FINAL JEE–MAIN EXAMINATION – JULY, 2022

(Held On Monday 25th July, 2022) **TIME : 9 : 00 AM to 12 : 00 NOON**

MATHEMATICS — TEST PAPER WITH SOLUTION

SECTION-A

1. The total number of functions,

$f : \{1, 2, 3, 4\} \cdot \{1, 2, 3, 4, 5, 6\}$

such that $f(1) + f(2) = f(3)$, is equal to :

(A) 60 (B) 90

(C) 108 (D) 126

Official Ans. by NTA (B)

Ans. (B)

Sol. $A = \{1, 2, 3, 4\}$

$B = \{1, 2, 3, 4, 5, 6\}$

Here f(3) can be 2, 3, 4, 5, 6

$f(3) = 2, (f(1), f(2)) \to (1,1) \to 6$ cases

$f(3) = 3, (f(1), f(2)) \to (1,2), (2,1)$

$\to 2 \times 6 = 12$ cases

$f(3) = 4, (f(1), f(2)) \to (1,3), (3,1), (2,2)$

$\to 3 \times 6 = 18$ cases

$f(3) = 5, (f(1), f(2)) \to (1,4), (4,1), (2,3), (3,2)$

$\to 4 \times 6 = 24$ cases

$f(3) = 6, (f(1), f(2)) \to (1,5),(5,1),(2,4),(4,2),(3,3)$

$\to 5 \times 6 = 30$ cases

Total number of cases = 6+ 12 + 18 + 24 + 30 = 90

2. If $\alpha, \beta, \gamma, \delta$ are the roots of the equation $x^4 + x^3 + x^2 + x + 1 = 0$, then

$\alpha^{2021} + \beta^{2021} + \gamma^{2021} + \delta^{2021}$ is equal to .

(A) –4 (B) –1

(C) 1 (D) 4

Official Ans. by NTA (B)

Ans. (B)

Sol. $\alpha, \beta, \gamma, \delta$ root of the equation

$x^4 + x^3 + x^2 + x + 1 = 0$

Which are 5^{th} roots of unity except 1.

then $\alpha^{2021} + \beta^{2021} + \gamma^{2021} + \delta^{2021} =$

$\alpha + \beta + \gamma + \delta = -1$

3. For $n \in N$, let $S_n = \left\{z \in C : |z - 3 + 2i| = \frac{n}{4}\right\}$ and

$T_n = \left\{z \in C : |z - 2 + 3i| = \frac{1}{n}\right\}$.

Then the number of elements in the set

$\{n \in N : S_n \cap T_n = \phi\}$ is :

(A) 0 (B) 2 (C) 3 (D) 4

Official Ans. by NTA (D)

Ans. (Bonus)

Sol. $S_n : |z - (3 - 2i)| = \frac{n}{4}$ is a circle center $C_1(3,-2)$

and radius n/4

$T_n : |z - (2 - 3i)| = \frac{1}{n}$ is a circle center C_2 (2,–3)

and radius 1/n

Here $S_n \cap T_n = \phi$

Both circles do not intersect each other

Case-1 : $C_1C_2 > n/4 + 1/n$

$\sqrt{2} > \frac{n}{4} + \frac{1}{n}$

then n = 1, 2, 3, 4

Case-2 : $C_1C_2 < \left|\frac{n}{4} - \frac{1}{n}\right|$

$\Rightarrow \sqrt{2} < \left|\frac{n^2 - 4}{4n}\right|$

$\Rightarrow$ n has infinite solutions for $n \in \mathbf{N}$

4. The number of $\theta \in (0, 4\pi)$ for which the system of linear equations

$3 (\sin 3\theta) x - y + z = 2$

$3 (\cos 2\theta) x + 4y + 3z = 3$

$6x + 7y + 7z = 9$

has no solution is :

(A) 6 (B) 7 (C) 8 (D) 9

Official Ans. by NTA (B)

Ans. (B)

Sol. The system of equation has no solution.

$$D=\begin{vmatrix}3\sin3\theta & -1 & 1\\ 3\cos2\theta & 4 & 3\\ 6 & 7 & 7\end{vmatrix}=0$$

