FINAL JEE–MAIN EXAMINATION – JUNE, 2022

(Held On Monday 27ᵗʰ June, 2022) **TIME : 3 : 00 PM to 6 : 00 PM**

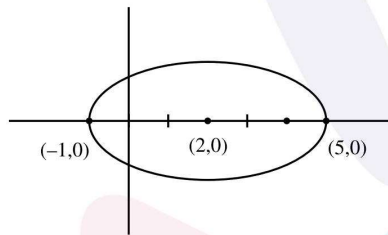| MATHEMATICS	TEST PAPER WITH SOLUTION

SECTION-A

1. The number of points of intersection of $|z-(4+3i)|=2$ and $|z|+|z-4|=6$, $z \in C$ is :

(A) 0 (B) 1

(C) 2 (D) 3

Official Ans. by NTA (C)

Ans. (C)

Sol. $C : (x-4)^2 + (y-3)^2 = 4$

$E : \dfrac{(x-2)^2}{9} + \dfrac{y^2}{5} = 1$

(–1,0) (2,0) (5,0)

Lower Extremity of vertical diameter of

circle $\to$ (4,1)

Put in ellipse $\Rightarrow \dfrac{(4-2)^2}{9} + \dfrac{1}{5} - 1$

$= \dfrac{4}{9} + \dfrac{1}{5} - 1$

$= \dfrac{29}{45} - 1 < 0$

Two Solutions

Answer (C)

2. Let $f(x) = \begin{vmatrix} a & -1 & 0 \\ ax & a & -1 \\ ax^2 & ax & a \end{vmatrix}$, $a \in R$. Then the sum of

which the squares of all the values of a for

$2f'(10) - f'(5) + 100 = 0$ is :

(A) 117 (B) 106

(C) 125 (D) 136

Official Ans. by NTA (C)

Ans. (C)

Sol. $f(x) = \begin{vmatrix} a & -1 & 0 \\ ax & a & -1 \\ ax^2 & ax & a \end{vmatrix}$

$f(x) = a \begin{vmatrix} 1 & -1 & 0 \\ x & a & -1 \\ x^2 & ax & a \end{vmatrix}$

$= a [1 (a^2 + ax) + 1 (ax + x^2)]$

$\Rightarrow f(x) = a (x + a)^2$

so, $f'(x) = 2a (x + a)$

as, $2f'(10) - f'(5) + 100 = 0$

$\Rightarrow 2 \times 2a (10 + a) - 2a (5 + a) + 100 = 0$

$\Rightarrow 40a + 4a^2 - 10a - 2a^2 + 100 = 0$

$2a^2 + 30a + 100 = 0$

$\Rightarrow a^2 + 15a + 50 = 0$

$(a + 10) (a + 5) = 0$

$a = -10$ or $a = -5$

Required $= (-10)^2 + (-5)^2 = 125$

3. Let for some real numbers α and β, $a = \alpha - i\beta$. If the system of equations $4ix + (1+i)y = 0$ and $8\left(\cos\dfrac{2\pi}{3} + i\sin\dfrac{2\pi}{3}\right)x + \bar{a}y = 0$ has more than one solution then $\dfrac{\alpha}{\beta}$ is equal to :

(A) $-2+\sqrt{3}$ (B) $2-\sqrt{3}$

(C) $2+\sqrt{3}$ (D) $-2-\sqrt{3}$

Official Ans. by NTA (B)

Ans. (B)

Sol. $a = \alpha - i\beta$; $\alpha \in R$; $\beta \in R$

$4ix + (1 + i) y = 0$ and

$8\left(\cos\dfrac{2\pi}{3} + i\sin\dfrac{2\pi}{3}\right)x + \overline{a}y = 0$

$\begin{vmatrix} 4i & 1+i \\ 8e^{i2\pi/3} & \overline{a} \end{vmatrix} = 0$

$\Rightarrow 4i\,\overline{a} - (1 + i)\,8e^{i2\pi/3} = 0$

$\Rightarrow 4i\,(\alpha + i\beta) - 8(1 + i)\left(\dfrac{-1 + i\sqrt{3}}{2}\right) = 0$

$\Rightarrow i\alpha - \beta + 1 + \sqrt{3} + i(1 - \sqrt{3}) = 0$

$\Rightarrow \beta = \sqrt{3} + 1$

$\quad \alpha = \sqrt{3} - 1$

So, $\dfrac{\alpha}{\beta} = \dfrac{\sqrt{3}-1}{\sqrt{3}+1} = 2 - \sqrt{3}$

4. Let A and B be two 3×3 matrices such that $AB = I$ and $|A| = \dfrac{1}{8}$ then $\left|\text{adj}(B\,\text{adj}(2A))\right|$ is equal to

(A) 16   (B) 32

(C) 64   (D) 128

Official Ans. by NTA (C)

    Ans. (C)

Sol. $AB = i$

$|\text{adj}(B\,\text{adj}(2A))| = |B\,\text{adj}(2A)|^2$

$\qquad = |B|^2 |\text{adj}(2A)|^2$

$= |B|^2 (|2A|^2)^2 = |B|^2 (2^6 |A|^2)^2$

$|A| = \dfrac{1}{8}$ and $|AB| = 1 \Rightarrow |A||B| = 1$

$\Rightarrow \dfrac{1}{8}|B| = 1$

$\Rightarrow |B| = 8$

required value $= 64$

5. Let $S = 2 + \dfrac{6}{7} + \dfrac{12}{7^2} + \dfrac{20}{7^3} + \dfrac{30}{7^4} + \dots$ then $4S$ is equal to

(A) $\left(\dfrac{7}{3}\right)^2$ (B) $\dfrac{7^3}{3^2}$

(C) $\left(\dfrac{7}{3}\right)^3$ (D) $\dfrac{7^2}{3^3}$

Official Ans. by NTA (C)

