

## FINAL JEE–MAIN EXAMINATION – JULY, 2022

(Held On Wednesday 27<sup>th</sup> July, 2022)

TIME : 9 : 00 AM to 12 : 00 NOON

### MATHEMATICS

### TEST PAPER WITH SOLUTION

#### SECTION-A

1. Let  $R_1$  and  $R_2$  be two relations defined on  $\mathbb{R}$  by  
 $a R_1 b \Leftrightarrow ab \geq 0$  and  $a R_2 b \Leftrightarrow a \geq b$ , then  
 (A)  $R_1$  is an equivalence relation but not  $R_2$   
 (B)  $R_2$  is an equivalence relation but not  $R_1$   
 (C) both  $R_1$  and  $R_2$  are equivalence relations  
 (D) neither  $R_1$  nor  $R_2$  is an equivalence relation

**Official Ans. by NTA (D)**

**Ans. (D)**

**Sol.**  $R_1 = \{xy \geq 0, x, y \in \mathbb{R}\}$

For reflexive  $x \times x \geq 0$  which is true.

For symmetric

If  $xy \geq 0 \Rightarrow yx \geq 0$

If  $x = 2, y = 0$  and  $z = -2$

Then  $x.y \geq 0$  &  $y.z \geq 0$  but  $x.z \geq 0$  is not true  
 $\Rightarrow$  not transitive relation.

$\Rightarrow R_1$  is not equivalence

$R_2$  if  $a \geq b$  it does not implies  $b \geq a$

$\Rightarrow R_2$  is not equivalence relation

$\Rightarrow D$

2. Let  $f, g: \mathbb{N} - \{1\} \rightarrow \mathbb{N}$  be functions defined by  
 $f(a) = \alpha$ , where  $\alpha$  is the maximum of the powers  
 of those primes  $p$  such that  $p^\alpha$  divides  $a$ , and  
 $g(a) = a + 1$ , for all  $a \in \mathbb{N} - \{1\}$ . Then, the  
 function  $f + g$  is  
 (A) one-one but not onto  
 (B) onto but not one-one  
 (C) both one-one and onto  
 (D) neither one-one nor onto

**Official Ans. by NTA (D)**

**Ans. (D)**

**Sol.**  $f: \mathbb{N} - \{1\} \rightarrow \mathbb{N}$   $f(a) = \alpha$

Where  $\alpha$  is max of powers of prime  $P$  such that  
 $p^\alpha$  divides  $a$ . Also  $g(a) = a + 1$

$\therefore f(2) = 1$   $g(2) = 3$

$f(3) = 1$   $g(3) = 4$

$f(4) = 2$   $g(4) = 5$

$f(5) = 1$   $g(5) = 6$

$\Rightarrow f(2) + g(2) = 4$

$(f(3) + g(3)) = 5$

$f(4) + g(4) = 7$

$f(5) + g(5) = 7$

$\therefore$  Many one  $f(x) + g(x)$  does not contain 1

$\Rightarrow$  into function

$\therefore$  Ans. (D) [neither one-one nor onto]

3. Let the minimum value  $v_0$  of  $v = |z|^2 + |z-3|^2 + |z-6i|^2$ ,  
 $z \in \mathbb{C}$  is attained at  $z = z_0$ . Then  $|2z_0^2 - \bar{z}_0^3 + 3|^2 + v_0^2$  is  
 equal to

(A) 1000 (B) 1024

(C) 1105 (D) 1196

**Official Ans. by NTA (A)**

**Ans. (A)**

**Sol.**  $z_0 = \left( \frac{0 + 3 + 0}{3}, \frac{0 + 6 + 0}{3} \right) = (1, 2)$

$v_0 = |1 + 2i|^2 + |1 + 2i - 3|^2 + |1 + 2i - 6i|^2 = 30$

Then  $|2z_0^2 - \bar{z}_0^3 + 3|^2 + v_0^2$

$= |2(1 + 2i)^2 - (1 - 2i)^3 + 3|^2 + 900$

$= |2(1 - 4 + 4i) - (1 - 4 - 4i)(1 - 2i) + 3|^2 + 900$

$= |8 + 6i|^2 + 900 = 100 + 900 = 1000$

4. Let  $A = \begin{pmatrix} 1 & 2 \\ -2 & -5 \end{pmatrix}$ . Let  $\alpha, \beta \in \mathbb{R}$  be such that

$\alpha A^2 + \beta A = 2I$ . Then  $\alpha + \beta$  is equal to -

- (A) -10
- (B) -6
- (C) 6
- (D) 10

**Official Ans. by NTA (D)**

**Ans. (D)**

**Sol.** Characteristic equation of matrix A

$$|A - \lambda I| = 0$$

$$\begin{vmatrix} 1-\lambda & 2 \\ -2 & -5-\lambda \end{vmatrix} = 0$$

$$\Rightarrow \lambda^2 + 4\lambda = 1$$

$$\Rightarrow A^2 + 4A = I$$

$$\Rightarrow 2A^2 + 8A = 2I \quad \dots\dots\dots (1)$$

$$\text{Given that } \alpha A^2 + \beta A = 2I \quad \dots\dots\dots (2)$$

Comparing equation (1) & (2) we get

$$\alpha = 2, \quad \beta = 8$$

$$\therefore \alpha + \beta = 10$$

Ans. (D) (10)

5. The remainder when  $(2021)^{2022} + (2022)^{2021}$  is divided by 7 is

- (A) 0
- (B) 1
- (C) 2
- (D) 6

**Official Ans. by NTA (A)**

**Ans. (A)**

**Sol.**  $(2021)^{2022} + (2022)^{2021}$   
 $= (2023 - 2)^{2022} + (2023 - 1)^{2021}$   
 $= 7n_1 + 2^{2022} + 7n_2 - 1$   
 $= 7(n_1 + n_2) + 8^{674} - 1$   
 $= 7(n_1 + n_2) + (7 - 1)^{674} - 1$

$$= 7(n_1 + n_2) + 7n_3 + 1 - 1$$

$$= 7(n_1 + n_2 + n_3)$$

$\therefore$  Given number is divisible by 7 hence remainder is zero

6. Suppose  $a_1, a_2, \dots, a_n, \dots$  be an arithmetic progression of natural numbers. If the ratio of the sum of the first five terms of the sum of first nine terms of the progression is 5 : 17 and  $110 < a_{15} < 120$ , then the sum of the first ten terms of the progression is equal to -