$21\sin3\theta+42\cos2\theta-42=0$

$\sin3\theta+2\cos2\theta-2=0$

Number of solution is 7 in $(0,4\pi)$

5. If $\lim\limits_{n\to\infty}\left(\sqrt{n^2-n-1}+n\alpha+\beta\right)=0$ then $8(\alpha+\beta)$ is equal to :

(A) 4 (B) –8

(C) –4 (D) 8

Official Ans. by NTA (C)

Ans. (C)

Sol. $\lim\limits_{n\to\infty} n\left(1-\frac{n+1}{n^2}\right)^{\frac{1}{2}}+\alpha n+\beta=0$

$$\lim_{n\to\infty} n\left\{1-\frac{1}{2}\left(\frac{n+1}{n^2}\right)+\frac{\left(\frac{1}{2}\right)\left(-\frac{1}{2}\right)}{2!}\left(\frac{n+1}{n^2}\right)^2+....\right\}+\alpha n+\beta=0$$

$\lim\limits_{n\to\infty} n-\frac{1}{2}+\frac{1}{n}+....+n\alpha+\beta=0$

$\alpha=-1,\beta=\frac{1}{2}$

$8(\alpha+\beta)=-4$

6. If the absolute maximum value of the function f(x) $=(x^2-2x+7)\,e^{(4x^3-12x^2-180x+31)}$ in the interval [–3, 0] is $f(\alpha)$, then :

(A) $\alpha=0$ (B) $\alpha=-3$

(C) $\alpha\in(-1,0)$ (D) $\alpha\in(-3,-1)$

Official Ans. by NTA (B)

Ans. (B)

Sol. $f'(x)=e^{(4x^3-12x^2-180x+31)}\left(12(x^2-2x+7)(x+3)(x-5)+2(x-1)\right)$

for $x\in[-3,0]$

$\Rightarrow f'(x)<0$

f(x) is decreasing function on [–3,0]

The absolute maximum value of the function f(x) is at x = –3

$\Rightarrow \alpha=-3$

7. The curve $y(x)=ax^3+bx^2+cx+5$ touches the x-axis at the point P(–2, 0) and cuts the y-axis at the point Q, where y' is equal to 3. Then the local maximum value of y(x) is :

(A) $\frac{27}{4}$ (B) $\frac{29}{4}$ (C) $\frac{37}{4}$ (D) $\frac{9}{2}$

Official Ans. by NTA (A)

Ans. (A)

Sol. $y(x)=ax^3+bx^2+cx+5$ is passing through (-2,0) then $8a-4b+2c=5$……(1)

$y'(x)=3ax^2+2bx+c$ touches x-axis at (-2,0)

$12a-4b+c=0$ ……(2)

again, for $x=0,\ y'(x)=3$

$c=3$ ………(3)

Solving eq. (1), (2) & (3) $a=-\frac{1}{2},\ b=-\frac{3}{4}$

$y'(x)=-\frac{3}{2}x^2-\frac{3}{2}x+3$

y(x) has local maxima at x = 1

$y(1)=\frac{27}{4}$

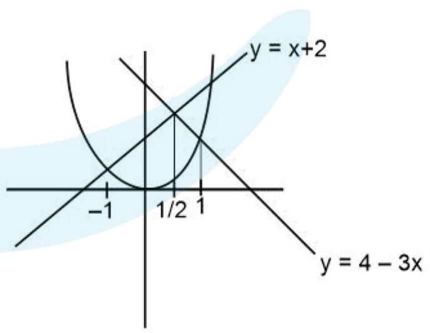
8. The area of the region given by $A=\{(x,y): x^2\le y\le \min\{x+2, 4-3x\}\}$ is :

(A) $\frac{31}{8}$ (B) $\frac{17}{6}$ (C) $\frac{19}{6}$ (D) $\frac{27}{8}$

Official Ans. by NTA (B)

Ans. (B)

Sol.



$$A=\int_{-1}^{\frac{1}{2}}(x+2-x^2)dx+\int_{\frac{1}{2}}^{1}(4-3x-x^2)dx=\frac{17}{6}$$

9. For any real number x, let [x] denote the largest integer less than equal to x. Let f be a real valued function defined on the interval [–10,10] by $f(x)=\begin{cases} x-[x], & \text{if}(x)\text{ is odd} \\ 1+[x]-x & \text{if}(x)\text{ is even}\end{cases}$