    Ans. (C)

Sol. $S = 2 + \dfrac{6}{7} + \dfrac{12}{7^2} + \dfrac{20}{7^3} + \dfrac{30}{7^4} + \dots$

Considering infinite sequence,

$S = 2 + \dfrac{6}{7} + \dfrac{12}{7^2} + \dfrac{20}{7^3} + \dfrac{30}{7^4} + \dots$

$\dfrac{S}{7} = \dfrac{2}{7} + \dfrac{6}{7^2} + \dfrac{12}{7^3} + \dfrac{20}{7^4} + \dots$

$\Rightarrow \dfrac{6S}{7} = 2 + \dfrac{4}{7} + \dfrac{6}{7^2} + \dfrac{8}{7^3} + \dfrac{10}{7^4} + \dots$

$\Rightarrow \dfrac{6S}{7^2} = \dfrac{2}{7} + \dfrac{4}{7^2} + \dfrac{6}{7^3} + \dfrac{8}{7^4} + \dots$

$\dfrac{6S}{7}\left(1 - \dfrac{1}{7}\right) = 2 + \dfrac{2}{7} + \dfrac{2}{7^2} + \dfrac{2}{7^3} + \dots$

$\Rightarrow \dfrac{6^2 S}{7^2} = \dfrac{2}{1 - \dfrac{1}{7}} = \dfrac{2}{6} \times 7$

$\Rightarrow S = \dfrac{2 \times 7^3}{6^3} \Rightarrow 4S = \dfrac{7^3}{3^3} = \left(\dfrac{7}{3}\right)^3$

6. If $a_1, a_2, a_3 \dots$ and $b_1, b_2, b_3 \dots$ are A.P. and $a_1 = 2$, $a_{10} = 3$, $a_1 b_1 = 1 = a_{10} b_{10}$ then $a_4 b_4$ is equal to

(A) $\dfrac{35}{27}$ (B) 1

(C) $\dfrac{27}{28}$ (D) $\dfrac{28}{27}$

Official Ans. by NTA (D)

    Ans. (D)

Sol. $a_1, a_2, a_3 \dots$ A.P. ; $a_1 = 2$; $a_{10} = 3$; $d_1 = \dfrac{1}{9}$

$b_1, b_2, b_3, \dots$ A.P. ; $b_1 = \dfrac{1}{2}$; $b_{10} = \dfrac{1}{3}$; $d_2 = \dfrac{-1}{54}$

[Using $a_1 b_1 = 1 = a_{10} b_{10}$; d_1 & d_2 are common differences respectively]

$a_4 \cdot b_4 = (2 + 3d_1)\left(\dfrac{1}{2} + 3d_2\right)$

$= \left(2 + \dfrac{1}{3}\right)\left(\dfrac{1}{2} - \dfrac{1}{18}\right)$

$= \left(\dfrac{7}{3}\right)\left(\dfrac{8}{18}\right) = \dfrac{28}{27}$

7. If m and n respectively are the number of local maximum and local minimum points of the function $f(x) = \displaystyle\int_0^{x^2} \dfrac{t^2 - 5t + 4}{2 + e^t}\,dt$, then the ordered pair (m, n) is equal to

(A) (3, 2)   (B) (2, 3)

(C) (2, 2)   (D) (3, 4)
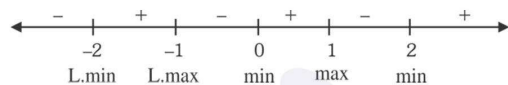
Official Ans. by NTA (B)

Ans. (B)

Sol. $m = L \cdot max$

$N = L \cdot min$

$f(x) = \int_0^{x^2} \frac{t^2 - 5t + 4}{2 + e^t} dt$

$f'(x) = \frac{(x^4 - 5x^2 + 4)2x}{2 + e^{x^2}} = \frac{2x(x^2 - 1)(x^2 - 4)}{2 + e^{x^2}}$