- (A) 290
- (B) 380
- (C) 460
- (D) 510

**Official Ans. by NTA (B)**

**Ans. (B)**

**Sol.**  $\frac{S_5}{S_9} = \frac{5}{17} \Rightarrow \frac{\frac{5}{2}(2a+4d)}{\frac{9}{2}(2a+8d)} = \frac{5}{17}$

$$\Rightarrow d = 4a$$

$$a_{15} = a + 14d = 57a$$

$$\text{Now, } 110 < a_{15} < 120$$

$$\Rightarrow 110 < 57a < 120$$

$$\Rightarrow a = 2 \therefore d = 8$$

$$S_{10} = \frac{10}{2}(2 \times 2 + 9 \times 8) = 380$$

7. Let  $f : \mathbb{R} \rightarrow \mathbb{R}$  be a function defined as

$$f(x) = a \sin\left(\frac{\pi[x]}{2}\right) + [2 - x], \quad a \in \mathbb{R}, \text{ where } [t]$$

is the greatest integer less than or equal to  $t$ . If

$\lim_{x \rightarrow -1} f(x)$  exists, then the value of  $\int_0^4 f(x) dx$  is

equal to :

- (A) -1
- (B) -2
- (C) 1
- (D) 2

**Official Ans. by NTA (B)**

**Ans. (B)**

**Sol.**  $\lim_{x \rightarrow -1^+} a \sin\left(\pi \frac{[x]}{2}\right) + [2-x] = -a + 2$

$\lim_{x \rightarrow -1^-} a \sin\left(\pi \frac{[x]}{2}\right) + [2-x] = 0 + 3 = 3$

$\lim_{x \rightarrow -1} f(x)$  exist when  $a = -1$

Now,

$\int_0^4 f(x) dx = \int_0^1 f(x) dx + \int_1^2 f(x) dx + \int_2^3 f(x) dx + \int_3^4 f(x) dx$

$= \int_0^1 (0+1) dx + \int_1^2 (-1+0) dx + \int_2^3 (0-1) dx + \int_3^4 (1-2) dx$

$= 1 - 1 - 1 - 1 = -2$

8.  $I = \int_{\pi/4}^{\pi/3} \left( \frac{8 \sin x - \sin 2x}{x} \right) dx$ . Then

(A)  $\frac{\pi}{2} < I < \frac{3\pi}{4}$

(B)  $\frac{\pi}{5} < I < \frac{5\pi}{12}$

(C)  $\frac{5\pi}{12} < I < \frac{\sqrt{2}}{3} \pi$

(D)  $\frac{3\pi}{4} < I < \pi$

**Official Ans. by NTA (C)**

**Ans. (C)**

**Sol.** Consider

$f(x) = 8 \sin x - \sin 2x$

$f'(x) = 8 \cos x - 2 \cos 2x$

$f''(x) = -8 \sin x + 4 \sin 2x$

$= -8 \sin x (1 - \cos x)$

$\therefore f''(x) < 0 \quad x \in \left( \frac{\pi}{4}, \frac{\pi}{3} \right)$

$\therefore f(x)$  is  $\downarrow$  function

$f'\left(\frac{\pi}{3}\right) < f'(x) < f'\left(\frac{\pi}{4}\right)$

$5 < f'(x) < \frac{8}{\sqrt{2}}$

$5 < f'(x) < 4\sqrt{2}$

$5x < f(x) < 4\sqrt{2}x$

$5 < \frac{f(x)}{x} < 4\sqrt{2}$

$\int_{\pi/4}^{\pi/3} 5 < \int \frac{f(x)}{x} < \int_{\pi/4}^{\pi/3} 4\sqrt{2}$

$\int_{\pi/4}^{\pi/3} 5 < \int \frac{8 \sin x - \sin 2x}{x} < \int_{\pi/4}^{\pi/3} 4\sqrt{2}$

$\frac{5\pi}{12} < I < \frac{\sqrt{2}\pi}{3}$

9. The area of the smaller region enclosed by the curves  $y^2 = 8x + 4$  and  $x^2 + y^2 + 4\sqrt{3}x - 4 = 0$  is equal to

(A)  $\frac{1}{3} (2 - 12\sqrt{3} + 8\pi)$

(B)  $\frac{1}{3} (2 - 12\sqrt{3} + 6\pi)$

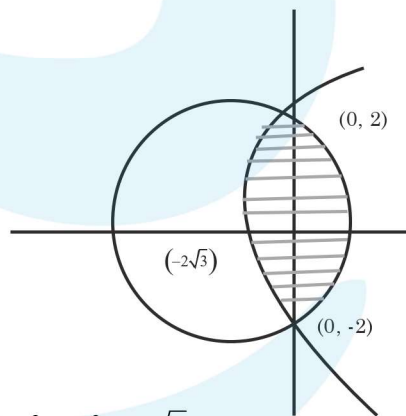
(C)  $\frac{1}{3} (4 - 12\sqrt{3} + 8\pi)$

(D)  $\frac{1}{3} (4 - 12\sqrt{3} + 6\pi)$

**Official Ans. by NTA (C)**

**Ans. (C)**

**Sol.**



$x^2 + y^2 + 4\sqrt{3}x - 4 = 0$

$y^2 = 8x + 4$

Point of intersections are (0, 2) & (0, -2)

Both are symmetric about x-axis

$Area = 2 \int_0^2 \left( \sqrt{16 - y^2} - 2\sqrt{3} \right) - \left( \frac{y^2 - 4}{8} \right) dy$

