite solutions

Hence λ can be any red number.

11. The set of all values of k > –1, for which the equation $(3x^2 + 4x + 3)^2 - (k+1)(3x^2 + 4x + 3)(3x^2 + 4x + 2) + k(3x^2 + 4x + 2)^2 = 0$ has real roots, is :

(1) $\left(1, \frac{5}{2}\right]$ (2) [2, 3)

(3) $\left[-\frac{1}{2}, 1\right)$ (4) $\left(\frac{1}{2}, \frac{3}{2}\right] - \{1\}$

Official Ans. by NTA (1)

Sol. $(3x^2 + 4x + 3)^2 - (k+1)(3x^2 + 4x + 3)(3x^2 + 4x + 2)$

$+ k\,(3x^2 + 4x + 2)^2 = 0$

Let $3x^2 + 4x + 3 = a$

and $3x^2 + 4x + 2 = b \Rightarrow b = a - 1$

Given equation becomes

$\Rightarrow a^2 - (k+1)\,ab + k\,b^2 = 0$

$\Rightarrow a\,(a - kb) - b\,(a - kb) = 0$

$\Rightarrow (a - kb)(a - b) = 0 \Rightarrow a = kb$ or $a = b$ (reject)

$\because a = kb$

$\Rightarrow 3x^2 + 4x + 3 = k\,(3x^2 + 4x + 2)$

$\Rightarrow 3\,(k - 1)\,x^2 + 4\,(k - 1)\,x + (2k - 3) = 0$

for real roots

$D \geq 0$

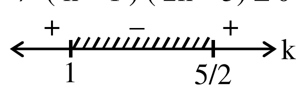
$\Rightarrow 16\,(k - 1)^2 - 4\,(3(k-1))\,(2k - 3) \geq 0$

$\Rightarrow 4\,(k - 1)\,\{4\,(k - 1) - 3\,(2k - 3)\} \geq 0$

$\Rightarrow 4\,(k - 1)\,\{-2k + 5\} \geq 0$

$\Rightarrow -4\,(k - 1)\,\{2k - 5\} \geq 0$

$\Rightarrow (k - 1)(2k - 5) \leq 0$



$\therefore \quad k \in \left[1, \frac{5}{2}\right]$

$\because \quad k \neq 1$

$\therefore \quad k \in \left(1, \frac{5}{2}\right]$ Ans.

12. A box open from top is made from a rectangular sheet of dimension a × b by cutting squares each of side x from each of the four corners and folding up the flaps. If the volume of the box is maximum, then x is equal to :

(1) $\frac{a+b-\sqrt{a^2+b^2-ab}}{12}$

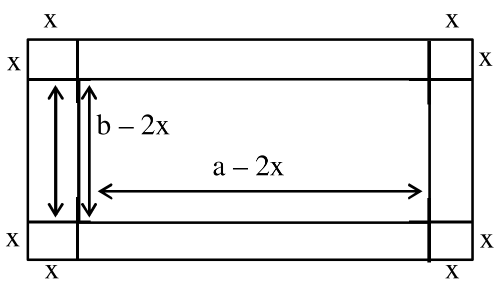
(2) $\frac{a+b-\sqrt{a^2+b^2+ab}}{6}$

(3) $\frac{a+b-\sqrt{a^2+b^2-ab}}{6}$

(4) $\frac{a+b+\sqrt{a^2+b^2-ab}}{6}$

Official Ans. by NTA (3)

Sol.



$V = \ell . b . h \quad = (a - 2x)(b - 2x)x$

$\Rightarrow V(x) = (2x - a)(2x - b)x$

$\Rightarrow V(x) = 4x^3 - 2(a + b)x^2 + abx$

$\Rightarrow \frac{d}{dx} v(x) = 12x^2 - 4(a + b)x + ab$

$\frac{d}{dx}(v(x)) = 0 \Rightarrow 12x^2 - 4(a + b)x + ab = 0 <^{\alpha}_{\beta}$

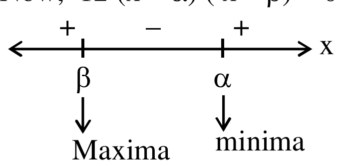
$\Rightarrow x = \frac{4(a+b) \pm \sqrt{16(a+b)^2 - 48ab}}{2(12)}$