Then the value of $\frac{\pi^2}{10}\int_{-10}^{10} f(x)\cos\pi x dx$ is :

(A) 4 (B) 2

(C) 1 (D) 0

Official Ans. by NTA (A)

Ans. (A)

Sol. f(x) is periodic function whose period is 2

$$\frac{\pi^2}{10}\int_{-10}^{10} f(x)\cos\pi x dx = \frac{\pi^2}{10}\times 10\int_0^2 f(x)\cos\pi x dx$$

$$=\pi^2\left(\int_0^1 (1-x)\cos\pi x dx + \int_1^2 (x-1)\cos\pi x dx\right)$$

Using by parts

$$=\pi^2\times\frac{4}{\pi^2}=4$$

10. The slope of the tangent to a curve C : y =y(x) at any point [x, y) on it is $\frac{2e^{2x}-6e^{-x}+9}{2+9e^{-2x}}$. If C passes through the points $\left(0,\frac{1}{2}+\frac{\pi}{2\sqrt{2}}\right)$ and $\left(\alpha,\frac{1}{2}e^{2\alpha}\right)$ then e^{α} is equal to :

(A) $\frac{3+\sqrt{2}}{3-\sqrt{2}}$

(B) $\frac{3}{\sqrt{2}}\left(\frac{3+\sqrt{2}}{3-\sqrt{2}}\right)$

(C) $\frac{1}{\sqrt{2}}\left(\frac{\sqrt{2}+1}{\sqrt{2}-1}\right)$

(D) $\frac{\sqrt{2}+1}{\sqrt{2}-1}$

Official Ans. by NTA (B)

Ans. (B)

Sol. $\frac{dy}{dx}=\frac{2e^{2x}-6e^{-x}+9}{2+9e^{-2x}}$

$$\frac{dy}{dx}=e^{2x}-\frac{6e^{x}}{2e^{2x}+9}$$

$$y=\frac{e^{2x}}{2}-\tan^{-1}\left(\frac{\sqrt{2}e^{x}}{3}\right)+c$$

If C passes through the point $\left(0,\frac{1}{2}+\frac{\pi}{2\sqrt{2}}\right)$

$$c=-\frac{\pi}{4}-\tan^{-1}\frac{\sqrt{2}}{3}$$

Again C passes through the point $\left(\alpha,\frac{1}{2}e^{2\alpha}\right)$

then $e^{\alpha}=\frac{3}{\sqrt{2}}\left(\frac{3+\sqrt{2}}{3-\sqrt{2}}\right)$

11. The general solution of the differential equation $(x-y^2)dx+y(5x+y^2)dy=0$ is :

(A) $(y^2+x)^4=C|(y^2+2x)^3|$

(B) $(y^2+2x)^4=C|(y^2+x)^3|$

(C) $|(y^2+x)^3|=C(2y^2+x)^4$

(D) $|(y^2+2x)^3|=C(2y^2+x)^4$

Official Ans. by NTA (A)

Ans. (A)

Sol. $(x-y^2)dx+y(5x+y^2)dy=0$

$$\frac{dy}{dx}=\frac{y^2-x}{y(5x+y^2)}.\ \text{Let } y^2=v$$

$$\frac{2ydy}{dx}=2\left(\frac{y^2-x}{5x+y^2}\right)$$

$$\frac{dv}{dx}=2\left(\frac{v-x}{5x+v}\right) \qquad v=kx$$

$$k+x\frac{dk}{dx}=2\left(\frac{kx-x}{5x+kx}\right)$$

$$x\frac{dk}{dx}=-\frac{(k^2+3k+2)}{k+5}$$

$$\int\frac{(5+k)}{(k+1)(k+2)}dk=\int-\frac{dx}{x}$$

$$\int\left(\frac{4}{k+1}-\frac{3}{k+2}\right)dk=-\int\frac{dx}{x}$$

$4\ln(k+1)-3\ln(k+2)=-\ln x+\ln c$

$$\frac{(k+1)^4}{(k+2)^3}=-\ln x+\ln c$$

$$c(y^2+2x)^3=(y^2+x)^4$$

12. A line, with the slope greater than one, passes through the point A(4, 3) and intersects the line x – y – 2 = 0 at the point B. If the length of the line segment AB is $\frac{\sqrt{29}}{3}$, then B also lies on the line :

