

FINAL JEE-MAIN EXAMINATION – JULY, 2022

(Held On Thursday 28th July, 2022)

TIME : 9 : 00 AM to 12 : 00 NOON

MATHEMATICS

TEST PAPER WITH SOLUTION

SECTION-A

1. Let the solution curve of the differential equation $xdy = (\sqrt{x^2 + y^2} + y)dx$, $x > 0$, intersect the line $x = 1$ at $y = 0$ and the line $x = 2$ at $y = \alpha$. Then the value of α is :

(A) $\frac{1}{2}$ (B) $\frac{3}{2}$ (C) $-\frac{3}{2}$ (D) $\frac{5}{2}$

Official Ans. by NTA (B)

Ans. (B)

Sol. $xdy = (\sqrt{x^2 + y^2} + y)dx$

$$xdy - ydx = \sqrt{x^2 + y^2} dx$$

$$\frac{xdy - ydx}{x^2} = \sqrt{1 + \frac{y^2}{x^2}} \cdot \frac{dx}{x}$$

$$\frac{d(y/x)}{\sqrt{1 + \left(\frac{y}{x}\right)^2}} = \frac{dx}{x}$$

$$\ln \left(\frac{y}{x} + \sqrt{\left(\frac{y}{x}\right)^2 + 1} \right) = \ln x + R$$

$$\frac{y + \sqrt{y^2 + x^2}}{x} = cx$$

$$y + \sqrt{y^2 + x^2} = cx^2$$

$$x = 1, y = 0 \Rightarrow 0 + 1 = C \Rightarrow C = 1$$

$$\text{Curve is } y + \sqrt{x^2 + y^2} = x^2$$

$$x = 2, y = \alpha$$

$$2 + \sqrt{4 + \alpha^2} = 4$$

$$4 + \alpha^2 = 16 + \alpha^2 = 8\alpha$$

$$\alpha = \frac{3}{2}$$

2. Considering only the principal values of the inverse trigonometric functions, the domain of the

function $f(x) = \cos^{-1} \left(\frac{x^2 - 4x + 2}{x^2 + 3} \right)$ is :

(A) $\left[-\infty, \frac{1}{4}\right]$ (B) $\left[-\frac{1}{4}, \infty\right)$

(C) $\left[-\frac{1}{3}, \infty\right)$ (D) $\left[-\infty, \frac{1}{3}\right)$

Official Ans. by NTA (B)

Ans. (B)

Sol.

$$\left| \frac{x^2 + 4x + 2}{x^2 + 3} \right| \leq 1$$

$$\Leftrightarrow (x^2 - 4x + 2)^2 \leq (x^2 + 3)^2$$

$$\Leftrightarrow (x^2 - 4x + 2)^2 - (x^2 + 3)^2 \leq 0$$

$$\Leftrightarrow (2x^2 - 4x + 5)(-4x - 1) \leq 0$$

$$\Leftrightarrow -4x - 1 \leq 0 \rightarrow x \geq -\frac{1}{4}$$

3. Let the vectors $\vec{a} = (1+t)\hat{i} + (1-t)\hat{j} + \hat{k}$,

$$\vec{b} = (1-t)\hat{i} + (1+t)\hat{j} + 2\hat{k} \text{ and } \vec{c} = t\hat{i} - t\hat{j} + \hat{k}, t \in \mathbb{R}$$

be such that for $\alpha, \beta, \gamma \in \mathbb{R}$, $\alpha\vec{a} + \beta\vec{b} + \gamma\vec{c} = \vec{0}$

$\Rightarrow \alpha = \beta = \gamma = 0$. Then, the set of all values of t is :

(A) a non-empty finite set

(B) equal to \mathbb{N}

(C) equal to $\mathbb{R} - \{0\}$

(D) equal to \mathbb{R}

Official Ans. by NTA (C)

Ans. (C)

Sol. By its given condition

: $\vec{a}, \vec{b}, \vec{c}$ are linearly independent vectors

$$\Rightarrow [\vec{a} \vec{b} \vec{c}] \neq 0 \quad \dots(i)$$

Now, $[\vec{a} \vec{b} \vec{c}]$

$$= \begin{vmatrix} 1+t & 1-t & 1 \\ 1-t & 1+t & 2 \\ t & -t & 1 \end{vmatrix}$$

$$C_2 \rightarrow C_1 + C_2$$

$$\begin{vmatrix} 1+t & 2 & 1 \\ 1-t & 2 & 2 \\ t & 0 & 1 \end{vmatrix}$$

$$= 2 \begin{vmatrix} 1+t & 1 & 1 \\ 1-t & 1 & 2 \\ t & 0 & 1 \end{vmatrix}$$

$$= 2[(1+t) - (1-t) + t]$$

$$= 2[3t] = 6t$$

$$[\vec{a} \vec{b} \vec{c}] \neq 0 \Rightarrow t \neq 0$$

4. Considering the principal values of the inverse trigonometric functions, the sum of all the solutions of the equation $\cos^{-1}(x) - 2\sin^{-1}(x) = \cos^{-1}(2x)$ is equal to :

- (A) 0 (B) 1 (C) $\frac{1}{2}$ (D) $-\frac{1}{2}$

Official Ans. by NTA (A)

Ans. (A)

Sol. $\cos^{-1} x = 2\sin^{-1} x = \cos^{-1} 2x$

$$\cos^{-1} x - 2\left(\frac{\pi}{2} - \cos^{-1} x\right) = \cos^{-1} 2x$$

$$\cos^{-1} x - \pi + 2\cos^{-1} x = \cos^{-1} 2x$$

$$3\cos^2 x = \pi + \cos^{-1} 2x \quad \dots(1)$$

$$\cos(3\cos^{-1} x) = \cos(\pi + \cos^{-1} 2x)$$

$$4x^3 - 3x = -2x$$

$$4x^3 = x \Rightarrow x = 0, \pm \frac{1}{2}$$

All satisfy the original equation

$$\text{sum} = -\frac{1}{2} \text{ to } +\frac{1}{2} = 0$$

5. Let the operations $*, \odot \in \{\wedge, \vee\}$. If

$(p*q)\odot(p\odot\sim q)$ is a tautology, then the ordered pair $(*, \odot)$ is :