$= \frac{(a+b) \pm \sqrt{a^2+b^2-ab}}{6}$

Let $x = \alpha = \frac{(a+b)+\sqrt{a^2+b^2-ab}}{6}$

$\beta = \frac{(a+b)-\sqrt{a^2+b^2-ab}}{6}$

Now, $12(x - \alpha)(x - \beta) = 0$



$\therefore x = \beta$

$= \frac{a+b-\sqrt{a^2+b^2-ab}}{b}$

13. The Boolean expression $(p \wedge q) \Rightarrow ((r \wedge q) \wedge p)$ is equivalent to :

(1) $(p \wedge q) \Rightarrow (r \wedge q)$ (2) $(q \wedge r) \Rightarrow (p \wedge q)$

(3) $(p \wedge q) \Rightarrow (r \vee q)$ (4) $(p \wedge r) \Rightarrow (p \wedge q)$

Official Ans. by NTA (1)

Sol. $(p \wedge q) \Rightarrow ((r \wedge q) \wedge p)$

$\sim (p \wedge q) \vee ((r \wedge q) \wedge p)$

$\sim (p \wedge q) \vee ((r \wedge p) \wedge (p \wedge q)$

$\Rightarrow [\sim(p \wedge q) \vee (p \wedge q)] \wedge (\sim(p \wedge q) \vee (r \wedge p))$

$\Rightarrow t \wedge [\sim (p \wedge q) \vee (r \wedge p)]$

$\Rightarrow \sim (p \wedge q) \vee (r \wedge p)$

$\Rightarrow (p \wedge q) \Rightarrow (r \wedge p)$

Aliter :

given statement says

" if p and q both happen then

p and q and r will happen"

it Simply implies

" If p and q both happen then

'r' too will happen "

i.e.

" if p and q both happen then r and p too will happen

i.e.

$(p \wedge q) \Rightarrow (r \wedge p)$

14. Let $\mathbb{Z}$ be the set of all integers,

$A = \{(x, y) \in \mathbb{Z} \times \mathbb{Z} : (x - 2)^2 + y^2 \leq 4\}$,

$B = \{(x, y) \in \mathbb{Z} \times \mathbb{Z} : x^2 + y^2 \leq 4\}$ and

$C = \{(x, y) \in \mathbb{Z} \times \mathbb{Z} : (x - 2)^2 + (y - 2)^2 \leq 4\}$

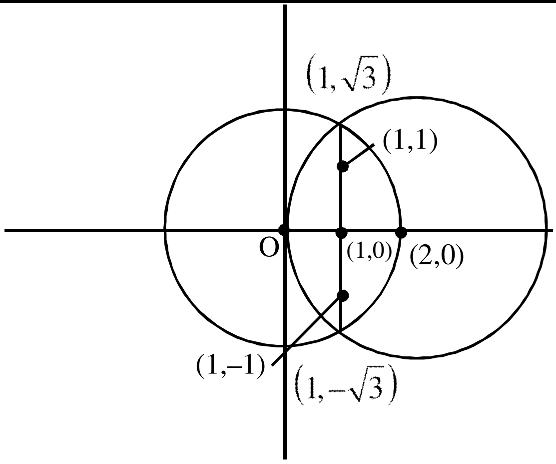
If the total number of relation from $A \cap B$ to $A \cap C$ is 2^p, then the value of p is :

(1) 16 (2) 25

(3) 49 (4) 9

Official Ans. by NTA (2)

Sol.



$(x-2)^2 + y^2 \le 4$

$x^2 + y^2 \le 4$

No. of points common in C_1 & C_2 is 5.

(0, 0) , (1, 0) , (2, 0) , (1, 1), (1, –1)

Similarly in C_2 & C_3 is 5.

No. of relations = $2^{5\times5} = 2^{25}$.