On solving  $Area = \frac{1}{3} [8\pi + 4 - 12\sqrt{3}]$

10. Let  $y = y_1(x)$  and  $y = y_2(x)$  be two distinct solutions of the differential equation  $\frac{dy}{dx} = x + y$ , with  $y_1(0) = 0$  and  $y_2(0) = 1$  respectively. Then, the number of points of intersection of  $y = y_1(x)$  and  $y = y_2(x)$  is
- (A) 0 (B) 1  
(C) 2 (D) 3

**Official Ans. by NTA (A)**

**Ans. (A)**

**Sol.**  $\frac{dy}{dx} = x + y \Rightarrow \frac{dy}{dx} - y = x$

$I f = e^{-x}$

$\therefore$  solution is  $ye^{-x} = \int xe^{-x} dx$

$\Rightarrow ye^{-x} = -xe^{-x} - e^{-x} + c$

$\Rightarrow y = -x - 1 + ce^x$

$y_1(0) = 0 \Rightarrow c = 1$

$\therefore y_1 = -x - 1 + e^x \dots(1)$

$y_2(0) = 1 \Rightarrow c = 2$

$\therefore y_2 = -x - 1 + 2e^x \dots(2)$

Now  $y_2 - y_1 = e^x > 0 \therefore y_2 \neq y_1$

$\therefore$  Number of points of intersection of  $y_1$  &  $y_2$  is zero.

11. Let P (a, b) be a point on the parabola  $y^2 = 8x$  such that the tangent at P passes through the centre of the circle  $x^2 + y^2 - 10x - 14y + 65 = 0$ . Let A be the product of all possible values of a and B be the product of all possible values of b. Then the value of A + B is equal to :
- (A) 0 (B) 25  
(C) 40 (D) 65

**Official Ans. by NTA (D)**

**Ans. (D)**

**Sol.** P(a, b) is point on  $y^2 = 8x$ , such that tangent at P pass through centre of  $x^2 + y^2 - 10x - 14y + 65 = 0$  i.e. (5, 7)  
Tangent at P( $at^2, 2at$ ) is  $ty = x + at^2$

A = 2 & it pass through (5, 7)

$7t = 5 + 2t^2$

$\Rightarrow t = 1, t = \frac{5}{2}$

$\therefore P(at^2, 2at) \Rightarrow (2, 4)$  when  $t = 1$

&  $\left(\frac{25}{2}, 10\right)$  when  $t = \frac{5}{2}$

$\therefore A = 2 \times \frac{25}{2} = 25$

B =  $4 \times 10 = 40 \therefore A + B = 65$

12. Let  $\vec{a} = \alpha\hat{i} + \hat{j} + \beta\hat{k}$  and  $\vec{b} = 3\hat{i} - 5\hat{j} + 4\hat{k}$  be two vectors, such that  $\vec{a} \times \vec{b} = -\hat{i} + 9\hat{j} + 12\hat{k}$ . Then the projection of  $\vec{b} - 2\vec{a}$  on  $\vec{b} + \vec{a}$  is equal to
- (A) 2 (B)  $\frac{39}{5}$   
(C) 9 (D)  $\frac{46}{5}$

**Official Ans. by NTA (D)**

**Ans. (Bonus)**

**Sol.** Let  $\vec{a} = \alpha\hat{i} + \hat{j} + \beta\hat{k}$ ,  $\vec{b} = 3\hat{i} - 5\hat{j} + 4\hat{k}$

$\vec{a} \times \vec{b} = -\hat{i} + 9\hat{j} + 12\hat{k}$

$\Rightarrow \begin{vmatrix} \hat{i} & \hat{j} & \hat{k} \\ \alpha & 1 & \beta \\ 3 & -5 & 4 \end{vmatrix}$

$\Rightarrow (4 + 5\beta)\hat{i} + (3\beta - 4\alpha)\hat{j} + (-5\alpha - 3)\hat{k}$

$= -\hat{i} + 9\hat{j} + 12\hat{k}$

$\therefore 4 + 5\beta = -1, 3\beta - 4\alpha = 9, -5\alpha - 3 = 12$

$\beta = -1, \alpha = -3$

$\therefore \vec{a} = -3\hat{i} + \hat{j} - \hat{k}, \vec{b} = 3\hat{i} - 5\hat{j} + 4\hat{k}$

$\therefore \vec{a} + \vec{b} = -4\hat{j} + 3\hat{k}$

$|\vec{a}|^2 = 11, |\vec{b}|^2 = 50$

$\vec{a} \cdot \vec{b} = -9 + (-5) - 4 = -18$



∴ Projectile of  $(\vec{b} - 2\vec{a})$  on  $\vec{a} + \vec{b}$  is

$$\frac{(\vec{b} - 2\vec{a}) \cdot (\vec{a} + \vec{b})}{|\vec{a} + \vec{b}|}$$

$$= \frac{|\vec{b}|^2 - 2|\vec{a}|^2 - (\vec{a} \cdot \vec{b})}{|\vec{a} + \vec{b}|} = \frac{50 - 22 - (-18)}{5} = \frac{46}{5}$$

Ans.  $\left(\frac{46}{5}\right)$

13. Let  $\vec{a} = 2\hat{i} - \hat{j} + 5\hat{k}$  and  $\vec{b} = \alpha\hat{i} + \beta\hat{j} + 2\hat{k}$ . If

$((\vec{a} \times \vec{b}) \times \hat{i}) \cdot \hat{k} = \frac{23}{2}$ , then  $|\vec{b} \times 2\hat{j}|$  is equal to

- (A) 4 (B) 5  
(C)  $\sqrt{21}$  (D)  $\sqrt{17}$

**Official Ans. by NTA (B)**

**Ans. (B)**

**Sol.**  $\vec{a} = 2\hat{i} - \hat{j} + 5\hat{k}$ ,  $\vec{b} = \alpha\hat{i} + \beta\hat{j} + 2\hat{k}$

$((\vec{a} \times \vec{b}) \times \hat{i}) \cdot \hat{k} = \frac{23}{2}$ , then  $|\vec{b} \times 2\hat{j}|$  is