(A) $2x + y = 9$ (B) $3x - 2y = 7$
(C) $x + 2y = 6$ (D) $2x - 3y = 3$

Official Ans. by NTA (C)

Ans. (C)

Sol. Let $B(x_1, x_1 - 2)$

$$\sqrt{(x_1-4)^2+(x_1-2-3)^2} = \frac{\sqrt{29}}{3}$$

Squaring on both side

$18x_1^{\,2} - 162x_1 + 340 = 0$

$x_1 = \frac{51}{9}$ or $x_1 = \frac{10}{3}$

$y_1 = \frac{33}{9}$ or $y_1 = \frac{4}{3}$

Option (C) will satisfy $\left(\frac{10}{3}, \frac{4}{3}\right)$

13. Let the locus of the centre (α, β), β > 0, of the circle which touches the circle $x^2 + (y - 1)^2 = 1$ externally and also touches the x-axis be L. Then the area bounded by L and the line y = 4 is :

(A) $\frac{32\sqrt{2}}{3}$ (B) $\frac{40\sqrt{2}}{3}$ (C) $\frac{64}{3}$ (D) $\frac{32}{3}$

Official Ans. by NTA (C)

Ans. (C)

Sol. $(\alpha - 0)^2 + (\beta - 1)^2 = (\beta + 1)^2$

$\alpha^2 = 4\beta$

$x^2 = 4y$

$$A = 2\int_0^4 \left(4 - \frac{x^2}{4}\right) dx = \frac{64}{3}$$

14. Let P be the plane containing the straight line $\frac{x-3}{9} = \frac{y+4}{-1} = \frac{z-7}{-5}$ and perpendicular to the plane containing the straight lines $\frac{x}{2} = \frac{y}{3} = \frac{z}{5}$ and $\frac{x}{3} = \frac{y}{7} = \frac{z}{8}$. If d is the distance of P from the point (2, –5, 11), then d^2 is equal to :

(A) $\frac{147}{2}$ (B) 96 (C) $\frac{32}{3}$ (D) 54

Official Ans. by NTA (D)

Ans. (C)

Sol. $a(x - 3) + b(y + 4) + c(z - 7) = 0$

P : $9a - b - 5c = 0$

$-11a - b + 5c = 0$

After solving DR's ∝ (1, –1, 2)

Equation of plane

$x - y + 2z = 21$

$d = \frac{8}{\sqrt{6}}$

$d^2 = \frac{32}{3}$

15. Let ABC be a triangle such that $\overrightarrow{BC} = \vec{a}$, $\overrightarrow{CA} = \vec{b}$, $\overrightarrow{AB} = \vec{c}$, $|\vec{a}| = 6\sqrt{2}$, $|\vec{b}| = 2\sqrt{3}$ and $\vec{b}\cdot\vec{c} = 12$

Consider the statements :

(S1) : $\left|\left(\vec{a}\times\vec{b}\right)+\left(\vec{c}\times\vec{b}\right)\right| - |\vec{c}| = 6(2\sqrt{2}-1)$

(S2) : $\angle ABC = \cos^{-1}\left(\sqrt{\frac{2}{3}}\right)$. Then

(A) both (S1) and (S2) are true
(B) only (S1) is true
(C) only (S2) is true
(D) both (S1) and (S2) are false

Official Ans. by NTA (D)

Ans. (C)

Sol. $\vec{a} + \vec{b} + \vec{c} = 0$

$\vec{b} + \vec{c} = -\vec{a}$

$|\vec{b}|^2 + |\vec{c}|^2 + 2\vec{b}\cdot\vec{c} = |\vec{a}|^2$

$|\vec{c}|^2 = 36$

$|\vec{c}| = 6$

S1 : $|\vec{a}\times\vec{b}+\vec{c}\times\vec{b}| - |\vec{c}|$

$|(\vec{a}+\vec{c})\times\vec{b}| - |\vec{c}|$

$|-\vec{b}\times\vec{b}| - |\vec{c}|$

$0 - 6 = -6$

S2 : $\vec{a} + \vec{b} + \vec{c} = 0$

$\vec{b} + \vec{c} = -\vec{a}$

$|\vec{a}|^2 + |\vec{b}|^2 - 2|\vec{a}||\vec{b}|\cos(\angle ACB) = |\vec{c}|^2$

$\cos(\angle ACB) = \sqrt{\frac{2}{3}}$

16. If the sum and the product of mean and variance of a binomial distribution are 24 and 128 respectively, then the probability of one or two successes is :