$= \frac{2x(x-1)(x+1)(x-2)(x+2)}{2 + e^{x^2}}$

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      −        +       −      +      −          +
  ──┼────────┼───────┼──────┼──────┼──────────►
   −2       −1       0      1      2
  L.min   L.max    min    max    min
```

so, $m = 2$ and $n = 3$

8. Let f be a differentiable function in $\left(0, \frac{\pi}{2}\right)$.

If $\int_{\cos x}^{1} t^2 f(t) dt = \sin^3 x + \cos x$ then $\frac{1}{\sqrt{3}} f'\left(\frac{1}{\sqrt{3}}\right)$ is equal to :

(A) $6 - 9\sqrt{2}$ (B) $6 - \frac{9}{\sqrt{2}}$

(C) $\frac{9}{2} - 6\sqrt{2}$ (D) $\frac{9}{\sqrt{2}} - 6$

Official Ans. by NTA (B)

Ans. (Bonus)

Sol. At right hand vicinity of x = 0 given equation does not satisfy

$\because$ LHS $= \int_{1^-}^{1} t^2 f(t) dt = 0$, RHS $= \lim_{x \to 0^+} (\sin^3 x + \cos x) = 1$

LHS ≠ RHS hence data given in question is wrong hence BONUS

Correct data should have been

$\int_{\cos x}^{1} t^2 f(t) dt = \sin^3 x + \cos x - 1$

Calculation for option

differentiating both sides

$-\cos^2 x \, f(\cos x).(-\sin x) = 3\sin^2 x.\cos x - \sin x$

$\Rightarrow f(\cos x) = 3 \tan x - \sec^2 x$

$\Rightarrow f'(\cos x)(-\sin x) = 3\sec^2 x - 2\sec^2 x \tan x$

$\Rightarrow f'(\cos x)\cos x = \frac{2}{\cos^2 x} - \frac{3}{\sin x. \cos x}$

When $\cos x = \frac{1}{\sqrt{3}}$; $\sin x = \frac{\sqrt{2}}{\sqrt{3}}$

$\therefore f'\left(\frac{1}{\sqrt{3}}\right)\frac{1}{\sqrt{3}} = 6 - \frac{9}{\sqrt{2}}$.

9. The integral $\int_0^1 \frac{1}{7^{\left[\frac{1}{x}\right]}} dx$, where [.] denotes the greatest integer function is equal to

(A) $1 + 6\log_e\left(\frac{6}{7}\right)$ (B) $1 - 6\log_e\left(\frac{6}{7}\right)$

(C) $\log_e\left(\frac{7}{6}\right)$ (D) $1 - 7\log_e\left(\frac{6}{7}\right)$

Official Ans. by NTA (A)

Ans. (A)

Sol. $\int_0^1 \frac{1}{7^{\left[\frac{1}{x}\right]}} dx = -\int_1^0 \frac{1}{7^{\left[\frac{1}{x}\right]}} dx$

$= (-1)\left[\int_1^{1/2} \frac{1}{7} dx + \int_{1/2}^{1/3} \frac{1}{7^2} dx + \int_{1/3}^{1/4} \frac{1}{7^3} dx + ...\infty\right]$

$= \left(\frac{1}{7} + \frac{1}{2.7^2} + \frac{1}{3.7^3} + ...\infty\right) - \left(\frac{1}{7.2} + \frac{1}{7^2.3} + \frac{1}{7^2.4} ...\infty\right)$

$= -\ln\left(1 - \frac{1}{7}\right) - 7\left(\frac{1}{7^2.2} + \frac{1}{7^3.3} + \frac{1}{7^4.4} + ...\infty\right)$

$\left[\text{as } \ln(1+x) = x - \frac{x^2}{2} + \frac{x^3}{3} - \frac{x^4}{4}....\infty\right]$

$\left[\text{as } \ln(1-x) = -\left(x + \frac{x^2}{2} + \frac{x^3}{3} + \frac{x^4}{4}....\infty\right)\right]$

$= -\ln\frac{6}{7} - 7\left(-\ln\left(1 - \frac{1}{7}\right) - \frac{1}{7}\right)$

$= 6\ln\frac{6}{7} + 1$

10. If the solution curve of the differential equation $\left((\tan^{-1} y) - x\right)dy = (1 + y^2)dx$ passes through the point (1, 0) then the abscissa of the point on the curve whose ordinate is tan (1) is :

(A) $2e$ (B) $\frac{2}{e}$

(C) 2 (D) $\frac{1}{e}$

Official Ans. by NTA (B)

Ans. (B)

Sol. $\frac{dx}{dy} + \frac{x}{1+y^2} = \frac{\tan^{-1} y}{1+y^2}$

I.f $= e^{\int \frac{1}{1+y^2}dy} = e^{\tan^{-1} y}$

$xe^{\tan^{-1} y} = \int \frac{\tan^{-1} y}{1+y^2}e^{\tan^{-1} y}dy$

$x \cdot e^{\tan^{-1} y} = \left(\tan^{-1} y - 1 \right) e^{\tan^{-1} y} + c$

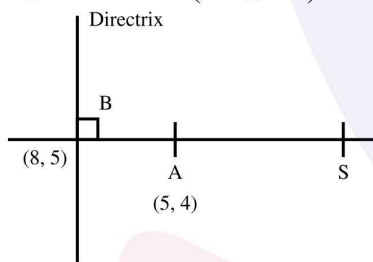$\therefore$ (1, 0) lies exit c = 2.

For y = tan 1 $\Rightarrow$ x = $\dfrac{2}{e}$

11. If the equation of the parabola, whose vertex is at (5, 4) and the directrix is $3x + y - 29 = 0$, is