$((\vec{a} \cdot \hat{i})\vec{b} - (\vec{b} \cdot \hat{i})\vec{a}) \cdot \hat{k} = \frac{23}{2}$

$(\vec{a} \cdot \hat{i})(\vec{b} \cdot \hat{i}) - (\vec{b} \cdot \hat{i})(\vec{a} \cdot \hat{k}) = \frac{23}{2}$

$2 \times 2 - \alpha \times 5 = \frac{23}{2} \Rightarrow 5\alpha = 4 - \frac{23}{2} \Rightarrow \alpha = \frac{-3}{2}$

$$\vec{b} \times 2\hat{j} = \begin{vmatrix} \hat{i} & \hat{j} & \hat{k} \\ \alpha & \beta & 2 \\ 0 & 2 & 0 \end{vmatrix} = -4\hat{i} + 2\alpha\hat{k}$$

∴  $|\vec{b} \times 2\hat{j}| = \sqrt{16 + 4\alpha^2} = \sqrt{16 + 4 \times \frac{9}{4}} = 5$

14. Let S be the sample space of all five digit numbers.

If  $p$  is the probability that a randomly selected number from S, is a multiple of 7 but not divisible by 5, then  $9p$  is equal to

- (A) 1.0146 (B) 1.2085  
(C) 1.0285 (D) 1.1521

**Official Ans. by NTA (C)**

**Ans. (C)**

**Sol.**  $n(S) = \text{all 5 digit nos} = 9 \times 10^4$

A : no is multiple of 7 but not divisible by 5

Smallest 5 digit divisible by 7 is 10003

Largest 5 digit divisible by 7 is 99995

∴  $99995 = 10003 + (n-1)7$   $n = 12857$

Numbers divisible by 35

$99995 = 10010 + (P-1)35 \Rightarrow P = 2572$

∴ Numbers divisible by 7 but not by 35 are

$12857 - 2572 = 10285$

∴  $P = \frac{10285}{90000}$  ∴  $9P = 1.0285$

Ans. (C) [1.0285]

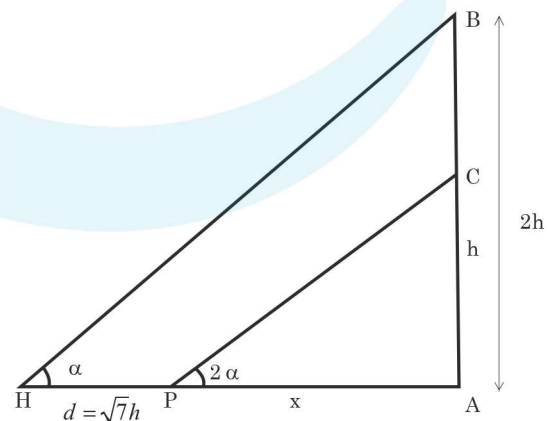
15. Let a vertical tower AB of height  $2h$  stands on a horizontal ground. Let from a point P on the ground a man can see upto height  $h$  of the tower with an angle of elevation  $2\alpha$ . When from P, he moves a distance  $d$  in the direction of  $\vec{AP}$ , he can see the top B of the tower with an angle of elevation  $\alpha$ . If  $d = \sqrt{7}h$ , then  $\tan \alpha$  is equal to

- (A)  $\sqrt{5} - 2$  (B)  $\sqrt{3} - 1$   
(C)  $\sqrt{7} - 2$  (D)  $\sqrt{7} - \sqrt{3}$

**Official Ans. by NTA (C)**

**Ans. (C)**

**Sol.**



$\tan 2\alpha = \frac{h}{x}$

$$\text{and } \tan \alpha = \frac{2h}{x + \sqrt{7}h}$$

$$\tan \alpha = \frac{2h}{h \cot 2\alpha + \sqrt{7}h}$$

$$\tan \alpha = \frac{2}{\frac{(1 - \tan^2 \alpha)}{2 \tan \alpha} + \sqrt{7}}$$

Put  $\tan \alpha = t$  & simplify  
 $\Rightarrow \tan \alpha = \sqrt{7} - 2$

16.  $(p \wedge r) \Leftrightarrow (p \wedge (\sim q))$  is equivalent to  $(\sim p)$  when r is
- (A) p (B)  $\sim p$   
 (C) q (D)  $\sim q$

**Official Ans. by NTA (C)**  
**Ans. (C)**

Sol. Given  $(p \wedge r) \Leftrightarrow (p \wedge (\sim q)) \equiv (\sim p)$

Taking  $r = q$

p	q	$\sim p$	$\sim q$	$p \wedge q$	$p \wedge \sim q$	$(p \wedge r) \Leftrightarrow (p \wedge (\sim q))$
T	T	F	F	T	F	F
T	F	F	T	F	T	F
F	T	T	F	F	F	T
F	F	T	T	F	F	T

So, clear  $(p \wedge r) \Leftrightarrow (p \wedge (\sim q)) \equiv (\sim p)$

17. If the plane P passes through the intersection of two mutually perpendicular planes  $2x + ky - 5z = 1$  and  $3kx - ky + z = 5$ ,  $k < 3$  and intercepts a unit length on positive x-axis, then the intercept made by the plane P on the y-axis is
- (A)  $\frac{1}{11}$  (B)  $\frac{5}{11}$   
 (C) 6 (D) 7