15. The area of the region bounded by the parabola $(y-2)^2 = (x-1)$, the tangent to it at the point whose ordinate is 3 and the x-axis is :

(1) 9 (2) 10 (3) 4 (4) 6

Official Ans. by NTA (1)

Sol. $y = 3 \Rightarrow x = 2$

Point is (2,3)

Diff. w.r.t x

$2(y-2)y' = 1$

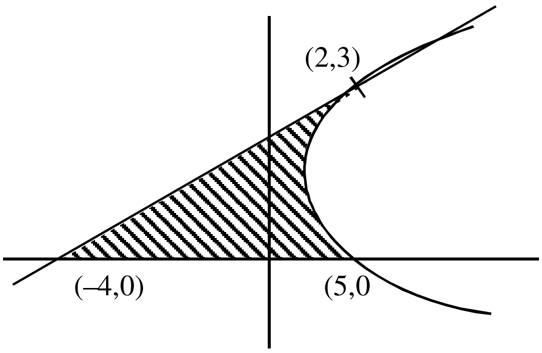
$\Rightarrow y' = \frac{1}{2(y-2)}$

$\Rightarrow y'_{(2,3)} = \frac{1}{2}$

$\Rightarrow \frac{y-3}{x-2} = \frac{1}{2} \Rightarrow x - 2y + 4 = 0$

Area $= \int_0^3 \left((y-2)^2 + 1 - (2y-4)\right) dy$

= 9 sq. units



16. If $y(x) = \cot^{-1}\left(\frac{\sqrt{1+\sin x}+\sqrt{1-\sin x}}{\sqrt{1+\sin x}-\sqrt{1-\sin x}}\right)$, $x \in \left(\frac{\pi}{2}, \pi\right)$, then $\frac{dy}{dx}$ at $x = \frac{5\pi}{6}$ is:

(1) $-\frac{1}{2}$ (2) –1 (3) $\frac{1}{2}$ (4) 0

Official Ans. by NTA (1)

Sol. $y(x) = \cot^{-1}\left[\frac{\cos\frac{x}{2}+\sin\frac{x}{2}+\sin\frac{x}{2}-\cos\frac{x}{2}}{\cos\frac{x}{2}+\sin\frac{x}{2}-\sin\frac{x}{2}+\cos\frac{x}{2}}\right]$

$y(x) = \cot^{-1}\left(\tan\frac{x}{2}\right) = \frac{\pi}{2} - \frac{x}{2}$

$y'(x) = \frac{-1}{2}$

17. Two poles, AB of length a metres and CD of length a + b (b ≠ a) metres are erected at the same horizontal level with bases at B and D. If BD = x and $\tan\angle ACB = \frac{1}{2}$, then:

(1) $x^2 + 2(a+2b)x - b(a+b) = 0$

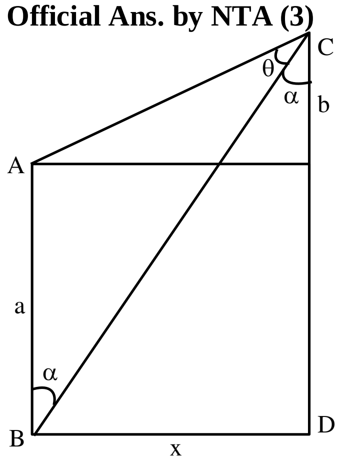
(2) $x^2 + 2(a+2b)x + a(a+b) = 0$

(3) $x^2 - 2ax + b(a+b) = 0$

(4) $x^2 - 2ax + a(a+b) = 0$

Official Ans. by NTA (3)

Sol.



$\tan\theta = \frac{1}{2}$

$\tan(\theta+\alpha) = \frac{x}{b}$, $\tan\alpha = \frac{x}{a+b}$

$\Rightarrow \frac{1}{2} + \frac{x}{a+b}$

$\Rightarrow \frac{\frac{1}{2}+\frac{x}{a+b}}{1-\frac{1}{2}\times\frac{x}{a+b}} = \frac{x}{b}$.

$\Rightarrow x^2 - 2ax + ab + b^2 = 0$

18. If $0 < x < 1$ and $y = \frac{1}{2}x^2 + \frac{2}{3}x^3 + \frac{3}{4}x^4 + \ldots$, then the value of e^{1+y} at $x = \frac{1}{2}$ is:

(1) $\frac{1}{2}e^2$ (2) $2e$

(3) $\frac{1}{2}\sqrt{e}$ (4) $2e^2$

Official Ans. by NTA (1)

Sol. $y = \left(1-\frac{1}{2}\right)x^2 + \left(1-\frac{1}{3}\right)x^3 + \ldots..$