$x^2 + ay^2 + bxy + cx + dy + k = 0$ then

a + b + c + d + k is equal to

(A) 575   (B) –575

(C) 576   (D) –576

Official Ans. by NTA (D)

       Ans. (D)

Sol. Vertex (5,4)

Directrix : 3x + y – 29 = 0

Co-ordinates of B (foot of directrix)

$\dfrac{x - 5}{3} = \dfrac{y - 4}{1} = -\left(\dfrac{15 + 4 - 29}{10} \right) = 1$

x = 8, y = 5

S = (2, 3) (focus)

Equation of parabola

PS = PM

so equation is

$x^2 + 9y^2 - 6xy + 134x - 2y - 711 = 0$

a + b + c + d + k = 9 – 6 + 134 – 2 – 711 = – 576

12. The set of values of k for which the circle

C : $4x^2 + 4y^2 - 12x + 8y + k = 0$ lies inside the

fourth quadrant and the point $\left(1, -\dfrac{1}{3} \right)$ lies on or

inside the circle C is :

(A) An empty set   (B) $\left(6, \dfrac{95}{9} \right]$

(C) $\left[\dfrac{80}{9}, 10 \right)$ (D) $\left(9, \dfrac{92}{9} \right]$
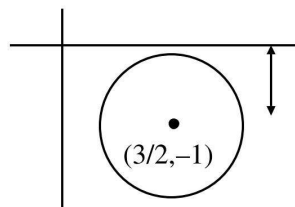
Official Ans. by NTA (D)

       Ans. (D)

Sol. C : $4x^2 + 4y^2 - 12x + 8y + k = 0$

$\Rightarrow x^2 + y^2 - 3x + 2y + \left(\dfrac{k}{4} \right) = 0$

Centre $\left(\dfrac{3}{2}, -1 \right)$; r = $\sqrt{\dfrac{13 - k}{2}}$ $\Rightarrow k \leq 13$ ... (1)

(i) Point $\left(1, \dfrac{-1}{3} \right)$ lies on or inside circle C

$\Rightarrow S_1 \leq 0 \Rightarrow k \leq \dfrac{92}{9}$ ... (2)

(ii) C lies in 4^{th} quadrant

r < 1

$\Rightarrow \dfrac{\sqrt{13 - k}}{2} < 1$

$\Rightarrow k < 9$ .... (3)

Hence (1) $\cap$ (2) $\cap$ (3) $\Rightarrow k \in \left(9, \dfrac{92}{9} \right]$

13. Let the foot of the perpendicular from the point (1, 2, 4) on the line $\dfrac{x + 2}{4} = \dfrac{y - 1}{2} = \dfrac{z + 1}{3}$ be P. Then the distance of P from the plane $3x + 4y + 12z + 23 = 0$
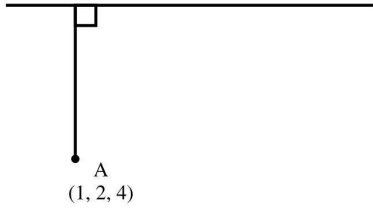
(A) 5   (B) $\dfrac{50}{13}$

(C) 4   (D) $\dfrac{63}{13}$

Official Ans. by NTA (A)

       Ans. (A)

Sol.

P $(4\lambda - 2, 2\lambda + 1, 3\lambda + 1)$ 4, 2, 3

A
(1, 2, 4)

$\dfrac{x+2}{4} = \dfrac{y-1}{2} = \dfrac{z+1}{3} = \lambda$

$(x, y, z) = (4\lambda - 2, 2\lambda + 1, 3\lambda - 1)$

$\overrightarrow{AP} = (4\lambda - 3)\ \hat{i} + (2\lambda - 1)\ \hat{j} + (3\lambda - 5)\ \hat{k}$

$\vec{b} = 4\hat{i} + 2\hat{j} + 3\hat{k}$

$\overrightarrow{AP} \cdot \vec{b} = 0$

$4(4\lambda - 3) + 2(2\lambda - 1) + 3(3\lambda - 5) = 0$

$29\lambda = 12 + 2 + 15 = 29$

$\lambda = 1$

P = (2, 3, 2)

$3x + 4y + 12z + 23 = 0$

$d = \left| \dfrac{6 + 12 + 24 + 23}{\sqrt{9 + 16 + 144}} \right|$

$d = \left| \dfrac{65}{13} \right| = 5$

14. The shortest distance between the lines $\dfrac{x-3}{2} = \dfrac{y-2}{3} = \dfrac{z-1}{-1}$ and $\dfrac{x+3}{2} = \dfrac{y-6}{1} = \dfrac{z-5}{3}$ is :

(A) $\dfrac{18}{\sqrt{5}}$ (B) $\dfrac{22}{3\sqrt{5}}$

(C) $\dfrac{46}{3\sqrt{5}}$ (D) $6\sqrt{3}$

Official Ans. by NTA (A)

Ans. (A)

Sol. $\dfrac{x-3}{2} = \dfrac{y-2}{3} = \dfrac{z-1}{-1}$

$\dfrac{x+3}{2} = \dfrac{y-6}{1} = \dfrac{z-5}{3}$

A = (3, 2, 1)   B = (−3, 6, 5)

$\overrightarrow{n_1} = 2\hat{i} + 3\hat{j} - \hat{k}$

$\overrightarrow{n_2} = 2\hat{i} + \hat{j} - 3\hat{k}$

$\overrightarrow{BA} = 6\hat{i} - 4\hat{j} - 4\hat{k}$

SHORTEST DISTANCE $= \dfrac{\left[\overrightarrow{BA}\ \overrightarrow{n_1}\ \overrightarrow{n_2} \right]}{\left| \overrightarrow{n_1} \times \overrightarrow{n_2} \right|}$

$\overrightarrow{n_1} \times \overrightarrow{n_2} = \begin{vmatrix} \hat{i} & \hat{j} & \hat{k} \\ 2 & 3 & -1 \\ 2 & 1 & 3 \end{vmatrix}$