**Official Ans. by NTA (D)**  
**Ans. (D)**

Sol. Two given planes mutually perpendicular

$$2(3k) + k(-k) + (-5)1 = 0$$

$$k = 1, 5$$

but  $k < 3$  So  $k = 1$

Plane passing through these planes is

$$2x + y - 5z - 1 + \lambda(3x - y + z - 5) = 0$$

$$\frac{x}{5\lambda + 1} + \frac{y}{5\lambda + 1} + \frac{z}{5\lambda + 1} = 1$$

$$\frac{x}{2 + 3\lambda} + \frac{y}{1 - \lambda} + \frac{z}{\lambda - 5} = 1$$

Given  $\frac{5\lambda + 1}{2 + 3\lambda} = 1 \Rightarrow \lambda = \frac{1}{2}$

So intercept on y-axis =  $\frac{5\lambda + 1}{1 - \lambda} = 7$

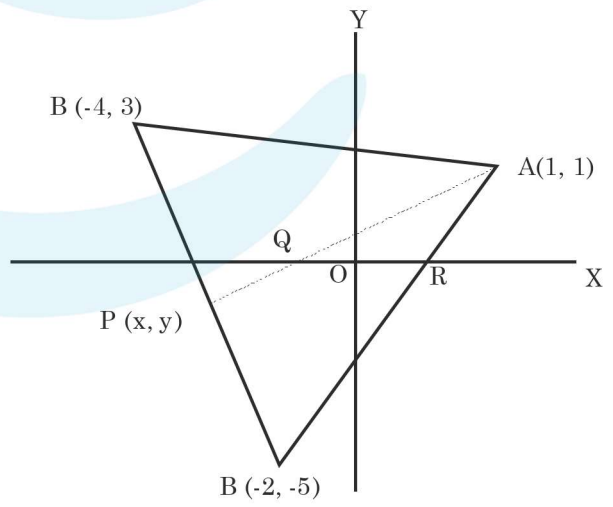
18. Let A(1, 1), B(-4, 3) C(-2, -5) be vertices of a triangle ABC, P be a point on side BC, and  $\Delta_1$  and  $\Delta_2$  be the areas of triangle APB and ABC. Respectively.

If  $\Delta_1 : \Delta_2 = 4 : 7$ , then the area enclosed by the lines AP, AC and the x-axis is

- (A)  $\frac{1}{4}$  (B)  $\frac{3}{4}$   
 (C)  $\frac{1}{2}$  (D) 1

**Official Ans. by NTA (C)**  
**Ans. (C)**

Sol.



$$\text{Given } \Delta_1 = \frac{1}{2} \begin{vmatrix} x & y & 1 \\ 1 & 1 & 1 \\ -4 & 3 & 1 \end{vmatrix}$$

$$\& \Delta_2 = \frac{1}{2} \begin{vmatrix} 1 & 1 & 1 \\ -4 & 3 & 1 \\ -2 & -5 & 1 \end{vmatrix}$$

$$\text{Given } \frac{\Delta_1}{\Delta_2} = \frac{4}{7} \Rightarrow \frac{-2x - 5y + 7}{36} = \frac{4}{7}$$

$$\Rightarrow 14x + 35y = -95 \dots(1)$$

$$\text{Equation of BC is } 4x + y = -13 \dots(2)$$

Solve equation (1) & (2)

$$\text{Point } P\left(\frac{-20}{7}, \frac{-11}{7}\right)$$

$$\text{Here point } Q\left(\frac{-1}{2}, 0\right) \& R\left(\frac{1}{2}, 0\right)$$

$$\text{So Area of triangle AQR} = \frac{1}{2} \times 1 \times 1 = \frac{1}{2}$$

19. If the circle  $x^2 + y^2 - 2gx + 6y - 19c = 0$ ,  $g, c \in \mathbb{R}$  passes through the point (6, 1) and its centre lies on the line  $x - 2cy = 8$ , then the length of intercept made by the circle on x-axis is

- (A)  $\sqrt{11}$  (B) 4  
(C) 3 (D)  $2\sqrt{23}$

**Official Ans. by NTA (D)**

**Ans. (D)**

**Sol.** Given circle  $x^2 + y^2 - 2gx + 6y - 19c = 0$

Passes through (6, 1)

$$12g + 19c = 43 \dots(1)$$

Centre (g, -3) lies on given line

$$\text{So, } g + 6c = 8 \dots(2)$$

Solve equation (1) & (2)

$$c = 1 \& g = 2$$

$$\text{equation of circle } x^2 + y^2 - 4x + 6y - 19 = 0$$

Length of intercept on x-axis

$$= 2\sqrt{g^2 - c} = 2\sqrt{23}$$

20. Let a function  $f : \mathbb{R} \rightarrow \mathbb{R}$  be defined as :

$$f(x) = \begin{cases} \int_0^x (5 - |t - 3|) dt, & x > 4 \\ x^2 + bx & , x \leq 4 \end{cases}$$

where  $b \in \mathbb{R}$ . If  $f$  is continuous at  $x = 4$ , then which of the following statements is NOT true ?

- (A)  $f$  is not differentiable at  $x = 4$   
(B)  $f'(3) + f'(5) = \frac{35}{4}$   
(C)  $f$  is increasing in  $\left(-\infty, \frac{1}{8}\right) \cup (8, \infty)$   
(D)  $f$  has a local minima at  $x = \frac{1}{8}$

**Official Ans. by NTA (C)**

**Ans. (C)**

**Sol.** Given  $f(x) = \begin{cases} \int_0^x (5 - |t - 3|) dt, & x > 4 \\ x^2 + bx & , x \leq 4 \end{cases}$

$f(x)$  is continuous at  $x = 4$

$$\text{So } \lim_{x \rightarrow 4^-} f(x) = \lim_{x \rightarrow 4^+} f(x) = f(4)$$

$$\text{So } 16 + 4b = \int_0^3 (2 - t) dt + \int_3^4 (8 - t) dt$$

$$\Rightarrow 16 + 4b = 15$$

$$\text{So } b = \frac{-1}{4}$$

At  $x = 4$

$$\text{LHD} = 2x + b = \frac{31}{4}$$

$$\text{RHD} = 5 - |x - 3| = 4$$

LHD  $\neq$  RHD

Option (A) is true

$$\text{and } f'(3) + f'(5) = \frac{23}{4} + 3 = \frac{35}{4}$$

Option (B) is true

$$\therefore f(x) = x^2 - \frac{x}{4} \text{ at } x \leq 4$$

$$f'(x) = 2x - \frac{1}{4}$$

This function is not increasing.

In the interval in  $x \in \left(-\infty, \frac{1}{8}\right)$

Option (C) is NOT TRUE.