- (A) (\vee, \wedge) (B) (\vee, \vee) (C) (\wedge, \wedge) (D) (\wedge, \vee)

Official Ans. by NTA (B)

Ans. (B)

Sol. Well check each option

For A $\pi = \vee$ of $\odot = \wedge$

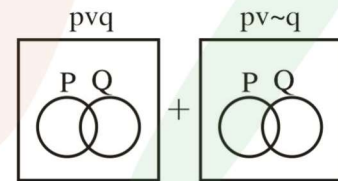
$$(pvq) \wedge (pv \sim q)$$

$$\equiv pv(q \wedge \sim q)$$

$$\equiv pv(c) \equiv p$$

For B : $* = \vee, \odot = \vee$

$$(pvq) \vee (pv \sim q) \equiv t \quad \text{using Venn Diagrams}$$



6. Let a vector \vec{a} has a magnitude 9. Let a vector \vec{b} be such that for every $(x, y) \in \mathbb{R} \times \mathbb{R} - \{(0, 0)\}$, the vector $(x\vec{a} + y\vec{b})$ is perpendicular to the vector $(6y\vec{a} - 18x\vec{b})$. Then the value of $|\vec{a} \times \vec{b}|$ is equal to:

- (A) $9\sqrt{3}$ (B) $27\sqrt{3}$ (C) 9 (D) 81

Official Ans. by NTA (B)

Ans. (B)

Sol. $|\vec{a}| = 9$ & $(x\vec{a} + y\vec{b}) \cdot (6y\vec{a} - 18x\vec{b}) = 0$

$$\Rightarrow 6xy|\vec{a}|^2 - 18x^2(\vec{a} \cdot \vec{b}) + 6y^2(\vec{a} \cdot \vec{b}) - 18xy|\vec{b}|^2 = 0$$

$$\Rightarrow 6xy(|\vec{a}|^2 - 3|\vec{b}|^2) + (\vec{a} \cdot \vec{b})(y^2 - 3x^2) = 0$$

This should hold $\forall x, y \in \mathbb{R} \times \mathbb{R}$

$$\therefore |\vec{a}|^2 = 3|\vec{b}|^2 \quad \& \quad (\vec{a} \cdot \vec{b}) = 0$$

$$\text{Now } |\vec{a} \times \vec{b}|^2 = |\vec{a}|^2 |\vec{b}|^2 - (\vec{a} \cdot \vec{b})^2$$

$$= |\vec{a}|^2 \cdot \frac{|\vec{a}|^2}{3}$$

$$\therefore |\vec{a} \times \vec{b}| = \frac{|\vec{a}|^2}{\sqrt{3}} = \frac{81}{\sqrt{3}} = 27\sqrt{3}$$

7. For $t \in (0, 2\pi)$, if ABC is an equilateral triangle with vertices $A(\sin t, -\cos t)$, $B(\cos t, \sin t)$ and $C(a, b)$ such that its orthocentre lies on a circle with centre $\left(1, \frac{1}{3}\right)$, then $(a^2 - b^2)$ is equal to :

- (A) $\frac{8}{3}$ (B) 8
(C) $\frac{77}{9}$ (D) $\frac{80}{9}$

Official Ans. by NTA (B)

Ans. (B)

Sol. $s \equiv \sin t, c \equiv \cos t$

Let orthocentre be (h, k)

Since it is an equilateral triangle hence orthocentre coincides with centroid.

$$\therefore a + s + c = 3h, b + s - c = 3k$$

$$\therefore (3h - a)^2 + (3k - b)^2 = (s + c)^2 + (s - c)^2 = 2(s^2 + c^2) = 2$$

$$\therefore \left(h - \frac{a}{3}\right)^2 + \left(k - \frac{b}{3}\right)^2 = \frac{2}{9},$$

circle centre at $\left(\frac{a}{3}, \frac{b}{3}\right)$

$$\text{Gives, } \frac{a}{3} = 1, \frac{b}{3} = \frac{1}{3} \Rightarrow a = 3, b = 1$$

$$\therefore a^2 - b^2 = 8$$

8. For $\alpha \in \mathbb{N}$, consider a relation R on \mathbb{N} given by $R = \{(x, y) : 3x + \alpha y \text{ is a multiple of } 7\}$. The

relation R is an equivalence relation if and only if :

- (A) $\alpha = 14$
(B) α is a multiple of 4
(C) 4 is the remainder when α is divided by 10
(D) 4 is the remainder when α is divided by 7

Official Ans. by NTA (D)

Ans. (D)

Sol. For R to be reflexive $\Rightarrow x R x$

$$\Rightarrow 3x + \alpha x = 7x \Rightarrow (3 + \alpha)x = 7K$$

$$\Rightarrow 3 + \alpha = 7\lambda \Rightarrow \alpha = 7\lambda - 3 = 7N + 4, K, \lambda, N \in \mathbb{I}$$

\therefore when α divided by 7, remainder is 4.

R to be symmetric $xRy \Rightarrow yRx$

$$3x + \alpha y = 7N_1, 3y + \alpha x = 7N_2$$

$$\Rightarrow (3 + \alpha)(x + y) = 7(N_1 + N_2) = 7N_3$$

Which holds when $3 + \alpha$ is multiple of 7

$$\therefore \alpha = 7N + 4 \text{ (as did earlier)}$$

R to be transitive

$$xRy \text{ \& } yRz \Rightarrow xRz.$$

$$3x + \alpha y = 7N_1 \text{ \& } 3y + \alpha z = 7N_2$$

and

$$3x + \alpha z = 7N_3$$

$$\therefore 3x + 7N_2 - 3y = 7N_3$$

$$\therefore 7N_1 - \alpha y + 7N_2 - 3y = 7N_3$$

$$\therefore 7(N_1 + N_2) - (3 + \alpha)y = 7N_3$$

$$\therefore (3 + \alpha)y = 7N$$

Which is true again when $3 + \alpha$ divisible by 7, i.e. when α divided by 7, remainder is 4.