$= 10\hat{i} - 8\hat{j} - 4\hat{k}$

$\left[\overrightarrow{BA}\ \overrightarrow{n_1}\ \overrightarrow{n_2} \right] = 60 + 32 + 16 = 108$

$\left| \overrightarrow{n_1} \times \overrightarrow{n_2} \right| = \sqrt{100 + 64 + 16} = \sqrt{180}$

S.D $= \dfrac{108}{\sqrt{180}} = \dfrac{108}{6\sqrt{5}} = \dfrac{18}{\sqrt{5}}$

15. Let $\vec{a}$ and $\vec{b}$ be the vectors along the diagonal of a parallelogram having area $2\sqrt{2}$. Let the angle between $\vec{a}$ and $\vec{b}$ be acute. $\left|\vec{a}\right| = 1$ and $\left|\vec{a}.\vec{b}\right| = \left|\vec{a} \times \vec{b}\right|$. If $\vec{c} = 2\sqrt{2}\left(\vec{a} \times \vec{b}\right) - 2\vec{b}$, then an angle between $\vec{b}$ and $\vec{c}$ is :
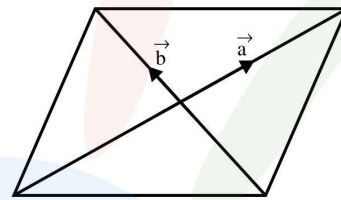
(A) $\dfrac{\pi}{4}$ (B) $-\dfrac{\pi}{4}$

(C) $\dfrac{5\pi}{6}$ (D) $\dfrac{3\pi}{4}$

Official Ans. by NTA (D)

Ans. (D)

Sol.

Area $= \dfrac{1}{2}\left|\vec{a} \times \vec{b}\right| = 2\sqrt{2} \Rightarrow \left|\vec{a} \times \vec{b}\right| = 4\sqrt{2}$

$\left|\vec{a}\right| = 1$ and $\left|\vec{a}.\vec{b}\right| = \left|\vec{a} \times \vec{b}\right|$

$\Rightarrow \cos\theta = \sin\theta$

$\Rightarrow \theta = \dfrac{\pi}{4}$

$\therefore \left|\vec{a} \times \vec{b}\right| = 4\sqrt{2} \Rightarrow \left|\vec{a}\right|\left|\vec{b}\right|\sin\dfrac{\pi}{4} = 4\sqrt{2}$

$\Rightarrow \left|\vec{b}\right| = 8$

Now, $\vec{c} = 2\sqrt{2}\left(\vec{a} \times \vec{b}\right) - 2\vec{b}$

$\left|\vec{c}\right| = \sqrt{(2\sqrt{2})^2 \left|\vec{a} \times \vec{b}\right|^2 + \left(2\left|\vec{b}\right|\right)^2} = 16\sqrt{2}$

Now, $\vec{b}.\vec{c} = -2\left|\vec{b}\right|^2$

$\Rightarrow 8 \times 16\sqrt{2} \times \cos\alpha = -2.64$

$\Rightarrow \cos\alpha = -\dfrac{1}{\sqrt{2}} \Rightarrow \alpha = \dfrac{3\pi}{4}$

16. The mean and variance of the data 4, 5, 6, 6, 7, 8, x, y where x < y are 6, and $\frac{9}{4}$ respectively. Then $x^4 + y^2$ is equal to

(A) 162 (B) 320
(C) 674 (D) 420

Official Ans. by NTA (B)

Ans. (B)

Sol. mean $\overline{x} = \frac{4+5+6+6+7+8+x+y}{8} = 6$

$\Rightarrow x + y = 48 - 36 = 12$

Variance

$= \frac{1}{8}(16+25+36+36+49+64+x^2+y^2) - 36 = \frac{9}{4}$

$\Rightarrow x^2 + y^2 = 80$
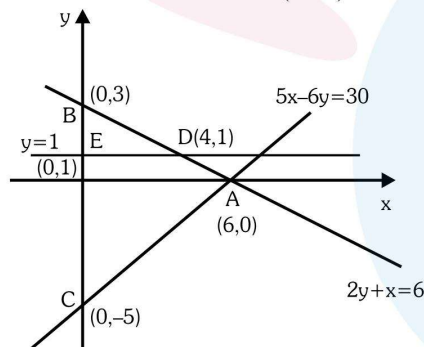
$\therefore$ x = 4; y = 8

$x^4 + y^2 = 256 + 64 = 320$

17. If a point A(x, y) lies in the region bounded by the y-axis, straight lines 2y + x = 6 and 5x – 6y = 30, then the probability that y < 1 is :

(A) $\frac{1}{6}$ (B) $\frac{5}{6}$

(C) $\frac{2}{3}$ (D) $\frac{6}{7}$

Official Ans. by NTA (B)

Ans. (B)

Sol. Required probability $= \frac{ar(ADEC)}{ar(ABC)}$

$= 1 - \frac{ar(BDE)}{ar(ABC)}$

$= 1 - \frac{\frac{1}{2} \times 2 \times 4}{\frac{1}{2} \times 8 \times 6} = 1 - \frac{1}{6} = \frac{5}{6}$

18. The value of $\cot\left(\sum_{n=1}^{50} \tan^{-1}\left(\frac{1}{1+n+n^2}\right)\right)$ is

(A) $\frac{26}{25}$ (B) $\frac{25}{26}$

(C) $\frac{50}{51}$ (D) $\frac{52}{51}$

Official Ans. by NTA (A)

Ans. (A)

Sol. $\tan^{-1}\frac{1}{1+n+n^2} = \tan^{-1}\left(\frac{(n+1)-n}{1+n(n+1)}\right)$

$= \tan^{-1}(n+1) - \tan^{-1}n$

so, $\sum_{n=1}^{50}(\tan^{-1}(n+1) - \tan^{-1}n)$

$= \tan^{-1}51 - \tan^{-1}1$

$\cot\left(\sum_{n=1}^{50}\tan^{-1}\left(\frac{1}{1+n+n^2}\right)\right) = \cot(\tan^{-1}51 + \tan^{-1}1)$

$= \frac{1}{\tan(\tan^{-1}51 - \tan^{-1}1)} = \frac{1+51\times1}{51-1} = \frac{52}{50} = \frac{26}{25}$

