

FINAL JEE-MAIN EXAMINATION – JANUARY, 2023

(Held On Wednesday 25th January, 2023)

TIME : 9 : 00 AM to 12 : 00 NOON

MATHEMATICS

TEST PAPER WITH SOLUTION

SECTION-A

61. Let M be the maximum value of the product of two positive integers when their sum is 66. Let the sample space $S = \left\{ x \in \mathbb{Z} : x(66 - x) \geq \frac{5}{9}M \right\}$ and the event $A = \{x \in S : x \text{ is a multiple of } 3\}$. Then P(A) is equal to

- (1) $\frac{15}{44}$ (2) $\frac{1}{3}$
 (3) $\frac{1}{5}$ (4) $\frac{7}{22}$

Official Ans. by NTA (2)

Ans. (2)

Sol. $M = 33 \times 33$
 $x(66 - x) \geq \frac{5}{9} \times 33 \times 33$
 $11 \leq x \leq 55$
 $A : \{12, 15, 18, \dots, 54\}$
 $P(A) = \frac{15}{45} = \frac{1}{3}$

62. Let \vec{a} , \vec{b} and \vec{c} be three non zero vectors such that $\vec{b} \cdot \vec{c} = 0$ and $\vec{a} \times (\vec{b} \times \vec{c}) = \frac{\vec{b} - \vec{c}}{2}$. If \vec{d} be a vector such that $\vec{b} \cdot \vec{d} = \vec{a} \cdot \vec{b}$, then $(\vec{a} \times \vec{b}) \cdot (\vec{c} \times \vec{d})$ is equal to

- (1) $\frac{3}{4}$ (2) $\frac{1}{2}$
 (3) $-\frac{1}{4}$ (4) $\frac{1}{4}$

Official Ans. by NTA (4)

Ans. (4)

Sol. $(\vec{a} \cdot \vec{c})\vec{b} - (\vec{a} \cdot \vec{b})\vec{c} = \frac{\vec{b} - \vec{c}}{2}$
 $\vec{a} \cdot \vec{c} = \frac{1}{2}, \vec{a} \cdot \vec{b} = \frac{1}{2}$
 $\therefore \vec{b} \cdot \vec{d} = \frac{1}{2}$

$$(\vec{a} \times \vec{b}) \cdot (\vec{c} \times \vec{d}) = \vec{a} \cdot (\vec{b} \times (\vec{c} \times \vec{d}))$$

$$= \vec{a} \cdot ((\vec{b} \cdot \vec{d})\vec{c} - (\vec{b} \cdot \vec{c})\vec{d})$$

$$= (\vec{a} \cdot \vec{c})(\vec{b} \cdot \vec{d}) = \frac{1}{4}$$

63. Let $y = y(x)$ be the solution curve of the differential equation $\frac{dy}{dx} = \frac{y}{x}(1 + xy^2(1 + \log_e x))$, $x > 0, y(1) = 3$. Then $\frac{y^2(x)}{9}$ is equal to :

- (1) $\frac{x^2}{5 - 2x^3(2 + \log_e x^3)}$
 (2) $\frac{x^2}{2x^3(2 + \log_e x^3) - 3}$
 (3) $\frac{x^2}{3x^3(1 + \log_e x^2) - 2}$
 (4) $\frac{x^2}{7 - 3x^3(2 + \log_e x^2)}$

Official Ans. by NTA (1)

Ans. (1)

Sol. $\frac{dy}{dx} - \frac{y}{x} = y^3(1 + \log_e x)$
 $\frac{1}{y^3} \frac{dy}{dx} - \frac{1}{xy^2} = 1 + \log_e x$
 Let $-\frac{1}{y^2} = t \Rightarrow \frac{2}{y^3} \frac{dy}{dx} = \frac{dt}{dx}$
 $\therefore \frac{dt}{dx} + \frac{2t}{x} = 2(1 + \log_e x)$
 I.F. = $e^{\int \frac{2}{x} dx} = x^2$
 $\frac{-x^2}{y^2} = \frac{2}{3} \left((1 + \log_e x)x^3 - \frac{x^3}{3} \right) + C$
 $y(1) = 3$
 $\frac{y^2}{9} = \frac{x^2}{5 - 2x^3(2 + \log_e x^3)}$