This function  $f(x)$  is also local minima at  $x = \frac{1}{8}$

### SECTION-B

1. For  $k \in \mathbb{R}$ , let the solutions of the equation  $\cos\left(\sin^{-1}\left(x \cot\left(\tan^{-1}\left(\cos\left(\sin^{-1}x\right)\right)\right)\right)\right) = k, 0 < |x| < \frac{1}{\sqrt{2}}$  be  $\alpha$  and  $\beta$ , where the inverse trigonometric functions take only principal values. If the solutions of the equation  $x^2 - bx - 5 = 0$  are  $\frac{1}{\alpha^2} + \frac{1}{\beta^2}$  and  $\frac{\alpha}{\beta}$ , then  $\frac{b}{k^2}$  is equal to \_\_\_\_\_.

**Official Ans. by NTA (12)**

**Ans. (12)**

**Sol.**  $\cos(\sin^{-1}x) = \cos(\cos^{-1}\sqrt{1-x^2}) = \sqrt{1-x^2}$

$$\cot(\tan^{-1}\sqrt{1-x^2}) = \cot \cot^{-1}\left(\frac{1}{\sqrt{1-x^2}}\right) = \frac{1}{\sqrt{1-x^2}}$$

$$\Rightarrow \cos\left(\sin^{-1}\left(\frac{x}{\sqrt{1-x^2}}\right)\right) = \frac{\sqrt{1-2x^2}}{\sqrt{1-x^2}}$$

$$\Rightarrow \frac{\sqrt{1-2x^2}}{\sqrt{1-x^2}} = k$$

$$\Rightarrow 1-2x^2 = k^2(1-x^2)$$

$$\Rightarrow (k^2-2)x^2 = k^2-1$$

$$x^2 = \frac{k^2-1}{k^2-2}$$

$$\alpha = \sqrt{\frac{k^2-1}{k^2-2}} \Rightarrow \alpha^2 = \frac{k^2-1}{k^2-2}$$

$$\beta = \sqrt{\frac{k^2-1}{k^2-2}} \Rightarrow \beta^2 = \frac{k^2-1}{k^2-2}$$

$$\frac{1}{\alpha^2} + \frac{1}{\beta^2} = 2\left(\frac{k^2-2}{k^2-1}\right) \& \frac{\alpha}{\beta} = -1$$

$$\text{Sum of roots} = \frac{1}{\alpha^2} + \frac{1}{\beta^2} + \frac{\alpha}{\beta} = b$$

$$\Rightarrow \frac{2(k^2-2)}{k^2-1} - 1 = b \dots (1)$$

$$\text{Product of roots} = \left(\frac{1}{\alpha^2} + \frac{1}{\beta^2}\right) \frac{\alpha}{\beta} = -5$$

$$\Rightarrow \frac{2(k^2-2)}{k^2-1} (-1) = -5$$

$$\Rightarrow 2k^2 - 4 = 5k^2 - 5$$

$$\Rightarrow 3k^2 = 1 \Rightarrow k^2 = \frac{1}{3} \dots \text{Put in (1)}$$

$$\Rightarrow b = \frac{2(k^2-2)}{k^2-1} - 1 = 5 - 1 = 4$$

$$\frac{b}{k^2} = \frac{4}{\frac{1}{3}} = 12$$

2. The mean and variance of 10 observations were calculated as 15 and 15 respectively by a student who took by mistake 25 instead of 15 for one observation. Then, the correct standard deviation is \_\_\_\_\_.

**Official Ans. by NTA (2)**

**Ans. (2)**

**Sol.**  $n = 10, \bar{x} = \frac{\sum x_i}{10} = 15$

$$6^2 = \frac{\sum x_i^2}{10} - (\bar{x})^2 = 15$$

$$\Rightarrow \sum_{i=1}^{10} x_i = 150$$



$$\Rightarrow \sum_{i=1}^9 x_i + 25 = 150$$

$$\Rightarrow \sum_{i=1}^9 x_i = 125$$

$$\Rightarrow \sum_{i=1}^9 x_i + 15 = 140$$

$$\text{Actual mean} = \frac{140}{10} = 14 = \bar{x}_{new}$$

$$\sum_{i=1}^9 \frac{x_i^2 + 25^2 - 15^2}{10} = 15$$

$$\Rightarrow \sum_{i=1}^9 x_i^2 + 625 = 2400$$

$$\sum_{i=1}^9 x_i^2 = 1775$$

$$\sum_{i=1}^9 x_i^2 + 15^2 = 2000 = \left( \sum_{i=1}^9 x_i^2 \right)_{actual}$$

$$6^2_{actual} = \frac{\left( \sum_{i=1}^9 x_i^2 \right)_{actual} - (\bar{x}_{new})^2}{10}$$

$$= \frac{2000}{10} - 14^2$$

$$= 200 - 196 = 4$$

$$(S.D.)_{actual} = 6 = 2$$

3. Let the line  $\frac{x-3}{7} = \frac{y-2}{-1} = \frac{z-3}{-4}$  intersect the plane containing the lines  $\frac{x-4}{1} = \frac{y+1}{-2} = \frac{z}{1}$  and  $4ax - y + 5z - 7a = 0 = 2x - 5y - z - 3$ ,  $a \in \mathbb{R}$  at the point  $P(\alpha, \beta, \gamma)$ . Then the value of  $\alpha + \beta + \gamma$  equals \_\_\_\_\_.