19. $\alpha = \sin 36°$ is a root of which of the following equation

(A) $10x^4 - 10x^2 - 5 = 0$ (B) $16x^4 + 20x^2 - 5 = 0$

(C) $16x^4 - 20x^2 + 5 = 0$ (D) $16x^4 - 10x^2 + 5 = 0$

Official Ans. by NTA (C)

Ans. (C)

Sol. $\cos 72° = \frac{\sqrt{5}-1}{4}$

$\Rightarrow 1 - 2\sin^2 36° = \frac{\sqrt{5}-1}{4}$

$\Rightarrow 4 - 8\alpha^2 = \sqrt{5} - 1$

$\Rightarrow 5 - 8\alpha^2 = \sqrt{5}$

$\Rightarrow (5 - 8\alpha^2)^2 = 5$

$\Rightarrow 25 + 64\alpha^4 - 80\alpha^2 = 5$

$\Rightarrow 64\alpha^4 - 80\alpha^2 + 20 = 0$

$\Rightarrow 16\alpha^4 - 20\alpha^2 + 5 = 0$

20. Which of the following statement is a tautology?
(A) $((\sim q) \wedge p) \wedge q$
(B) $((\sim q) \wedge p) \wedge (p \wedge (\sim p))$
(C) $((\sim q) \wedge p) \vee (p \vee (\sim p))$
(D) $(p \wedge q) \wedge (\sim(p \wedge q))$

Official Ans. by NTA (C)

Ans. (C)

Sol. (A) $(\sim q \wedge p) \wedge q = (\sim q \wedge q) \wedge p = f$

(B) $(\sim q \wedge p) \wedge (p \wedge \sim p) = \sim q \wedge (p \wedge \sim p) = f$

(C) $(\sim q \wedge p) \vee (p \vee \sim p) = (\sim q \wedge p) \vee (t) = t$

(D) $(p \wedge q) \wedge (\sim(p \wedge q)) = f$

SECTION-B

1. Let S = {1, 2, 3, 4, 5, 6, 7, 8, 9, 10}. Define

$f : S \to S$ as $f(n) = \begin{cases} 2n, & \text{if } n = 1, 2, 3, 4, 5 \\ 2n - 11 & \text{if } n = 6, 7, 8, 9, 10 \end{cases}$.

Let g : S $\to$ S be a function such that

$fog(n) = \begin{cases} n+1 & \text{,if n is odd} \\ n-1 & \text{,if n is even} \end{cases}$, then

g(10) ((g(1) + g(2) + g(3) + g(4) + g(5)) is equal to:

Official Ans. by NTA (190)

Ans. (190)

Sol. $f^{-1}(n) = \begin{cases} \dfrac{n}{2} & ; \quad n = 2, 4, 6, 8, 10 \\ \dfrac{n+11}{2} & ; \quad n = 1, 3, 5, 7, 9 \end{cases}$

$f\big(g(n)\big) = \begin{cases} n+1 & ; \quad n \in \text{odd} \\ n-1 & ; \quad n \in \text{even} \end{cases}$

$\Rightarrow \quad g(n) = \begin{cases} f^{-1}(n+1) & ; \quad n \in \text{odd} \\ f^{-1}(n-1) & ; \quad n \in \text{even} \end{cases}$

$\therefore \quad g(n) = \begin{cases} \dfrac{n+1}{2} & ; \quad n \in \text{odd} \\ \dfrac{n+10}{2} & ; \quad n \in \text{even} \end{cases}$

g(10) · [g(1) + g(2) + g(3) + g(4) + g(5)]

= 10 · [1 + 6 + 2 + 7 + 3] = 190

2. Let α, β be the roots of the equation $x^2 - 4\lambda x + 5 = 0$ and α, γ be the roots of the equation $x^2 - \left(3\sqrt{2} + 2\sqrt{3}\right)x + 7 + 3\lambda\sqrt{3} = 0$.

If $\beta + \gamma = 3\sqrt{2}$, then $(\alpha + 2\beta + \gamma)^2$ is equal to:

Official Ans. by NTA (98)

Ans. (98)

Sol. $x^2 - 4\lambda x + 5 = 0 \Big\langle {\alpha \atop \beta}$

$x^2 - \left(3\sqrt{2} + 2\sqrt{3}\right)x + \left(7 + 3\lambda\sqrt{3}\right) = 0 \Big\langle {\alpha \atop \gamma}$

$\alpha + \beta = 4\lambda$

$\alpha + \gamma = 3\sqrt{2} + 2\sqrt{3}$

$\beta + \lambda = 3\sqrt{2}$ $\qquad\qquad \alpha\gamma = 7 + 3\lambda\sqrt{3}$

$\therefore \quad \alpha = 2\lambda + \sqrt{3}$ $\qquad\qquad \alpha\beta = 5$

$\beta = 2\lambda - \sqrt{3}$ $\qquad\qquad 4\lambda^2 = 8 \Rightarrow \lambda = \sqrt{2}$

$\therefore \quad (\alpha + 2\beta + \lambda)^2 = \left(4\alpha + 3\sqrt{2}\right)^2 = \left(7\sqrt{2}\right)^2 = 98$

3. Let A be a matrix of order 2 × 2, whose entries are from the set {0, 1, 2, 3, 4, 5}. If the sum of all the entries of A is a prime number p, 2 < p < 8, then the number of such matrices A is :

Official Ans. by NTA (180)

Ans. (180)

Sol. Let $A = \begin{bmatrix} a & b \\ c & d \end{bmatrix}$; a, b, c, d ∈ {0, 1, 2, 3, 4, 5}