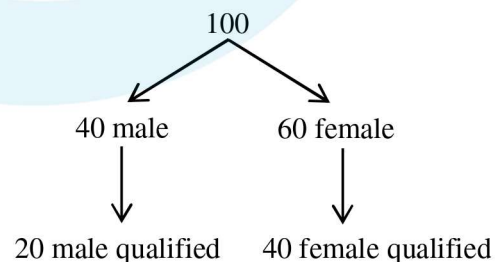
9. Out of 60% female and 40% male candidates appearing in an exam, 60% candidates qualify it. The number of females qualifying the exam is twice the number of males qualifying it. A candidate is randomly chosen from the qualified candidates. The probability, that the chosen candidate is a female, is :

- (A) $\frac{3}{4}$ (B) $\frac{11}{16}$
(C) $\frac{23}{32}$ (D) $\frac{13}{16}$

Official Ans. by NTA (A)

Ans. (Bonus)

Sol.



$$\text{Probability that chosen candidate is female} = \frac{40}{60} = \frac{2}{3}$$

10. If $y = y(x)$, $x \in \left(0, \frac{\pi}{2}\right)$ be the solution curve of the differential equation

$$(\sin^2 2x) \frac{dy}{dx} + (8 \sin^2 2x + 2 \sin 4x)y = 2e^{-4x} (2 \sin 2x + \cos 2x), \quad \text{with } y\left(\frac{\pi}{4}\right) = e^{-\pi},$$

then $y\left(\frac{\pi}{6}\right)$ is equal to :

- (A) $\frac{2}{\sqrt{3}} e^{-2\pi/3}$ (B) $\frac{2}{\sqrt{3}} e^{2\pi/3}$
 (C) $\frac{1}{\sqrt{3}} e^{-2\pi/3}$ (D) $\frac{1}{\sqrt{3}} e^{2\pi/3}$

Official Ans. by NTA (A)

Ans. (A)

Sol. Given differential equation can be re-written as

$$\frac{dy}{dx} + (8 + 4 \cot 2x)y = \frac{2e^{-4x}}{\sin^2 2x} (2 \sin x + \cos 2x)$$

which is a linear diff. equation.

$$\text{I.f.} = e^{\int (8+4\cot 2x) dx} = e^{8x+2C\cot(\sin 2x)} = e^{8x} \cdot \sin^2 2x$$

\therefore solution is

$$y(e^{8x} \cdot \sin^2 2x) = \int 2e^{4x} (2 \sin 2x + \cos 2x) dx + C = e^{4x} \cdot \sin 2x + C$$

$$\text{Given } y\left(\frac{\pi}{4}\right) = e^{-\pi} \Rightarrow C = 0$$

$$\therefore y = \frac{e^{-4x}}{\sin 2x}$$

$$\therefore y\left(\frac{\pi}{6}\right) = \frac{e^{-4 \cdot \frac{\pi}{6}}}{\sin\left(2 \cdot \frac{\pi}{6}\right)} = \frac{2}{\sqrt{3}} e^{-\frac{2\pi}{3}}$$

11. If the tangents drawn at the points P and Q on the parabola $y^2 = 2x - 3$ intersect at the point R(0, 1), then the orthocentre of the triangle PQR is :

- (A) (0, 1) (B) (2, -1)
 (C) (6, 3) (D) (2, 1)

Official Ans. by NTA (B)

Ans. (B)

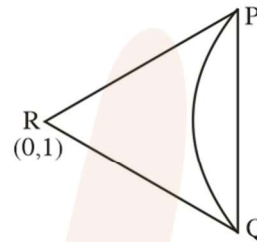
Sol. $y^2 = 2x - 3$... (i)

Equation of chord of contact

$$PQ : r = 0$$

$$yx_1 = (x + 0) - 3$$

$$y = x - 3$$
 ... (2)



from (1) and (2)

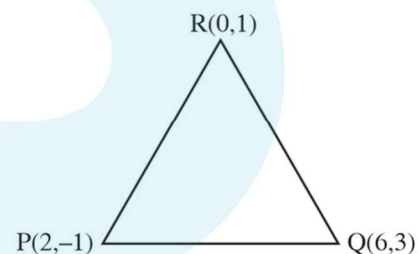
$$(x \cdot 3)^2 = 2x - 3$$

$$x^2 - 8x + 12 = 0$$

$$(x - 2)(x - 6) = 0$$

$$x = 2 \text{ or } 6$$

$$y = -1 \text{ or } 3$$



$$MPQ = \frac{1}{4} = 1$$

$$MQR = \frac{2}{6} = \frac{1}{3}$$

$$MPR = \frac{2}{6} = \frac{1}{3}$$

$$MPR = \frac{2}{-2} = -1$$

$$MPQ \times MPR = - \Rightarrow PQ \perp PR$$

$$\text{Orthocentre} = P(2, -1)$$

12. Let C be the centre of the circle $x^2 + y^2 - x + 2y = \frac{11}{4}$ and P be a point on the circle. A line passes through the point C, makes an angle of $\frac{\pi}{4}$ with the line CP and intersects the circle at the points Q and R. Then the area of the triangle PQR (in unit²) is :
- (A) 2 (B) $2\sqrt{2}$
 (C) $8\sin\left(\frac{\pi}{8}\right)$ (D) $8\cos\left(\frac{\pi}{8}\right)$

Official Ans. by NTA (B)

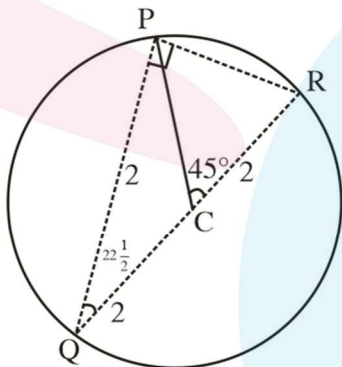
Ans. (B)