(A) $\frac{33}{2^{32}}$ (B) $\frac{33}{2^{29}}$ (C) $\frac{33}{2^{28}}$ (D) $\frac{33}{2^{27}}$

Official Ans. by NTA (C)

Ans. (C)

Sol. $np + npq = 24$ ...(1)

$np \cdot npq = 128$ ...(2)

Solving (1) and (2) :

We get $p = \frac{1}{2}, q = \frac{1}{2}, n = 32.$

Now,

$P(X = 1) + P(X = 2)$

$= {}^{32}C_1\, pq^{31} + {}^{32}C_2\, p^2 q^{30}$

$= \frac{33}{2^{28}}$

17. If the numbers appeared on the two throws of a fair six faced die are α and β, then the probability that $x^2 + \alpha x + \beta > 0$, for all $x \in R$, is :

(A) $\frac{17}{36}$ (B) $\frac{4}{9}$ (C) $\frac{1}{2}$ (D) $\frac{19}{36}$

Official Ans. by NTA (A)

Ans. (A)

Sol. $x^2 + \alpha x + \beta > 0, \forall x \in R$

$D = \alpha^2 - 4\beta < 0$

$\alpha^2 < 4\beta$

Total cases = $6 \times 6 = 36$

Fav. cases = $\beta = 1, \alpha = 1$

$\beta = 2, \alpha = 1, 2$

$\beta = 3, \alpha = 1, 2, 3$

$\beta = 4, \alpha = 1, 2, 3$

$\beta = 5, \alpha = 1, 2, 3, 4$

$\beta = 6, \alpha = 1, 2, 3, 4$

Total favourable cases = 17

$P(x) = \frac{17}{36}$

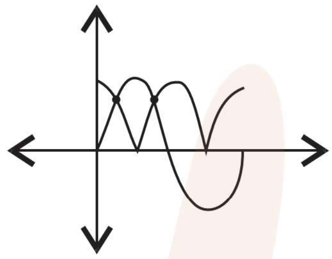
18. The number of solutions of $|\cos x| = \sin x$, such that $-4\pi \le x \le 4\pi$ is :

(A) 4 (B) 6 (C) 8 (D) 12

Official Ans. by NTA (C)

Ans. (C)

Sol.



2 solutions in $(0, 2\pi)$

So, 8 solutions in $[-4\pi, 4\pi]$

19. A tower PQ stands on a horizontal ground with base Q on the ground. The point R divides the tower in two parts such that QR = 15 m. If from a point A on the ground the angle of elevation of R is 60° and the part PR of the tower subtends an angle of 15° at A, then the height of the tower is :

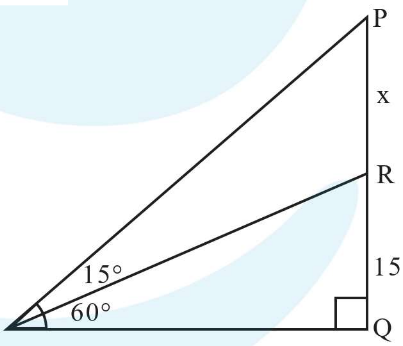
(A) $5(2\sqrt{3}+3)$ m (B) $5(\sqrt{3}+3)$ m

(C) $10(\sqrt{3}+1)$ m (D) $10(2\sqrt{3}+1)$ m

Official Ans. by NTA (A)

Ans. (A)

Sol.