a + b + c + d = p, p ∈ {3, 5, 7}

Case-(i)

a + b + c + d = 3; a, b, c, d ∈ {0, 1, 2, 3}

No. of ways = $^{3+4-1}C_{4-1} = \,^6C_3 = 56$ .... (1)

Case-(ii)

a + b + c + d = 5; a, b, c, d ∈ {0, 1, 2, 3, 4, 5}

No. of ways = $^{5+4-1}C_{4-1} = \,^8C_3 = 56$ .... (2)

Case-(iii)

a + b + c + d = 7

No. of ways = total ways when a, b, c, d ∈ {0, 1, 2, 3, 4, 5, 6, 7}– total ways when a, b, c, d ∉ {6, 7}

No of ways $= \,^{7+4-1}C_{4-1} = \left(\dfrac{\lfloor 4}{\lfloor 3} + \dfrac{\lfloor 4}{\lfloor 2}\right)$

$= \,^{10}C_3 - 16 = 104$ .... (3)

Hence total no. of ways = 180

4. If the sum of the coefficients of all the positive powers of x, in the binomial expansion of $\left(x^n + \dfrac{2}{x^5}\right)^7$ is 939, then the sum of all the possible integral values of n is :

Official Ans. by NTA (57)

Ans. (57)

Sol. coefficients and there cumulative sum are :

Coefficient	Commulative sum
$x^{7n} \to {}^7C_0$	1
$x^{6n-5} \to 2 \cdot {}^7C_1$	$1 + 14$
$x^{5n-10} \to 2^2 \cdot {}^7C_2$	$1 + 14 + 84$
$x^{4n-15} \to 2^3 \cdot {}^7C_3$	$1 + 14 + 84 + 280$
$x^{3n-20} \to 2^4 \cdot {}^7C_4$	$1 + 4 + 84 + 280 + 560 = 939$
$x^{2n-25} \to 2^5 \cdot {}^7C_5$	

$3n - 20 \geq 0 \cap 2n - 25 < 0 \cap n \in I$

$\therefore$ $7 \leq n \leq 12$

Sum $= 7 + 8 + 9 + 10 + 11 + 12 = 57$

5. Let $[t]$ denote the greatest integer $\leq t$ and $\{t\}$ denote the fractional part of t. Then integral value of α for which the left hand limit of the function

$f(x) = [1 + x] + \dfrac{\alpha^{2[x]+\{x\}} + [x] - 1}{2[x] + \{x\}}$ at $x = 0$ is equal to

$\alpha - \dfrac{4}{3}$ is _____

Official Ans. by NTA (3)

Ans. (3)

Sol. $f(x) = [1+x] + \dfrac{\alpha^{2[x]+\{x\}} + [x] - 1}{2[x] + \{x\}}$

$\displaystyle\lim_{x \to 0^-} f(x) = \alpha - \dfrac{4}{3} \Rightarrow 0 + \dfrac{\alpha^{-1} - 2}{-1} = \alpha - \dfrac{4}{3}$

$\Rightarrow 2 - \dfrac{1}{\alpha} = \alpha - \dfrac{4}{3}$

$\Rightarrow \alpha + \dfrac{1}{\alpha} = \dfrac{10}{3}$

$\Rightarrow \alpha = 3; \alpha \in I$

6. If $y(x) = (x^{x^x})$, $x > 0$ then $\dfrac{d^2 x}{dy^2} + 20$ at $x = 1$ is equal to:

Official Ans. by NTA (16)

Ans. (16)

Sol. $y = (x) = (x^x)^x$

$\ln y(x) = x^2 \cdot \ln x$

$\dfrac{1}{y(x)} \cdot y'(x) = \dfrac{x^2}{x} + 2x \cdot \ln x$

$y'(x) = y(x) [x + 2x \, \ln x]$

$y(1) = 1; \; y'(1) = 1$

$y''(x) = y'(x) [x + 2x \cdot \ln (x)]$

$+ y(x) [1 + 2(1 + \ln x)]$

$y''(1) = 1 [1 + 0] + 1 (1 + 2) = 4$

$\dfrac{d^2 y}{dx^2} = - \left(\dfrac{dy}{dx}\right)^3 \cdot \dfrac{d^2 x}{dy^2}$
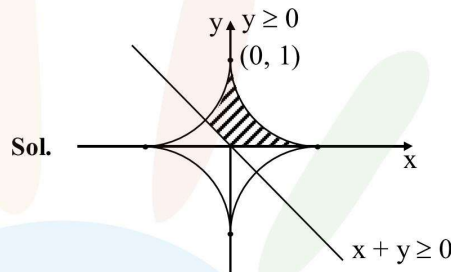
$\Rightarrow 4 = - \dfrac{d^2 x}{dy^2}$

$\dfrac{d^2 x}{dy^2} = -4$

Ans. $-4 + 20 = 16$

7. If the area of the region $\left\{(x,y) : x^{\frac{2}{3}} + y^{\frac{2}{3}} \leq 1, x + y \geq 0, y \geq 0\right\}$ is A, then $\dfrac{256A}{\pi}$ is

Official Ans. by NTA (36)

Ans. (36)

Sol.

$A = \dfrac{3}{2} \displaystyle\int_0^1 (1 - x^{2/3})^{3/2} dx$

Let $x = \sin^3 \theta$

$A = \dfrac{3}{2} \displaystyle\int_0^{\pi/2} \left(1 - \sin^2 \theta\right)^{3/2} \cdot 3 \sin^2 \theta \cos \theta \, d\theta$

$= \dfrac{3}{2} \displaystyle\int_0^{\pi/2} 3 \sin^2 \theta \cos^4 \theta \, d\theta$

$= \dfrac{9}{2} \displaystyle\int_0^{\pi/2} \sin^2 \theta \cos^4 \theta \, d\theta$

$A = \dfrac{9}{2} \times \dfrac{1 \cdot 3 \cdot 1}{(2+4)(4)(2)} \cdot \dfrac{\pi}{2}$

$\Rightarrow A = \dfrac{9\pi}{64} \Rightarrow \dfrac{64A}{\pi} = 9$

$\Rightarrow \dfrac{256A}{\pi} = 36$ **Ans.**