Sol. $x^2 + y^2 - x + 2y = \frac{11}{4}$

$$\left(x - \frac{1}{2}\right)^2 + (y+1)^2 = (2)^2$$

Or ΔPQR

$$PR = QK \sin 2 \geq \frac{1}{3}$$



$$= 4 \cdot 6 \sin \frac{\pi}{8}$$

$$PQ = QR \cos 22 \frac{1}{2}$$

$$= 4 \cos \frac{\pi}{8}$$

$$\text{As } \Delta PQR = \frac{1}{2} PR \times PQ$$

$$= \frac{1}{2} \left(4^2 \sin \frac{\pi}{6}\right) \left(4 \cos \frac{\pi}{8}\right)$$

$$= 4 \sin \frac{\pi}{4} = \frac{4}{\sqrt{2}} = 2\sqrt{2}$$

13. The remainder when $7^{2022} + 3^{2022}$ is divided by 5 is:
 (A) 0 (B) 2 (C) 3 (D) 4

Official Ans. by NTA (C)

Ans. (C)

Sol. $7^{2022} + 3^{2022}$

$$= (49)^{1011} + (9)^{1011}$$

$$= (50-1)^{1011} + (10-1)^{1011}$$

$$= 5\lambda - 1 + 5K - 1$$

$$= 5m - 2$$

$$\text{Remainder} = 5 - 2 = 3$$

14. Let the matrix $A = \begin{bmatrix} 0 & 1 & 0 \\ 0 & 0 & 1 \\ 1 & 0 & 0 \end{bmatrix}$ and the matrix

$$B_0 = A^{49} + 2A^{98}. \text{ If } B_n = \text{Adj}(B_{n-1}) \text{ for all } n \geq 1,$$

then $\det(B_4)$ is equal to :

- (A) 3^{28} (B) 3^{30} (C) 3^{32} (D) 3^{36}

Official Ans. by NTA (C)

Ans. (C)

Sol. $A^2 = \begin{bmatrix} 0 & 1 & 0 \\ 0 & 0 & 1 \\ 1 & 0 & 0 \end{bmatrix} \begin{bmatrix} 0 & 1 & 0 \\ 0 & 0 & 1 \\ 1 & 0 & 0 \end{bmatrix}$

$$= \begin{bmatrix} 0 & 0 & 1 \\ 1 & 0 & 0 \\ 0 & 1 & 0 \end{bmatrix}$$

$$a \leftrightarrow R_2$$

$$- \begin{bmatrix} 1 & 0 & 0 \\ 0 & 0 & 1 \\ 0 & 1 & 0 \end{bmatrix}$$

$$R_2 \leftrightarrow R_3$$

$$\begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix} = I$$

$$B_0 = A^{49} + 2A^{98}$$

$$= A + 2I$$

$$B_n = \text{Adj}(B_n - 1)$$

$$B_4 = \text{Adj}(\text{Adj}(\text{Adj}(\text{Adj} B_0)))$$

$$= |B_0|^{(n-1)^4}$$

$$= |B_0|^{16}$$

$$B_0 = \begin{bmatrix} 0 & 1 & 0 \\ 0 & 0 & 1 \\ 1 & 0 & 0 \end{bmatrix} + \begin{bmatrix} 2 & 0 & 0 \\ 0 & 2 & 0 \\ 0 & 0 & 2 \end{bmatrix}$$

$$= \begin{bmatrix} 2 & 1 & 0 \\ 0 & 2 & 1 \\ 1 & 0 & 2 \end{bmatrix}$$

$$= 2(4-0) - 1(0-1)$$

$$= 9$$

$$B_4(9)^{16} = (3)^{32}$$

15. Let $S_1 = \left\{ z_1 \in \mathbb{C} : |z_1 - 3| = \frac{1}{2} \right\}$ and

$S_2 = \left\{ z_2 \in \mathbb{C} : |z_2 - |z_2 + 1|| = |z_2 + |z_2 - 1|| \right\}$. Then,

for $z_1 \in S_1$ and $z_2 \in S_2$, the least value of $|z_2 - z_1|$ is :

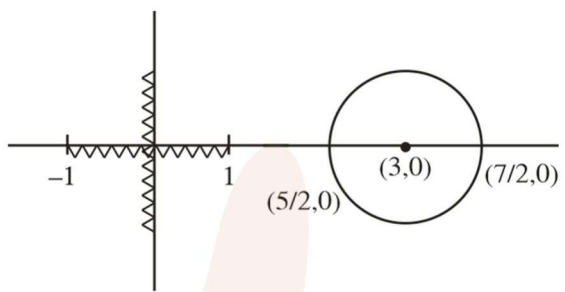
- (A) 0 (B) $\frac{1}{2}$ (C) $\frac{3}{2}$ (D) $\frac{5}{2}$

Official Ans. by NTA (C)

Ans. (C)

Sol. $|z_2 + |z_2 - 1||^2 = |z_2 - |z_2 + 1||^2$
 $\Rightarrow |z_2 + |z_2 - 1||(\bar{z}_2 + |z_2 - 1|) = (z_2 - |z_2 + 1|)(\bar{z}_2 - (z_2 + 1))$
 $\Rightarrow z_2|\bar{z}_2 + |z_2 - 1| - (\bar{z}_2 - |z_2 + 1|) + \bar{z}_2(|z_2 - 1| + |z_2 + 1|)$
 $= |z_2 + 1|^2 = |z_2 - 1|^2$
 $\Rightarrow [z_2 + \bar{z}_2](|z_2 - 1|) + (z_2 + 1) = 2(z_2 + \bar{z}_2)$
 $\Rightarrow (z_2 + \bar{z}_2)(|z_2 - 1| + |z_2 + 1| - 2) = 0$
 $\therefore z_2 + \bar{z}_2 = 0$ or $|z_2 - 1| + |z_2 + 1| - 2 = 0$
 $\therefore z_2$ lie on imaginary axis. Or on real axis with in $[-1, 1]$

Also $|z_1 - 3| = \frac{1}{2}$ lie on circle having centre 3 and radius $\frac{1}{2}$.