**Official Ans. by NTA (12)**

**Ans. (12)**

**Sol.** Equation of plane

$$4ax - y + 5z - 7a + \lambda (2x - 5y - z - 3) = 0$$

this satisfy (4, -1, 0)

$$16a + 1 - 7a + \lambda(8 + 5 - 3) = 0$$

$$9a + 1 + 10\lambda = 0 \quad \dots(1)$$

Normal vector of the plane A is  $(4a + 2\lambda, -1 - 5\lambda, 5 - \lambda)$  vector along the line which contained the plane A is

$$i - 2j + k$$

$$\therefore 4a + 2\lambda + 2 + 10\lambda + 5 - \lambda = 0$$

$$11\lambda + 4a + 7 = 0 \dots\dots(2)$$

Solve (1) and (2) to get  $a = 1$ ,  $\lambda = -1$

Now equation of plane

$$x + 2y + 3z - 2 = 0$$

Let the point in the line  $\frac{x-3}{7} = \frac{y-2}{-1} = \frac{z-3}{-4} = t$

is  $(7t + 3, -t + 2, -4t + 3)$  satisfy the equation of plane A

$$7t + 3 - 2t + 4 + 9 - 12t - 2 = 0$$

$$t = 2$$

$$\text{So } \alpha + \beta + \gamma = 2t + 8 = 12$$

4. An ellipse  $E: \frac{x^2}{a^2} + \frac{y^2}{b^2} = 1$  passes through the vertices of the hyperbola  $H: \frac{x^2}{49} - \frac{y^2}{64} = -1$ . Let the major and minor axes of the ellipse E coincide with the transverse and conjugate axes of the hyperbola H. Let the product of the eccentricities of E and H be  $\frac{1}{2}$ . If  $l$  is the length of the latus rectum of the ellipse E, then the value of  $113l$  is equal to \_\_\_\_\_.

**Official Ans. by NTA (1552)**

**Ans. (1552)**

**Sol.** Hyp:  $\frac{y^2}{64} - \frac{x^2}{49} = 1$

An ellipse  $E: \frac{x^2}{a^2} + \frac{y^2}{b^2} = 1$  passes through the

vertices of the hyperbola  $H: \frac{x^2}{49} - \frac{y^2}{64} = -1$ .

So  $b^2 = 64$

$$e_H = \sqrt{1 + \frac{a^2}{b^2}} = \sqrt{1 + \frac{49}{64}}$$

Ellipse  $\frac{x^2}{a^2} + \frac{y^2}{b^2} = 1$

$$e_E = \sqrt{1 - \frac{a^2}{b^2}} = \sqrt{1 - \frac{a^2}{64}}$$

$$b = 8, \sqrt{\frac{1-a^2}{64}} \times \frac{\sqrt{113}}{8} = \frac{1}{2} \Rightarrow \sqrt{64-a^2} \times \sqrt{113} = 32$$

$$(64 - a^2) = \frac{32^2}{113}$$

$$\Rightarrow a^2 = 64 - \frac{32^2}{113}$$

$$l = \frac{2a^2}{b} = \frac{2}{8} \left( 64 - \frac{32^2}{113} \right) = \frac{1552}{113}$$

$$113l = 1552$$

5. Let  $y = y(x)$  be the solution curve of the differential equation

$$\sin(2x^2) \log_e(\tan x^2) dy + \left( 4xy - 4\sqrt{2}x \sin\left(x^2 - \frac{\pi}{4}\right) \right) dx = 0,$$

$0 < x < \sqrt{\frac{\pi}{2}}$ , which passes through the point

$$\left( \sqrt{\frac{\pi}{6}}, 1 \right). \text{ Then } \left| y\left(\sqrt{\frac{\pi}{3}}\right) \right| \text{ is equal to } \underline{\hspace{2cm}}.$$

**Official Ans. by NTA (1)**

**Ans. (1)**

**Sol.**  $\sin(2x^2) \ln(\tan x^2) dy + \left( 4xy - 4\sqrt{2}x \sin\left(x^2 - \frac{\pi}{4}\right) \right) dx = 0$

$$\ln(\tan x^2) dy + \frac{4xy dx}{\sin(2x^2)} - \frac{4\sqrt{2}x \sin\left(x^2 - \frac{\pi}{4}\right)}{\sin(2x^2)} dx = 0$$

$$d(y \cdot \ln(\tan x^2)) - 4\sqrt{2}x \frac{(\sin x^2 - \cos x^2)}{\sqrt{2} - 2 \sin x^2 \cos x^2} dx = 0$$

$$d(y \ln(\tan x^2)) - \frac{4x(\sin x^2 - \cos x^2)}{(\sin x^2 + \cos^2) - 1} dx = 0$$

$$\Rightarrow \int d(y \ln(\tan x^2)) + 2 \int \frac{dt}{t^2 - 1} = \int 0$$

$$\Rightarrow y \ln(\tan x^2) + 2 \cdot \frac{1}{2} \ln \left| \frac{t-1}{t+1} \right| = c$$

$$y \ln(\tan x^2) + \ln \left( \frac{\sin x^2 + \cos x^2 - 1}{\sin x^2 + \cos x^2 + 1} \right) = c$$

Put  $y = 1$  and  $x = \sqrt{\frac{\pi}{6}}$

$$1 \ln \left( \frac{1}{\sqrt{3}} \right) + \ln \left( \frac{\frac{1}{2} + \frac{\sqrt{3}}{2} - 1}{\frac{1}{2} + \frac{\sqrt{3}}{2} + 1} \right) = c$$

Now  $x = \sqrt{\frac{\pi}{3}} \Rightarrow y(\ln \sqrt{3}) + \ln \left( \frac{\frac{1}{2} + \frac{\sqrt{3}}{2} - 1}{\frac{1}{2} + \frac{\sqrt{3}}{2} + 1} \right) = \ln \left( \frac{1}{\sqrt{3}} \right) + \ln \left( \frac{\sqrt{3}-1}{\sqrt{3}+3} \right)$

$$y(\ln \sqrt{3}) = \ln \left( \frac{1}{\sqrt{3}} \right)$$

$$\Rightarrow y = -1$$

$$|y| = 1$$

6. Let  $M$  and  $N$  be the number of points on the curve  $y^5 - 9xy + 2x = 0$ , where the tangents to the curve are parallel to  $x$ -axis and  $y$ -axis, respectively. Then the value of  $M + N$  equals \_\_\_\_\_.