8. Let v be the solution of the differential equation $(1 - x^2) dy = \left(xy + (x^3 + 2)\sqrt{1 - x^2}\right) dx$, $-1 < x < 1$

and $y(0) = 0$ if $\displaystyle\int_{-\frac{1}{2}}^{\frac{1}{2}} \sqrt{1 - x^2} \, y(x) \, dx = k$ then k^{-1} is equal to :

Official Ans. by NTA (320)

Ans. (320)

Sol. $(1-x^2)\dfrac{dy}{dx} = xy + (x^3+2)\sqrt{1-x^2}$

$\Rightarrow \dfrac{dy}{dx} + \left(\dfrac{-x}{1-x^2}\right)y = \dfrac{x^3+2}{\sqrt{1-x^2}}$

IF $= e^{\int \frac{-x}{1-x^2}dx} = \sqrt{1-x^2}$

$y(x)\cdot\sqrt{1-x^2} = \dfrac{x^4}{4} + 2x + c$

$y(0) = 0 \Rightarrow c = 0$

$\sqrt{1-x^2}\,y(x) = \dfrac{x^4}{4} + 2x$

required value $= \displaystyle\int_{-1/2}^{1/2}\left(\dfrac{x^4}{4}+2x\right)dx - \dfrac{1}{4}\cdot 2\int_{0}^{1/2}x^4 dx$

$= \dfrac{1}{10}\left(x^5\right)_0^{1/2} = \dfrac{1}{320}$

$k^{-1} = 320$

$\dfrac{x}{-25} = \dfrac{y}{50} = \dfrac{1}{-25}$

$\dfrac{x-1}{\cos\theta} = \dfrac{y+2}{\sin\theta} = \pm 5$

$y = -2 + 5\left(-\dfrac{4}{5}\right) = -6$

$x = 1 + 5\left(\dfrac{3}{5}\right) = 4$

Req. distance

$\left|\dfrac{5(4)-12(-6)+51}{13}\right|$
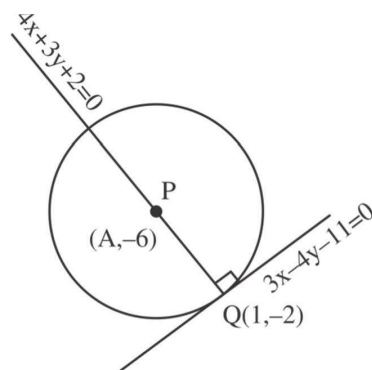
$= \left|\dfrac{20+72+51}{13}\right|$

$= \dfrac{143}{13} = 11$

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9. Let a circle C of radius 5 lie below the x-axis. The line $L_1 = 4x + 3y - 2$ passes through the centre P of the circle C and intersects the line $L_2 : 3x - 4y - 11 = 0$ at Q. The line L_2 touches C at the point Q. Then the distance of P from the line $5x - 12y + 51 = 0$ is
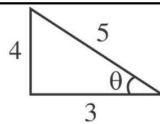
Official Ans. by NTA (11)

Ans. (11)

Sol.

$4x + 3y + 2 = 0$

$3x - 4y - 11 = 0$

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10. Let $S = \{E, E_2 \ldots E_8\}$ be a sample space of random experiment such that $P(E_n) = \dfrac{n}{36}$ for every $n = 1, 2\ldots 8$. Then the number of elements in the set $\left\{A \subset S : P(A) \geq \dfrac{4}{5}\right\}$ is _____

Official Ans. by NTA (19)

Ans. (19)

Sol. $P(A') < \dfrac{1}{5} = \dfrac{36}{180}$

5 times the sum of missing number should be less than 36.

If 1 digit is missing = 7

If 2 digit is missing = 9

If 3 digit is missing = 2

If 0 digit is missing = 1

Alternate

A is subset of S hence

A can have elements:

type 1 : { }

type 2: $\{E_1\}, \{E_2\}, \ldots \{E_8\}$

type 3: $\{E_1, E_2\}, \{E_1, E_3\} \ldots \{E_1, E_8\}$

$\vdots$
$\vdots$

type 6: $\{E_1, E_2, \ldots E_5\}, \ldots \{E_4, E_5, E_6, E_7, E_8\}$

type 7: $\{E_1, E_2, \ldots E_6\}, \ldots \{E_3, E_4, \ldots E_8\}$

type 8: $\{E_1, E_2, \ldots E_7\}\{E_2, E_3, \ldots E_8\}$

type 9: $\{E_1, E_2, \ldots E_8\}$

As $P(A) \geq \dfrac{4}{5}$;

Note : Type 1 to Type 4 elements can not be in set A as maximum probability of type 4 elements.

$\{E_5, E_6, E_7, E_8\}$ is $\dfrac{5}{36} + \dfrac{6}{36} + \dfrac{7}{36} + \dfrac{8}{36} = \dfrac{13}{18} < \dfrac{4}{5}$

Now for Type 5 acceptable elements let's call probability as P_5

$P_5 = \dfrac{n_1 + n_2 + n_3 + n_4 + n_5}{36} \leq \dfrac{4}{5}$

$\Rightarrow n_1 + n_2 + n_3 + n_4 + n_5 \geq 28.8$

Hence, 2 possible ways $\{E_5, E_6, E_7, E_8, \mathbf{E_3 \text{ or } E_4}\}$

$P_6 = n_1 + n_2 + n_3 + n_4 + n_5 + n_6 \geq 28.8$

$\Rightarrow$ 9 possible ways

$P_7 \Rightarrow n_1 + n_2 + \ldots + n_7 \geq 288$

$\Rightarrow$ 7 possible ways

$P_8 \Rightarrow n_1 + n_2 + \ldots + n_8 \geq 28.8$

$\Rightarrow$ 1 possible way

Total = 19