$= (x^2 + x^3 + x^4 + \ldots\ldots.) - \left(\frac{x^2}{2} + \frac{x^3}{3} + \frac{x^4}{4} + \ldots.\right)$

$= \frac{x^2}{1-x} + x - \left(x + \frac{x^2}{2} + \frac{x^2}{3} + \ldots\right)$

$= \frac{x}{1-x} + \ell n(1-x)$

$x = \frac{1}{2} \Rightarrow y = 1 - \ell n2$

$e^{1+y} = e^{1+1-\ell n2}$

$= e^{2-\ell n2} = \frac{e^2}{2}$

19. The value of the integral $\int_0^1 \frac{\sqrt{x}\,dx}{(1+x)(1+3x)(3+x)}$ is:

(1) $\frac{\pi}{8}\left(1-\frac{\sqrt{3}}{2}\right)$ (2) $\frac{\pi}{4}\left(1-\frac{\sqrt{3}}{6}\right)$

(3) $\frac{\pi}{8}\left(1-\frac{\sqrt{3}}{6}\right)$ (4) $\frac{\pi}{4}\left(1-\frac{\sqrt{3}}{2}\right)$

Official Ans. by NTA (1)

Sol. $I = \int_0^1 \frac{\sqrt{x}}{(1+x)(1+3x)(3+x)}dx$

Let $x = t^2 \Rightarrow dx = 2t.dt$

$I = \int_0^1 \frac{t(2t)}{(t^2+1)(1+3t^2)(3+t^2)}dt$

$I = \int_0^1 \frac{(3t^2+1)-(t^2+1)}{(3t^2+1)(t^2+1)(3+t^2)}dt$

$I = \int_0^1 \frac{dt}{(t^2+1)(3+t^2)} - \int_0^1 \frac{dt}{(1+3t^2)(3+t^2)}$

$= \frac{1}{2}\int_0^1 \frac{(3+t^2)-(t^2+1)}{(t^2+1)(3+t^2)}dt + \frac{1}{8}\int_0^1 \frac{(1+3t^2)-3(3+t^2)}{(1+3t^2)(3+t^2)}dt$

$= \frac{1}{2}\int_0^1 \frac{dt}{1+t^2} - \frac{1}{2}\int_0^1 \frac{dt}{t^2+3} + \frac{1}{8}\int_0^1 \frac{dt}{t^2+3} - \frac{3}{8}\int_0^1 \frac{dt}{(1+3t^2)}$

$= \frac{1}{2}\int_0^1 \frac{dt}{t^2+1} - \frac{3}{8}\int_0^1 \frac{dt}{t^2+3} - \frac{3}{8}\int_0^1 \frac{dt}{1+3t^2}$

$= \frac{1}{2}\left(\tan^{-1}(t)\right)_0^1 - \frac{3}{8\sqrt{3}}\left(\tan^{-1}\left(\frac{t}{\sqrt{3}}\right)\right)_0^1$

$- \frac{3}{8\sqrt{3}}\left(\tan^{-1}(\sqrt{3}t)\right)_0^1$

$= \frac{1}{2}\left(\frac{\pi}{4}\right) - \frac{\sqrt{3}}{8}\left(\frac{\pi}{6}\right) - \frac{\sqrt{3}}{8}\left(\frac{\pi}{3}\right)$

$= \frac{\pi}{8} - \frac{\sqrt{3}}{16}\pi$

$= \frac{\pi}{8}\left(1-\frac{\sqrt{3}}{2}\right)$

20. If $\lim_{x\to\infty}\left(\sqrt{x^2-x+1}-ax\right) = b$, then the ordered pair (a, b) is:

(1) $\left(1,\frac{1}{2}\right)$ (2) $\left(1,-\frac{1}{2}\right)$

(3) $\left(-1,\frac{1}{2}\right)$ (4) $\left(-1,-\frac{1}{2}\right)$

Official Ans. by NTA (2)]