$\frac{15}{AQ} = \tan 60°$ ...(1)

$\frac{15+x}{AQ} = \tan 75°$ ...(2)

$\frac{(1)}{(2)} \Rightarrow x = 10\sqrt{3}$

So, $PQ = 5(2\sqrt{3}+3)$ m

20. Which of the following statements is a tautology ?

(A) $((\sim p) \vee q) \Rightarrow p$ (B) $p \Rightarrow ((\sim p) \vee q)$

(C) $((\sim p) \vee q) \Rightarrow q$ (D) $q \Rightarrow ((\sim p) \vee q)$

Official Ans. by NTA (D)

Ans. (D)

Sol.

p	q	~p	~q	~pvq
T	T	F	F	T
T	F	F	T	F
F	T	T	F	T
F	F	T	T	T

options	1	2	3	4
	T	T	T	T
	T	F	T	T
	F	T	T	T
	F	T	F	T

SECTION-B

1. Let $A = \begin{bmatrix} 2 & -1 & -1 \\ 1 & 0 & -1 \\ 1 & -1 & 0 \end{bmatrix}$ and $B = A - I$. If $\omega = \frac{\sqrt{3}i - 1}{2}$, then the number of elements in the set $\{n \in \{1, 2, \ldots, 100\} : A^n + (\omega B)^n = A + B\}$ is equal to ______.

Official Ans. by NTA (17)

Ans. (17)

Sol. $A = \begin{bmatrix} 2 & -1 & -1 \\ 1 & 0 & -1 \\ 1 & -1 & 0 \end{bmatrix} \Rightarrow A^2 = A \Rightarrow A^n = A.$

$\forall\, n \in \{1, 2, \ldots, 100\}$

Now, $B = A - I = \begin{bmatrix} 1 & -1 & -1 \\ 1 & -1 & -1 \\ 1 & -1 & -1 \end{bmatrix}$

$B^2 = -B$

$\Rightarrow B^3 = -B^2 = B$

$\Rightarrow B^5 = B$

$\Rightarrow B^{99} = B$

Also, $\omega^{3k} = 1$

So, n = common of $\{1, 3, 5, \ldots, 99\}$ and $\{3, 6, 9, \ldots, 99\} = 17$

2. The letters of the word 'MANKIND' are written in all possible orders and arranged in serial order as in an English dictionary. Then the serial number of the word 'MANKIND' is ______.

Official Ans. by NTA (1492)

Ans. (1492)

Sol.

M	A	N	K	I	N	D

$\left(\frac{4 \times 6!}{2!}\right) + (5! \times 0) + \left(\frac{4! \times 3}{2!}\right) + (3! \times 2) + (2! \times 1) + (1! \times 1) + (0! \times 0) + 1 = 1492$

3. If the maximum value of the term independent of t in the expansion of $\left(t^2 x^{\frac{1}{5}} + \frac{(1-x)^{\frac{1}{10}}}{t}\right)^{15}$, $x \geq 0$, is K, then 8K is equal to ______.

Official Ans. by NTA (6006)

Ans. (6006)

Sol. $\left(t^2 x^{\frac{1}{5}} + \frac{(1-x)^{\frac{1}{10}}}{t}\right)^{15}$

$T_{r+1} = {}^{15}C_r \left(t^2 x^{\frac{1}{5}}\right)^{15-r} \cdot \frac{(1-x)^{\frac{r}{10}}}{t^r}$

For independent of t,

$30 - 2r - r = 0$

$\Rightarrow r = 10$

So, Maximum value of ${}^{15}C_{10}\, x(1 - x)$ will be at $x = \frac{1}{2}$

i.e. 6006

4. Let a, b be two non-zero real numbers. If p and r are the roots of the equation $x^2 - 8ax + 2a = 0$ and q and s are the roots of the equation $x^2 + 12bx + 6b = 0$, such that $\frac{1}{p}, \frac{1}{q}, \frac{1}{r}, \frac{1}{s}$ are in A.P., then $a^{-1} - b^{-1}$ is equal to ______.

Official Ans. by NTA (38)

Ans. (38)

Sol. $x^2 - 8ax + 2a = 0$ $\quad$ $x^2 + 12bx + 6b = 0$

$p + r = 8a$ $\quad$ $q + s = -12b$

$pr = 2a$ $\quad$ $qs = 6b$

$\frac{1}{p}+\frac{1}{r} = 4$ $\quad$ $\frac{1}{q}+\frac{1}{s} = -2$

$\frac{2}{q} = 4$ $\quad$ $\frac{2}{r} = -2$

$q = \frac{1}{2}$ $\quad$ $r = -1$

$p = \frac{1}{5}$ $\quad$ $s = \frac{-1}{4}$

Now, $\frac{1}{a}-\frac{1}{b}=\frac{2}{pr}-\frac{6}{qs} = 38$

5. Let $a_1 = b_1 = 1$, $a_n = a_{n-1} + 2$ and $b_n = a_n + b_{n-1}$ for every natural number $n \geq 2$. Then $\sum_{n=1}^{15} a_n \cdot b_n$ is equal to ________.