**Official Ans. by NTA (2)**

**Ans. (2)**

**Sol.**  $y^5 - 9xy + 2x = 0$

$$5y^4 \frac{dy}{x} - 9x \frac{dy}{dx} - 9y + 2 = 0$$

$$\frac{dy}{dx} (5y^4 - 9x) = 9y - 2$$

$$\frac{dy}{dx} = \frac{9y-2}{5y^4-9x} = 0 \text{ (for horizontal tangent)}$$

$$y = \frac{2}{9} \Rightarrow \text{Which does not satisfy the original equation} \Rightarrow M = 0.$$

Now  $5y^4 - 9x = 0$  (for vertical tangent)

$$5y^4 (9y - 2) - 9y^5 = 0$$

$$y^4 [45y - 10 - 9y] = 0$$

$$y = 0 \text{ (Or) } 36y = 10$$

$$y = \frac{5}{18}$$

$$y = 0 \Rightarrow x = 0 \text{ \& } y = \frac{5}{18} \Rightarrow x =$$

$$(0, 0) \quad \left(x, \frac{5}{18}\right)$$

$$N = 2$$

$$M + N = 0 + 2 = 2$$

7. Let  $f(x) = 2x^2 - x - 1$  and

$S = \{n \in \mathbb{Z} : |f(n)| \leq 800\}$ . Then, the value of

$\sum_{n \in S} f(n)$  is equal to \_\_\_\_\_.

**Official Ans. by NTA (10620)**

**Ans. (10620)**

**Sol.**  $f(x) = 2x^2 - x - 1$

$$|f(x)| \leq 800$$

$$2n^2 - n - 801 \leq 0$$

$$n^2 - \frac{1}{2}n - \frac{801}{2} \leq 0$$

$$\left(n - \frac{1}{4}\right)^2 - \frac{801}{2} - \frac{1}{16} \leq 0$$

$$\left(n - \frac{1}{4}\right)^2 - \frac{6409}{16} \leq 0$$

$$\left(n - \frac{1}{4} - \frac{\sqrt{6409}}{4}\right) \left(n - \frac{1}{4} + \frac{\sqrt{6409}}{16}\right) \leq 0$$

$$\frac{1 - \sqrt{6409}}{4} \leq n \leq \frac{1 + \sqrt{6409}}{4}$$

$$n = \{-19, -18, -17, \dots, 0, 1, 2, \dots, 20\}$$

$$\sum_{n \in S} f(x) = \sum (2x^2 - x - 1)$$

$$= 2[19^2 + 18^2 + \dots + 1^2 + 1^2 + 2^2 + \dots + 19^2 + 20^2]$$

$$= 4[1^2 + 2^2 + \dots + 19^2] + 2[20^2] - 20 - 40$$

$$= \frac{4 \times 19 \times 20 \times (2 \times 19 + 1)}{6} + 2 \times 400 - 60$$

$$= \frac{4 \times 19 \times 20 \times 39}{6} + 800 - 60 - 9880 + 800 - 60$$

$$= 10620$$

8. Let  $S$  be the set containing all  $3 \times 3$  matrices with entries from  $\{-1, 0, 1\}$ . The total number of matrices  $A \in S$  such that the sum of all the diagonal elements of  $A^T A$  is 6 is \_\_\_\_\_.

**Official Ans. by NTA (5376)**

**Ans. (5376)**

**Sol.**  $Tr(AA^T) = 6$

$$AA^T = \begin{bmatrix} a & d & g \\ b & e & h \\ c & f & i \end{bmatrix} \begin{bmatrix} a & b & c \\ d & e & f \\ g & h & i \end{bmatrix}$$

Now given  $a^2 + d^2 + g^2 + b^2 + e^2 + h^2 + c^2 + f^2 + i^2 = 6$

$$\Rightarrow {}^9 C_3 \times 2^6$$

$$= 5376$$

9. If the length of the latus rectum of the ellipse  $x^2 + 4y^2 + 2x + 8y - \lambda = 0$  is 4, and  $l$  is the length of its major axis, then  $\lambda + l$  is equal to \_\_\_\_\_.

**Official Ans. by NTA (75)**

**Ans. (75)**

**Sol.**  $\lambda + l = 75$

$$x^2 + 4y^2 + 2x + 8y - \lambda = 0$$

$$\frac{(x+1)^2}{\lambda+5} + \frac{(y+1)^2}{\frac{\lambda+5}{4}} = 1$$

$$\therefore \frac{2b^2}{a} = 4$$

$$\frac{2(\lambda+5)}{4} = 4(\sqrt{\lambda+5})$$

$$\Rightarrow \lambda = 59$$

$$\lambda \neq -5$$

$$l = 2a = 2\sqrt{\lambda+5} = 2\sqrt{65} = 16$$

$$\Rightarrow \lambda + l = 59 + 16 = 75$$

10. Let  $S = \{z \in \mathbb{C} : z^2 + \bar{z} = 0\}$ . Then  $\sum_{z \in S} (\operatorname{Re}(z) + \operatorname{Im}(z))$  is equal to \_\_\_\_\_.

**Official Ans. by NTA (0)**

**Ans. (0)**

**Sol.**  $S = \{z \in \mathbb{C} : z^2 + \bar{z} = 0\}$

$$\text{Let } z = x + iy$$

$$z^2 = x^2 - y^2 + 2ixy$$

$$\bar{z} = x - iy$$

$$z^2 + \bar{z} = x^2 - y^2 + x + i(2xy - y) = 0$$

$$\Rightarrow x^2 + x - y^2 = 0 \text{ \& } 2xy - y = 0$$

$$y = 0 \text{ or } x = \frac{1}{2}$$

$$\text{If } y = 0; x = 0, -1$$

$$\text{If } x = \frac{1}{2}; y = \frac{\sqrt{3}}{2}, \frac{-\sqrt{3}}{2}$$

$$\sum_{z \in S} (\operatorname{Re}(z) + \operatorname{Im}(z)) = \left(0 - 1 + \frac{1}{2} + \frac{1}{2}\right) + 0 + 0 + \frac{\sqrt{3}}{2} - \frac{\sqrt{3}}{2}$$