Sol. (2)

$$\lim_{x\to\infty}\left(\sqrt{x^2-x+1}\right)-ax=b \qquad (\infty-\infty)$$

$\Rightarrow a>0$

Now, $\lim_{x\to\infty}\frac{(x^2-x+1-a^2x^2)}{\sqrt{x^2-x+1}+ax}=b$

$\Rightarrow \lim_{x\to\infty}\frac{(1-a^2)x^2-x+1}{\sqrt{x^2-x+1}+ax}=b$

$\Rightarrow \lim_{x\to\infty}\frac{(1-a^2)x^2-x+1}{x\left(\sqrt{1-\frac{1}{x}+\frac{1}{x^2}}+a\right)}=b$

$\Rightarrow 1-a^2=0 \Rightarrow a=1$

Now, $\lim_{x\to\infty}\frac{-x+1}{x\left(\sqrt{1-\frac{1}{x}+\frac{1}{x^2}}+a\right)}=b$

$\Rightarrow \frac{-1}{1+a}=b \Rightarrow b=-\frac{1}{2}$

$(a,b)=\left(1,-\frac{1}{2}\right)$

SECTION-B

1. Let S be the sum of all solutions (in radians) of the equation $\sin^4\theta+\cos^4\theta-\sin\theta\cos\theta=0$ in $[0, 4\pi]$. Then $\frac{8S}{\pi}$ is equal to ________.

Official Ans. by NTA (56)

Sol. Given equation

$\sin^4\theta+\cos^4\theta-\sin\theta\cos\theta=0$

$\Rightarrow 1-\sin^2\theta\cos^2\theta-\sin\theta\cos\theta=0$

$\Rightarrow 2-(\sin 2\theta)^2-\sin 2\theta=0$

$\Rightarrow (\sin 2\theta)^2+(\sin 2\theta)-2=0$

$\Rightarrow (\sin 2\theta+2)(\sin 2\theta-1)=0$

$\Rightarrow \sin 2\theta=1$ or $\boxed{\sin 2\theta=-2}$ (not possible)

$\Rightarrow 2\theta=\frac{\pi}{2},\frac{5\pi}{2},\frac{9\pi}{2},\frac{13\pi}{2}$

$\Rightarrow \theta=\frac{\pi}{4},\frac{5\pi}{4},\frac{9\pi}{4},\frac{13\pi}{4}$

$\Rightarrow S=\frac{\pi}{4}+\frac{5\pi}{4}+\frac{9\pi}{4}+\frac{13\pi}{4}=7\pi$

$\Rightarrow \frac{8S}{\pi}=\frac{8\times 7\pi}{\pi}=56.00$

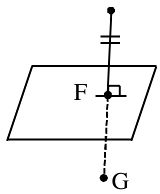
2. Let S be the mirror image of the point Q(1, 3, 4) with respect to the plane $2x-y+z+3=0$ and let R $(3, 5, \gamma)$ be a point of this plane. Then the square of the length of the line segment SR is __________.

Official Ans. by NTA (72)

Sol. Since R $(3,5,\gamma)$ lies on the plane $2x-y+z+3=0$.

Therefore, $6-5+\gamma+3=0$

$\Rightarrow \gamma=-4$



Now,

dr's of line QS

are $2, -1, 1$

equation of line QS is

$\frac{x-1}{2}=\frac{y-3}{-1}=\frac{z-4}{1}=\lambda$ (say)

$\Rightarrow F(2\lambda+1, -\lambda+3, \lambda+4)$

F lies in the plane

$\Rightarrow 2(2\lambda+1)-(-\lambda+3)+(\lambda+4)+3=0$

$\Rightarrow 4\lambda+2+\lambda-3+\lambda+7=0$

$\Rightarrow 6\lambda+6=0 \Rightarrow \lambda=-1.$

$\Rightarrow F(-1,4,3)$

Since, F is mid-point of QS.

Therefore, co-ordinated of S are $(-3,5,2)$.

So, $SR=\sqrt{36+0+36}=\sqrt{72}$

$SR^2=72.$

3. The probability distribution of random variable X is given by:

X	1	2	3	4	5
P(X)	K	2K	2K	3K	K

Let $p=P(1<X<4 \mid X<3)$. If $5p=\lambda K$, then λ equal to ________.