Official Ans. by NTA (27560)

Ans. (27560)

Sol. $a_1 = b_1 = 1$

$a_2 = a_1 + 2 = 3$

$a_3 = a_2 + 2 = 5$

$a_4 = a_2 + 2 = 7$

$\Rightarrow a_n = 2n - 1$

$b_2 = a_1 + b_1 = 4$

$b_3 = a_3 + b_2 = 9$

$b_4 = a_4 + b_3 = 16$

$b_n = n^2$

$\sum_{n=1}^{15} a_n b_n$

$\sum_{n=1}^{15} (2n-1)n^2$

$\sum_{n=1}^{15} (2n^3 - n^2)$

$= 2\frac{n^2(n+1)^2}{4} - \frac{n(n+1)(2n+1)}{6}$

Put n = 15

$= \frac{2\times225\times16\times16}{4} - \frac{15\times16\times31}{6}$

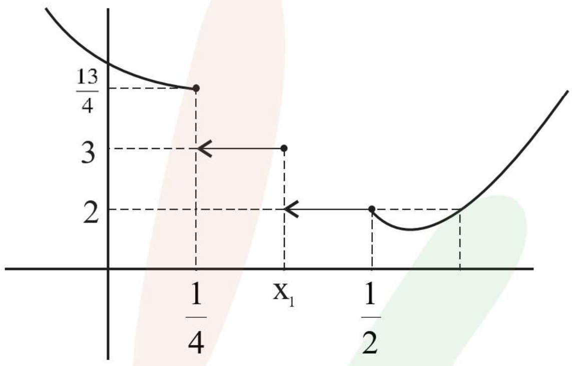
$= 27560$

6. Let $f(x) = \begin{cases} |4x^2-8x+5|, & \text{if } 8x^2-6x+1 \geq 0 \\ [4x^2-8x+5], & \text{if } 8x^2-6x+1 < 0 \end{cases}$, where $[\alpha]$ denotes the greatest integer less than or equal to α. Then the number of points in R where f is not differentiable is ________.

Official Ans. by NTA (3)

Ans. (3)

Sol.



ND at $\frac{1}{4}, x_1, \frac{1}{2}$

7. If $\lim_{n\to\infty} \frac{(n+1)^{k-1}}{n^{k+1}}$ [(nk + 1) + (nk + 2) + … + (nk + n)] = 33. $\lim_{n\to\infty} \frac{1}{n^{k+1}} \cdot [1^k + 2^k + 3^k + ... + n^k]$, then the integral value of k is equal to ________.

Official Ans. by NTA (5)

Ans. (5)

Sol. LHS

$\lim_{n\to\infty} \frac{(n+1)^{k-1}}{n^{k+1}}[nk \cdot n + 1 + 2 + ... + n]$

$= \lim_{n\to\infty} \frac{(n+1)^{k-1}}{n^{k+1}} \cdot \left[n^2k + \frac{n(n+1)}{2}\right]$

$= \lim_{n\to\infty} \frac{(n+1)^{k-1} \cdot n^2 \left(k + \frac{\left(1+\frac{1}{n}\right)}{2}\right)}{n^{k+1}}$

$\Rightarrow \lim_{n\to\infty} \left(1+\frac{1}{n}\right)\left(k + \frac{\left(1+\frac{1}{n}\right)}{2}\right)$

$\Rightarrow \left(k + \frac{1}{2}\right)$

RHS

$\Rightarrow \lim_{n\to\infty} \frac{1}{n^{k+1}}(1^k + 2^k + ... + n^k) = \frac{1}{k+1}$

LHS = RHS

$\Rightarrow k + \frac{1}{2} = 33 \cdot \frac{1}{k+1}$

$\Rightarrow (2k + 1)(k + 1) = 66$

$\Rightarrow (k – 5)(2k + 13) = 0$

$\Rightarrow k = 5 \text{ or } -\frac{13}{2}$

8. Let the equation of two diameters of a circle $x^2 + y^2 – 2x + 2fy + 1 = 0$ be $2px – y = 1$ and $2x + py = 4p$. Then the slope $m \in (0, \infty)$ of the tangent to the hyperbola $3x^2 – y^2 = 3$ passing through the centre of the circle is equal to _______.