Clearly $|z_1 - z_2|_{\min} = \frac{5}{2} - 1 = \frac{3}{2}$

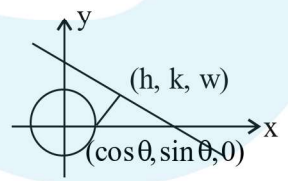
16. The foot of the perpendicular from a point on the circle $x^2 + y^2 = 1, z = 0$ to the plane $2x + 3y + z = 6$ lies on which one of the following curves ?

- (A) $(6x + 5y - 12)^2 + 4(3x + 7y - 8)^2 = 1, z = 6 - 2x - 3y$
 (B) $(5x + 6y - 12)^2 + 4(3x + 5y - 9)^2 = 1, z = 6 - 2x - 3y$
 (C) $(6x + 5y - 14)^2 + 9(3x + 5y - 7)^2 = 1, z = 6 - 2x - 3y$
 (D) $(5x + 6y - 14)^2 + 9(3x + 7y - 8)^2 = 1, z = 6 - 2x - 3y$

Official Ans. by NTA (B)

Ans. (B)

Sol.



$$\frac{h - \cos \theta}{2} = \frac{k - \sin \theta}{3} = \frac{w - 0}{1}$$

$$= \frac{-1(2 \cos \theta + 3 \sin \theta - 6)}{14}$$

$$h = \cos \frac{-2(2 \cos \theta + 3 \sin \theta - 6)}{14}$$

$$= \frac{10 \cos \theta - 6 \sin \theta + 12}{14}$$

$$k = \sin \theta - \frac{3}{14}(2 \cos \theta + 3 \sin \theta - 6)$$

$$k = \frac{5 \sin \theta - 6 \cos \theta + 18}{14}$$

Elementary $\sin \theta$ and $\cos \theta$

$$(5h + 6k - 12)^2 + 4(3h + 5k - 9)^2 = 1$$

17. If the minimum value of $f(x) = \frac{5x^2}{2} + \frac{\alpha}{x^5}$, $x > 0$, is

- 14, then the value of α is equal to :
 (A) 32 (B) 64
 (C) 128 (D) 256

Official Ans. by NTA (C)

Ans. (C)

Sol. $\frac{x^2}{2} + \frac{x^2}{2} + \frac{x^2}{2} + \frac{x^2}{2} + \frac{x^2}{2} + \frac{\alpha}{2x^5} + \frac{\alpha}{2x^5}$
 $\geq 7 \left(\frac{\alpha^2}{2^7} \right)^{\frac{1}{7}}$
 $\frac{7 \cdot (\alpha)^{2/7}}{2} = 14$
 $(\alpha^2)^{1/7} = 2^2$
 $\alpha = (2^2)^{7/2} = 2^7$
 $\alpha = 128$

18. Let α, β and γ be three positive real numbers. Let $f(x) = \alpha x^5 + \beta x^3 + \gamma x$, $x \in \mathbb{R}$ and $g: \mathbb{R} \rightarrow \mathbb{R}$ be such that $g(f(x)) = x$ for all $x \in \mathbb{R}$. If $a_1, a_2, a_3, \dots, a_n$ be in arithmetic progression with mean zero, then

the value of $f \left(g \left(\frac{1}{n} \sum_{i=1}^n f(a_i) \right) \right)$ is equal to :
 (A) 0 (B) 3
 (C) 9 (D) 27

Official Ans. by NTA (A)

Ans. (A)

Sol. Consider a case when $\alpha = \beta = 0$ then
 $f(x) = yx$
 $g(x) = \frac{x}{y}$
 $\frac{1}{n} \sum_{i=1}^n f(a_i) \Rightarrow \frac{y}{n} (a_1 + a_2 + \dots + a_n)$
 $= 0$
 $\Rightarrow f(g(0)) \Rightarrow f(0)$
 $\Rightarrow 0$

19. Consider the sequence a_1, a_2, a_3, \dots such that

$a_1 = 1, a_2 = 2$ and $a_{n+2} = \frac{2}{a_{n+1}} + a_n$ for $n = 1, 2, 3, \dots$

If $\left(\frac{a_1 + \frac{1}{a_2}}{a_3} \right) \cdot \left(\frac{a_2 + \frac{1}{a_3}}{a_4} \right) \cdot \left(\frac{a_3 + \frac{1}{a_4}}{a_5} \right) \dots \left(\frac{a_{30} + \frac{1}{a_{31}}}{a_{32}} \right) = 2^\alpha \binom{61}{31}$,

- then α is equal to :
 (A) -30 (B) -31
 (C) -60 (D) -61

Official Ans. by NTA (C)

Ans. (C)

Sol. $a_{n+2} a_{n+1} - a_{n+1} a_n = 2$

Series will satisfy
 $a_1 a_2, a_2 a_3, a_3 a_4, a_4 a_5,$
 1.2 2.2 2.3 2.4

$$\frac{a_n + \frac{1}{a_{n+1}}}{a_{n+2}} = \frac{a_{n+2} - \frac{1}{a_{n+1}}}{a_{n+2}}$$

$$= 1 - \frac{1}{a_{n+1} a_{n+2}}$$

$$= 1 - \frac{1}{2(r+1)}$$

$$= \frac{2r+1}{2(r+1)}$$

Now proof is given by

$$= \prod_{r=1}^{30} \frac{(2r+1)}{2(r+1)}$$

$$= \frac{(1 \cdot 3 \cdot 5 \cdot \dots \cdot 61)}{2^{30} \cdot (2 \cdot 3 \cdot \dots \cdot 31)}$$

$$\Rightarrow \frac{(1 \cdot 3 \cdot 5 \cdot \dots \cdot 61)}{31 \cdot 2^{30}} \times \frac{2^{30} \times 30}{2^{30} \times 30}$$

$$= \frac{61}{2^{60} \cdot 31 \cdot 30}$$

$\alpha = -60$

20. The minimum value of the twice differentiable function $f(x) = \int_0^x e^{x-t} f'(t) dt - (x^2 - x + 1)e^x$, $x \in \mathbb{R}$, is :

- (A) $-\frac{2}{\sqrt{e}}$ (B) $-2\sqrt{e}$
 (C) $-\sqrt{e}$ (D) $\frac{2}{\sqrt{e}}$