Official Ans. by NTA (30)

Sol. $\sum P(X) = 1 \Rightarrow k + 2k + 2k + 3k + k = 1$

$\Rightarrow k = \frac{1}{9}$

Now, $p = P\left(\frac{kX < 4}{X < 3}\right) = \frac{P(X=2)}{P(X<3)} = \frac{\frac{2k}{9k}}{\frac{k}{9k} + \frac{2k}{9k}} = \frac{2}{3}$

$\Rightarrow p = \frac{2}{3}$

Now, $5p = \lambda k$

$\Rightarrow (5)\left(\frac{2}{3}\right) = \lambda(1/9)$

$\Rightarrow \lambda = 30$

4. Let z_1 and z_2 be two complex numbers such that $\arg(z_1 - z_2) = \frac{\pi}{4}$ and z_1, z_2 satisfy the equation $|z - 3| = \text{Re}(z)$. Then the imaginary part of $z_1 + z_2$ is equal to __________.

Official Ans. by NTA (6)

Sol. $|z - 3| = \text{Re}(z)$

let $Z = x = iy$

$\Rightarrow (x - 3)^2 + y^2 = x^2$

$\Rightarrow x^2 + 9 - 6x + y^2 = x^2$

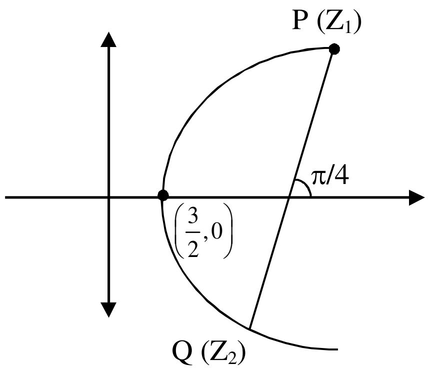
$\Rightarrow y^2 = 6x - 9$

$\Rightarrow y^2 = 6\left(x - \frac{3}{2}\right)$

$\Rightarrow z_1$ and z_2 lie on the parabola mentioned in eq.(1)

$\arg(z_1 - z_2) = \frac{\pi}{4}$

$\Rightarrow$ Slope of PQ = 1.



Let $P\left(\frac{3}{2} + \frac{3}{2}t_1^2, 3t_1\right)$ and $Q\left(\frac{3}{2} + \frac{3}{2}t_2^2, 3t_2\right)$

Slope of PQ = $\frac{3(t_2 - t_1)}{\frac{3}{2}(t_2^2 - t_1^2)} = 1$

$\Rightarrow \frac{2}{t_2 + t_2} = 1$

$\Rightarrow t_2 + t_1 = 2$

$\text{Im}(z_1 + z_2) = 3t_1 + 3t_2 = 3(t_1 + t_2) = 3\ (2)$

Ans. 6.00

Aliter :

Let $z_1 = x_1 + iy_1$; $z_2 = x_2 + iy_2$

$z_1 - z_2 = (x_1 - x_2) + i(y_1 - y_2)$

$\therefore \arg(z_1 - z_2) = \frac{\pi}{4} \Rightarrow \tan^{-1}\left(\frac{y_1 - y_2}{x_1 - x_2}\right) = \frac{\pi}{4}$

$y_1 - y_2 = x_1 - x_2$ ______(1)

$|z_1 - 3| = \text{Re}(z_1) \Rightarrow (x_1 - 3)^2 + y_1^2 = x_1^2$ __ (2)

$|z_2 - 3| = \text{Re}(z_2) \Rightarrow (x_2 - 3)^2 + y_2^2 = x_2^2$ __(2)

sub (2) & (3)

$(x_1 - 3)^2 - (x_2 - 3)^2 + y_1^2 - y_2^2 = x_1^2 - x_2^2$

$(x_1 - x_2)(x_1 + x_2 - 6) + (y_1 - y_2)(y_1 + y_2)$

$= (x_1 - x_2)(x_1 + x_2)$

$x_1 + x_2 - 6 + y_1 + y_2 = x_1 + x_2 \Rightarrow y_1 + y_2 = 6.$

5. Let S = {1, 2, 3, 4, 5, 6, 9}. Then the number of elements in the set $T = \{A \subseteq S : A \neq \phi$ and the sum of all the elements of A is not a multiple of 3} is __________.