Official Ans. by NTA (2)

Ans. (2)

Sol. $2p + f – 1 = 0$ …(1)

$2 – pf – 4p = 0$ …(2)

$2 = p(f + 4)$

$p = \frac{2}{f+4}$

$2p = 1 – f$

$\frac{4}{f+4} = 1 - f$

$f^2 + 3f = 0$

$f = 0$ or $–3$

Hyperbola $3x^2 – y^2 = 3$, $x^2 - \frac{y^2}{3} = 1$

$y = mx \pm\sqrt{m^2 - 3}$

It passes (1, 0)

$o = m \pm\sqrt{m^2 - 3}$

m tends ∞

It passes (1, 3)

$3 = m \pm\sqrt{m^2 - 3}$

$(3 – m)^2 = m^2 – 3$

$m = 2$

9. The sum of diameters of the circles that touch (i) the parabola $75x^2 = 64(5y – 3)$ at the point $\left(\frac{8}{5}, \frac{6}{5}\right)$ and (ii) the y-axis, is equal to _______.

Official Ans. by NTA (10)

Ans. (10)

Sol. $x^2 = \frac{64.5}{75}\left(y - \frac{3}{5}\right)$

equation of tangent at $\left(\frac{8}{5}, \frac{6}{5}\right)$

$x \cdot \frac{8}{5} = \frac{64}{15}\left(\frac{y + \frac{6}{5}}{2} - \frac{3}{5}\right)$

$3x – 4y = 0$

equation of family of circle is

$\left(x - \frac{8}{5}\right)^2 + \left(y - \frac{6}{5}\right)^2 + \lambda(3x - 4y) = 0$

It touches y axis so $f^2 = c$

$x^2 + y^2 + x\left(3\lambda - \frac{16}{5}\right) + y\left(-4\lambda - \frac{12}{5}\right) + 4 = 0$

$\frac{\left(4\lambda + \frac{12}{5}\right)^2}{4} = 4$

$\lambda = \frac{2}{5}$ or $\lambda = -\frac{8}{5}$

$\lambda = \frac{2}{5}$, $r = 1$

$\lambda = -\frac{8}{5}$, $r = 4$

$d_1 + d_2 = 10$

10. The line of shortest distance between the lines $\frac{x-2}{0} = \frac{y-1}{1} = \frac{z}{1}$ and $\frac{x-3}{2} = \frac{y-5}{2} = \frac{z-1}{1}$ makes an angle of $\cos^{-1}\left(\sqrt{\frac{2}{27}}\right)$ with the plane P : $ax – y – z = 0$, $(a > 0)$. If the image of the point (1, 1, –5) in the plane P is (α, β, γ), then $\alpha + \beta – \gamma$ is equal to _______.

Official Ans. by NTA (3)

Ans. (Bonus)

Sol. DR's of line of shortest distance

$\begin{vmatrix} \hat{i} & \hat{j} & \hat{k} \\ 0 & 1 & 1 \\ 2 & 2 & 1 \end{vmatrix} = -\hat{i} + 2\hat{j} - 2\hat{k}$

angle between line and plane is $\cos^{-1}\sqrt{\frac{2}{27}} = \alpha$

$\cos\alpha = \sqrt{\frac{2}{27}}$, $\sin\alpha = \frac{5}{3\sqrt{3}}$

DR's normal to plane (1, –1, –1)

$\sin\alpha = \left|\frac{-a - 2 + 2}{\sqrt{4+4+1}\sqrt{a^2+1+1}}\right| = \frac{5}{3\sqrt{3}}$

$\sqrt{3}\,|a| = 5\sqrt{a^2 + 2}$

$3a^2 = 25a^2 + 50$

No value of (a)