Official Ans. by NTA (A)

Ans. (A)

Sol. $f(x) = e^x \cdot \int_0^x \frac{f'(t)}{e^t} dt$
 $f'(x) = e^x \cdot \int_0^x \frac{f'(t)}{e^t} dt + e^x \cdot \frac{f'(x)}{e^x}$
 $-[(2x-1) \cdot e^x + (x^2 - x + 1) \cdot e^x]$
 $\int_0^x \frac{f'(t)}{e^t} dt = x^2 + x$
 $\frac{f'(x)}{e^x} = 2x + 1$
 $f'(x) = (2x + 1) \cdot e^x$
 $f'(x) = 0 \Rightarrow x = -\frac{1}{2}$
 $f(x) = (2x + 1) \cdot e^x - 2e^x + C$
 $\left. \begin{array}{l} \\ \\ \\ \end{array} \right\} f(0) = -1$
 $-1 = 1 - 2 + C$
 $C = 0$
 $f(x) = e^x(2x - 1)$
 $f\left(-\frac{1}{2}\right) = \frac{-2}{\sqrt{e}}$

SECTION-B

1. Let S be the set of all passwords which are six to eight characters long, where each character is either an alphabet from {A, B, C, D, E} or a number from {1, 2, 3, 4, 5} with the repetition of characters allowed. If the number of passwords in S whose at least one character is a number from {1, 2, 3, 4, 5} is $\alpha \times 5^6$, then α is equal to _____.

Official Ans. by NTA (7073)

Ans. (7073)

Sol. Required no. = Total - no character from {1, 2, 3, 4, 5}
 $= (10^6 - 5^6) + (10^7 - 5^7) + (10^8 - 5^8)$
 $= 10^6(1 + 10 + 100) - 5^6(1 + 5 + 25)$
 $= 10^6 \times 111 - 5^6 \times 31$
 $= 2^6 \times 5^6 \times 111 - 5^6 \times 31$
 $= 5^6(2^6 \times 111 - 31)$
 $= 5^6 \times \underbrace{7073}_\alpha$
 $\therefore \alpha = 7073$

2. Let P(-2, -1, 1) and Q($\frac{56}{17}, \frac{43}{17}, \frac{111}{17}$) be the vertices of the rhombus PRQS. If the direction ratios of the diagonal RS are $\alpha, -1, \beta$, where both α and β are integers of minimum absolute values, then $\alpha^2 + \beta^2$ is equal to _____.

Official Ans. by NTA (450)

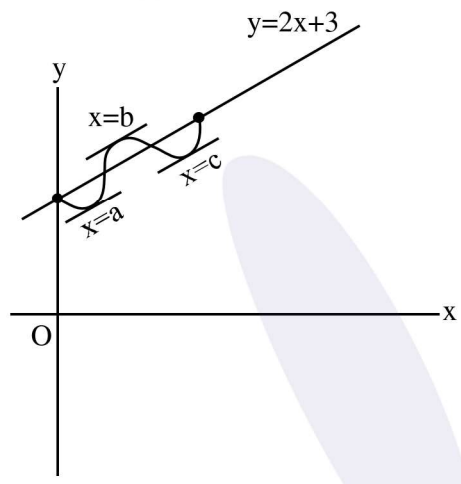
Ans. (450)

Sol. $RS \equiv (\alpha, -1, \beta)$
 DR of PQ $\equiv \left(\frac{56}{17} + 2, \frac{43}{17} + 1, \frac{111}{17} - 1\right)$
 $\equiv \left(\frac{90}{17}, \frac{60}{17}, \frac{94}{17}\right)$
 $\frac{90}{17}\alpha + \frac{60}{17}(-1) + \frac{94}{17}\beta = 0$
 $90\alpha + 94\beta = 60$
 $\beta = \frac{60 - 90\alpha}{94}$
 $\beta = \frac{30(2 - 3\alpha)}{94}$
 $\beta = -30 \frac{(3\alpha - 2)}{94}$
 $\beta = \frac{-15}{47}(3\alpha - 2)$
 $\Rightarrow \frac{\beta}{-15} = \frac{3\alpha - 2}{47}$
 $\Rightarrow \beta = -15, \alpha = -15$
 $\alpha^2 + \beta^2 = 225 + 225$
 $= 450$

3. Let $f : [0, 1] \rightarrow \mathbb{R}$ be a twice differentiable function in $(0, 1)$ such that $f(0) = 3$ and $f(1) = 5$. If the line $y = 2x + 3$ intersects the graph of f at only two distinct points in $(0, 1)$, then the least number of points $x \in (0, 1)$, at which $f''(x) = 0$, is _____.

Official Ans. by NTA (2)
Ans. (2)

Sol.



$f'(a) = f'(b) = f'(c) = 2$
 $\Rightarrow f''(x)$ is zero

for atleast $x_1 \in (a, b)$ & $x_2 \in (b, c)$

4. If $\int_0^{\sqrt{3}} \frac{15x^3}{\sqrt{1+x^2} + \sqrt{(1+x^2)^3}} dx = \alpha\sqrt{2} + \beta\sqrt{3}$, where

α, β are integers, then $\alpha + \beta$ is equal to

Official Ans. by NTA (10)
Ans. (10)

Sol.