Official Ans. by NTA (80)

Sol. 3n type $\rightarrow 3, 6, 9 = P$

3n – 1 type $\rightarrow 2, 5 = Q$

3n – 2 type $\rightarrow 1, 4 = R$

number of subset of S containing one element which are not divisible by 3 = ${}^2C_1 + {}^2C_1 = 4$

number of subset of S containing two numbers whose some is not divisible by 3

$= {}^3C_1 \times {}^2C_1 + {}^3C_1 \times {}^2C_1 + {}^2C_2 + {}^2C_2 = 14$

number of subsets containing 3 elements whose sum is not divisible by 3

$= {}^3C_2 \times {}^4C_1 + ({}^2C_2 \times {}^2C_1)2 + {}^3C_1({}^2C_2 + {}^2C_2) = 22$

number of subsets containing 4 elements whose sum is not divisible by 3

$= {}^3C_3 \times {}^4C_1 + {}^3C_2({}^2C_2 + {}^2C_2) + ({}^3C_1{}^2C_1 \times {}^2C_2)2$

= 4 + 6 + 12 = 22.

number of subsets of S containing 5 elements whose sum is not divisible by 3.

$= {}^3C_3({}^2C_2 + {}^2C_2) + ({}^3C_2{}^2C_1 \times {}^2C_2) \times 2$ = 2 + 12 = 14

number of subsets of S containing 6 elements whose sum is not divisible by 3 = 4

$\Rightarrow$ Total subsets of Set A whose sum of digits is not divisible by 3 = 4 + 14 + 22 + 22 + 14 + 4 = 80.

6. Let A (secθ, 2tanθ) and B (secϕ, 2tanϕ), where θ + ϕ = π/2, be two points on the hyperbola $2x^2 - y^2 = 2$. If (α, β) is the point of the intersection of the normals to the hyperbola at A and B, then $(2\beta)^2$ is equal to ___________.

Official Ans. by NTA (36)

ALLEN Ans. (Bonus)

Sol. Since, point A (sec θ, 2 tan θ) lies on the hyperbola

$2x^2 - y^2 = 2$

Therefore, $2\sec^2\theta - 4\tan^2\theta = 2$

$\Rightarrow 2 + 2\tan^2\theta - 4\tan^2\theta = 2$

$\Rightarrow \tan\theta = 0 \Rightarrow \theta = 0$

Similarly, for point B, we will get ϕ = 0.

but according to question $\theta + \phi = \frac{\pi}{2}$

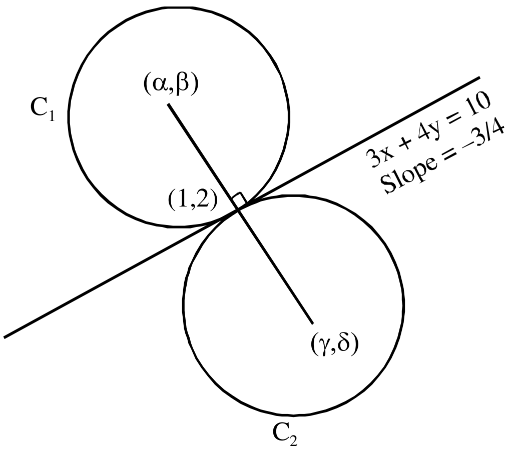
which is not possible.

Hence it must be a 'BONUS'.

7. Two circles each of radius 5 units touch each other at the point (1, 2). If the equation of their common tangent is 4x + 3y = 10, and $C_1(\alpha, \beta)$ and $C_2(\gamma, \delta)$, $C_1 \neq C_2$ are their centres, then $|(\alpha + \beta)(\gamma + \delta)|$ is equal to ___________.

Official Ans. by NTA (40)

Sol. Slope of line joining centres of circles $= \frac{4}{3} = \tan\theta$



$\Rightarrow \cos\theta = \frac{3}{5}, \sin\theta = \frac{4}{5}$

Now using parametric form

$\frac{x-1}{\cos\theta} = \frac{y-2}{\sin\theta} = \pm 5$

⊕ $(x, y) = (1 + 5\cos\theta, 2 + 5\sin\theta)$

$(\alpha, \beta) = (4, 6)$

Θ $(x, y) = (\gamma, \delta) = (1 - 5\cos\theta, 2 - 5\sin\theta)$

$(\gamma, s) = (-2, -2)$

$\Rightarrow |(\alpha + \beta)(\gamma + \delta)| = |10x - 4| = 40$

8. $3 \times 7^{22} + 2 \times 10^{22} - 44$ when divided by 18 leaves the remainder _________.