Put $1 + x^2 = t^2$
 $2x dx = 2t dt$
 $X dx = t dt$
 $\therefore \int_1^2 \frac{15(t^2 - 1)t dt}{\sqrt{t^2 + t^3}}$
 $15 \int_1^2 \frac{t(t^2 - 1)}{t\sqrt{1+t}} dt$
 Put $1 + t = u^2$
 $dt = 2u du$
 $15 \int_{\frac{\sqrt{3}}{2}}^{\sqrt{3}} \frac{(u^2 - 1)^2 - 1}{u} \times 2u du$
 $30 \int_{\frac{\sqrt{3}}{2}}^{\sqrt{3}} (u^4 - 2u^2) du$
 $30 \left(\frac{u^5}{5} - \frac{2u^3}{3} \right)_{\frac{\sqrt{3}}{2}}^{\sqrt{3}}$

$$30 \left[\frac{1}{5} (\sqrt{3}^5 - \sqrt{2}^5) - \frac{2}{3} (\sqrt{3}^3 - \sqrt{2}^3) \right]$$

$$30 \left[\frac{1}{5} (9\sqrt{3} - 4\sqrt{2}) - \frac{2}{3} (3\sqrt{3} - 2\sqrt{2}) \right]$$

$$30 \left[-\frac{1}{5} \times \sqrt{3} + \frac{8}{15} \sqrt{2} \right]$$

$$-6\sqrt{3} + 16\sqrt{2} = \alpha\sqrt{2} + \beta\sqrt{3}$$

$$\alpha = 16, \beta = -6$$

$$\therefore \alpha + \beta = 10$$

5. Let $A = \begin{bmatrix} 1 & -1 \\ 2 & \alpha \end{bmatrix}$ and $B = \begin{bmatrix} \beta & 1 \\ 1 & 0 \end{bmatrix}$, $\alpha, \beta \in \mathbb{R}$. Let α_1 be the value of α which satisfies $(A+B)^2 = A^2 + \begin{bmatrix} 2 & 2 \\ 2 & 2 \end{bmatrix}$ and α_2 be the value of α which satisfies $(A+B)^2 = B^2$. Then $|\alpha_1 - \alpha_2|$ is equal to _____.

Official Ans. by NTA (2)

Ans. (2)

Sol.

$$A+B = \begin{bmatrix} \beta+1 & 0 \\ 3 & \alpha \end{bmatrix}$$

$$(A+B)^2 = \begin{bmatrix} \beta+1 & 0 \\ 3 & \alpha \end{bmatrix} \begin{bmatrix} \beta+1 & 0 \\ 3 & \alpha \end{bmatrix}$$

$$= \begin{bmatrix} (\beta+1)^2 & 0 \\ 3(\beta+1) + 3\alpha & \alpha^2 \end{bmatrix}$$

$$A^2 = \begin{bmatrix} 1 & -1 \\ 2 & \alpha \end{bmatrix} \begin{bmatrix} 1 & -1 \\ 2 & \alpha \end{bmatrix}$$

$$= \begin{bmatrix} -1 & -1-\alpha \\ 2+2\alpha & \alpha^2-2 \end{bmatrix}$$

$$\therefore \begin{bmatrix} 1 & -\alpha+1 \\ 2\alpha+4 & \alpha^2 \end{bmatrix} = \begin{bmatrix} (\beta+1)^2 & 0 \\ 3(\alpha+\beta+1) & \alpha^2 \end{bmatrix}$$

$$\alpha = 1 = \alpha_1$$

$$B^2 = \begin{bmatrix} \beta & 1 \\ 1 & 0 \end{bmatrix} \begin{bmatrix} \beta & 1 \\ 1 & 0 \end{bmatrix}$$

$$= \begin{bmatrix} \beta^2+1 & \beta \\ \beta & 1 \end{bmatrix} = \begin{bmatrix} (\beta+1)^2 & 0 \\ 3(\beta+1) + 3\alpha & \alpha^2 \end{bmatrix}$$

$$\therefore \beta = 0, \alpha = -1 = \alpha_2$$

$$|\alpha_1 - \alpha_2| = |1 - (-1)| = 2$$

6. For $p, q \in \mathbb{R}$, consider the real valued function $f(x) = (x - p)^2 - q$, $x \in \mathbb{R}$ and $q > 0$. Let a_1, a_2, a_3 and a_4 be in an arithmetic progression with mean p and positive common difference. If $|f(a_i)| = 500$ for all $i = 1, 2, 3, 4$, then the absolute difference between the roots of $f(x) = 0$ is

Official Ans. by NTA (50)

Ans. (50)

Sol. $f(x) = 0 \Rightarrow (x - p)^2 - q = 0$.

Roots are $p + \sqrt{q}$, $p - \sqrt{q}$ absolute difference between roots $2\sqrt{q}$.

Now, $|f(a_i)| = 500$

Let a_1, a_2, a_3, a_4 are a_1 $a + d$, $a + 2d$, $a + 3d$

$|f(a_4)| = 500$

$|(a_1 - p)^2 - q| = 500$

$\Rightarrow (a_1 - p)^2 - q = 500$

$\Rightarrow \frac{9}{4}d^2 - q = 500$ _____ (1)

and $|f(a_1)|^2 = |f(a_2)|^2$

$((a_1 - p)^2 - q)^2 = ((a_2 - p)^2 - q)^2$

$\Rightarrow ((a_1 - p)^2 - (a_2 - p)^2) ((a_1 - p)^2 - q + (a_2 - p)^2 - q) = 0$

$\Rightarrow \frac{9}{4}d^2 - q + \frac{d^2}{4} - q = 0$

$2q = \frac{10d^2}{4} \Rightarrow q = \frac{5d^2}{4}$

$\Rightarrow d^2 = \frac{4q}{5}$

From equation (1) $\frac{9}{4} \cdot \frac{4q}{5} - q = 500$

$\frac{4q}{5} = 500$

$\frac{4q}{5} = 500$

and $2\sqrt{q} = 2 \times \frac{50}{2} = 50$

7. For the hyperbola $H : x^2 - y^2 = 1$ and the ellipse

$E : \frac{x^2}{a^2} + \frac{y^2}{b^2} = 1$, $a > b > 0$, let the

- (1) eccentricity of E be reciprocal of the eccentricity of H , and

- (2) the line $y = \sqrt{\frac{5}{2}}x + K$ be a common tangent of E and H .

Then $4(a^2 + b^2)$ is equal to _____.

Official Ans. by NTA (3)

Ans. (3)

Sol. $e_E = \sqrt{1 - \frac{b^2}{a^2}}$, $e_H = \sqrt{2}$

If $\Rightarrow e_E = \frac{1}{e_H}$

$\Rightarrow \frac{a^2 - b^2}{a^2} = \frac{1}{2}$

$2a^2 - 2b^2 = a^2$

$a^2 = 2b^2$

and $y = \sqrt{\frac{5}{2}}x + k$ is tangent to ellipse then

$K^2 = a^2 \times \frac{5}{2} + b^2 = \frac{3}{2}$

$6b^2 = \frac{3}{2} \Rightarrow b^2 = \frac{1}{4}$ and $a^2 = \frac{1}{2}$

$\therefore 4(a^2 + b^2) = 3$

8. Let $x_1, x_2, x_3, \dots, x_{20}$ be in geometric progression

with $x_1 = 3$ and the common ratio $\frac{1}{2}$. A new data

is constructed replacing each x_i by $(x_i - i)^2$. If \bar{x} is the mean of new data, then the greatest integer less than or equal to \bar{x} is _____.

Official Ans. by NTA (142)

Ans. (142)

Sol.
$$\sum x_0^1 = \frac{3\left(1 - \left(\frac{1}{2}\right)\right)^{20}}{1 - \frac{1}{2}} = 6\left(1 - \frac{1}{2^{20}}\right)$$

$$= \sum_{i=1}^{20} (x_{i-1})^2$$

$$= \sum_{i=1}^{20} (x_i)^2 + (i)^2 - 2x_i i$$

Now
$$= \sum_{i=1}^{20} (x_i)^2 = \frac{9\left(1 - \left(\frac{1}{4}\right)\right)^{20}}{1 - \frac{1}{4}} = 12\left(1 - \frac{1}{2^{40}}\right)$$

$$\sum_{i=1}^{20} i^2 = \frac{1}{6} \times 20 \times 21 \times 41 = 2870$$

$$\sum_{i=1}^{20} x_i i = s = 3 + 2.3 \frac{1}{2} + 3.3 \frac{1}{2^2} + 4.3 \frac{1}{2^3} + \dots \text{AGP}$$

$$= 6\left(2 - \frac{22}{2^{20}}\right)$$

$$\bar{x} = \frac{12 - \frac{12}{2^{40}} + 2870 - 12\left(2 - \frac{22}{2^{20}}\right)}{20}$$

$$\bar{x} = \frac{2858}{20} + \left(\frac{-12}{2^{40}} + \frac{22}{2^{20}}\right) \times \frac{1}{20}$$

$$[\bar{x}] = 142$$

9.
$$\lim_{x \rightarrow 0} \left(\frac{(x+2\cos x)^3 + 2(x+2\cos x)^2 + 3\sin(x+2\cos x)}{(x+2)^3 + 2(x+2)^2 + 3\sin(x+2)} \right)^{\frac{100}{x}}$$

is equal to _____.

Official Ans. by NTA (1)

Ans. (1)

Sol.

$$\lim_{x \rightarrow 0} \left(\frac{(x+2\cos x)^3 + 2(x+2\cos x)^2 + 3\sin(x+2\cos x)}{(x+2)^3 + 2(x+2)^2 + 3\sin(x+2)} \right)^x$$

Form 1^∞

$$= e^{\lim_{x \rightarrow 0} \left[\left(\frac{(x+2\cos x)^3 + 2(x+2\cos x)^2 + 3\sin(x+2\cos x)}{(x+2)^3 + 2(x+2)^2 + 3\sin(x+2)} \right)^{-1} \right] \times \frac{100}{x}}$$

$$= e^{\lim_{x \rightarrow 0} \left[\frac{100}{x} \left(\frac{(x+2\cos x)^3 + 2(x+2\cos x)^2 + 3\sin(x+2\cos x) - ((x+2)^3 + 2(x+2)^2 + 3\sin(x+2))}{(x+2)^3 + 2(x+2)^2 + 3\sin(x+2)} \right) \right]}$$

$$= e^{\lim_{x \rightarrow 0} \frac{100}{x} \left[\frac{(x+2\cos x)^3 + (x+2)^3 + 2(x+2\cos x)^2 - 2(x+2)^2 + 3\sin(x+2\cos x) - 3\sin(x+2)}{8+8+3\sin^2} \right]}$$

$$= e^{\frac{100}{16+3\sin^2} \lim_{x \rightarrow 0} \frac{3(x+2\cos x)^2 \times (1+2\sin x) - 3(x+2)^2 - 4(x+2\cos x)}{x(1-2\sin x) - 4(x+2) + 3\cos(x+2\cos x) \times (1-2\sin x) - 3\cos(x+2)}}$$

$$= e^{\frac{100}{16+3\sin^2} \left(\frac{12 - 3(4) + 8 \times 1 - 8 + 3\cos 2 - 3\cos 2}{1} \right)}$$

Using L'H rule.

$$= e^0 = 1$$

10. The sum of all real values of x for which

$$\frac{3x^2 - 9x + 17}{x^2 + 3x + 10} = \frac{5x^2 - 7x + 19}{3x^2 + 5x + 12} \text{ is equal to } \underline{\hspace{2cm}}.$$

Official Ans. by NTA (6)

Ans. (6)

Sol.
$$\frac{3x^2 - 9x + 17}{x^2 + 3x + 10} = \frac{5x^2 - 7x + 19}{3x^2 + 5x + 12}$$

$$\frac{x^2 + 3x + 10 + 2x^2 - 12x + 7}{x^2 + 3x + 10} = \frac{3x^2 + 5x + 12 + 2x^2 - 12x + 7}{3x^2 + 5x + 12}$$

$$1 + \frac{2x^2 - 12x + 7}{x^2 + 3x + 10} = 1 + \frac{2x^2 - 12x + 7}{3x^2 + 5x + 12}$$

$$(2x^2 - 12x + 7) \left(\frac{1}{x^2 + 3x + 10} - \frac{1}{3x^2 + 5x + 12} \right) = 0$$

$$2x^2 - 12x + 7 = 0 \text{ OR } 3x^2 + 5x + 12 = x^2 + 3x + 10$$

$$x = \frac{12 \pm \sqrt{D}}{4}$$

$$2x^2 + 2x + 2 = 0$$

$$x^2 + x + 1 = 0$$

Sum of Roots = 6

No solution.