Official Ans. by NTA (15)

Sol. $3(1 + 6)^{22} + 2\cdot(1 + 9)^{22} - 44 = (3 + 2 - 44) = 18.I$

$= -39 + 18.I$

$= (54 - 39) + 18(I - 3)$

$= 15 + 18 I_1$

$\Rightarrow$ Remainder = 15.

9. An online exam is attempted by 50 candidates out of which 20 are boys. The average marks obtained by boys is 12 with a variance 2. The variance of marks obtained by 30 girls is also 2. The average marks of all 50 candidates is 15. If μ is the average marks of girls and σ^2 is the variance of marks of 50 candidates, then $\mu + \sigma^2$ is equal to _________.

Official Ans. by NTA (25)

Sol. $\sigma_b^2 = 2$ (variance of boys) n_1 = no. of boys

$\bar{x}_b = 12$ n_2 = no. of girls

$\sigma_g^2 = 2$

$\bar{x}_g = \frac{50 \times 15 - 12 \times \sigma_b}{30} = \frac{750 - 12 \times 20}{30} = 17 = \mu$

variance of combined series

$\sigma^2 = \frac{n_1\sigma_b^2 + n_2\sigma_g^2}{n_1 + n_2} + \frac{n_1 \cdot n_2}{(n_1 + n_2)^2}(\bar{x}_b - \bar{x}_g)^2$

$\sigma^2 = \frac{20 \times 2 + 30 \times 2}{20 + 30} + \frac{20 \times 30}{(20 + 30)^2}(12 - 17)^2$

$\sigma^2 = 8.$

$\Rightarrow \mu + \sigma^2 = 17 + 8 = 25$

10. If $\int \frac{2e^x + 3e^{-x}}{4e^x + 7e^{-x}} dx = \frac{1}{14}(ux + v \log_e(4e^x + 7e^{-x})) + C$, where C is a constant of integration, then u + v is equal to __________.

Official Ans. by NTA (7)

Sol. $\int \frac{2e^x}{4e^x + 7e^{-x}} dx \ + 3\int \frac{e^{-x}}{4e^x + 7e^{-x}} dx$

$= \int \frac{2e^{2x}}{4e^{2x} + 7} dx + 3\int \frac{e^{-2x}}{4 + 7e^{-2x}} dx$

Let $4e^{2x} + 7 = T$ Let $4 + 7e^{-2x} = t$

$8e^{2x} dx = dT$ $-14e^{-2x} dx = dt$

$2e^{2x} dx = \frac{dT}{4}$ $e^{-2x} dx = -\frac{dt}{14}$

$\int \frac{dT}{4T} - \frac{3}{14}\int \frac{dt}{t}$

$= \frac{1}{4}\log T - \frac{3}{14}\log t + C$

$= \frac{1}{4}\log(4e^{2x} + 7) - \frac{3}{14}\log\left(4 + 7e^{-2x}\right) + C$

$= \frac{1}{14}\left[\frac{1}{2}\log\left(4e^x + 7e^{-x}\right) + \frac{13}{2}x\right] + C$

$u = \frac{13}{2}, v = \frac{1}{2} \Rightarrow u + v = 7$

Aliter :

$2e^x + 3e^{-x} = A(4e^x + 7e^{-x}) + B(4e^x - 7e^{-x}) + \lambda$

$2 = 4A + 4B$; $3 = 7A - 7B$; $\lambda = 0$

$A + B = \frac{1}{2}$

$A - B = \frac{3}{7}$

$A = \frac{1}{2}\left(\frac{1}{2} + \frac{3}{7}\right) = \frac{7+6}{28} = \frac{13}{28}$

$B = A - \frac{3}{7} = \frac{13}{28} - \frac{3}{7} = \frac{13-12}{28} = \frac{1}{28}$

$\int \frac{13}{28} dx + \frac{1}{28}\int \frac{4e^x - 7e^{-x}}{4e^x + 7e^{-x}} dx$

$\frac{13}{28}x + \frac{1}{28}\ell n|4e^x + 7e^{-x}| + C$

$u = \frac{13}{2}; v = \frac{1}{2}$

$\Rightarrow u + v = 7$