OR

72. Let $z_1 = 2 + 3i$ and $z_2 = 3 + 4i$. The set
- $$S = \left\{ z \in \mathbb{C} : |z - z_1|^2 - |z - z_2|^2 = |z_1 - z_2|^2 \right\}$$
- represents a
- (1) straight line with sum of its intercepts on the coordinate axes equals 14
 - (2) hyperbola with the length of the transverse axis 7
 - (3) straight line with the sum of its intercepts on the coordinate axes equals -18
 - (4) hyperbola with eccentricity 2

Official Ans. by NTA (1)

Ans. (1)

Sol. $(x-2)^2 + (y-3)^2 - ((x-3)^2 - (y-4)^2) = 1+1$
 $\Rightarrow x + y = 7$

73. The distance of the point $(6, -2\sqrt{2})$ from the common tangent $y = mx + c$, $m > 0$, of the curves $x = 2y^2$ and $x = 1 + y^2$ is

(1) $\frac{1}{3}$

(2) 5

(3) $\frac{14}{3}$

(4) $5\sqrt{3}$

Official Ans. by NTA (2)

Ans. (2)

Sol. For

$$y^2 = \frac{x}{2}, T: y = mx + \frac{1}{8m}$$

For tangent to $y^2 + 1 = x$

$$\Rightarrow \left(mx + \frac{1}{8m} \right)^2 + 1 = x$$

$$D = 0 \Rightarrow m = \frac{1}{2\sqrt{2}}$$

$$\therefore T: x - 2\sqrt{2}y + 1 = 0$$

$$d = \left| \frac{6+8+1}{\sqrt{9}} \right| = 5$$

74. Let S_1 and S_2 be respectively the sets of all $a \in \mathbb{R} - \{0\}$ for which the system of linear equations

$$ax + 2ay - 3az = 1$$

$$(2a + 1)x + (2a + 3)y + (a + 1)z = 2$$

$$(3a + 5)x + (a + 5)y + (a + 2)z = 3$$

has unique solution and infinitely many solutions. Then

(1) $n(S_1) = 2$ and S_2 is an infinite set

(2) S_1 is an infinite set and $n(S_2) = 2$

(3) $S_1 = \Phi$ and $S_2 = \mathbb{R} - \{0\}$

(4) $S_1 = \mathbb{R} - \{0\}$ and $S_2 = \Phi$

Official Ans. by NTA (4)

Ans. (4)

Sol. $\Delta = \begin{vmatrix} a & 2a & -3a \\ 2a+1 & 2a+3 & a+1 \\ 3a+5 & a+5 & a+2 \end{vmatrix}$

$$= a(15a^2 + 31a + 36) = 0 \Rightarrow a = 0$$

$$\Delta \neq 0 \text{ for all } a \in \mathbb{R} - \{0\}$$

$$\text{Hence } S_1 = \mathbb{R} - \{0\} \quad S_2 = \Phi$$

75. Let $f(x) = \int \frac{2x}{(x^2+1)(x^2+3)} dx$.

If $f(3) = \frac{1}{2}(\log_e 5 - \log_e 6)$, then $f(4)$ is equal to

(1) $\frac{1}{2}(\log_e 17 - \log_e 19)$

(2) $\log_e 17 - \log_e 18$

(3) $\frac{1}{2}(\log_e 19 - \log_e 17)$

(4) $\log_e 19 - \log_e 20$

Official Ans. by NTA (1)

Allen Ans. (1)

Sol. Put $x^2 = t$

$$\int \frac{dt}{(t+1)(t+3)} = \frac{1}{2} \int \left(\frac{1}{t+1} - \frac{1}{t+3} \right) dt$$

$$f(x) = \frac{1}{2} \ln \left(\frac{x^2+1}{x^2+3} \right) + C$$

$$f(3) = \frac{1}{2}(\ln 10 - \ln 12) + C$$

$$\Rightarrow C = 0$$

$$f(4) = \frac{1}{2} \ln \left(\frac{17}{19} \right)$$

76. The statement $(p \wedge (\sim q)) \Rightarrow (p \Rightarrow (\sim q))$ is

- (1) equivalent to $(\sim p) \vee (\sim q)$
- (2) a tautology
- (3) equivalent to $p \vee q$
- (4) a contradiction

Official Ans. by NTA (2)

Ans. (2)

Sol. $(p \wedge \sim q) \rightarrow (p \rightarrow \sim q)$
 $\equiv (\sim(p \wedge \sim q)) \vee (\sim p \vee \sim q)$
 $\equiv (\sim p \vee q) \vee (\sim p \vee \sim q)$
 $\equiv \sim p \vee t \equiv t$

77. Let $f : (0,1) \rightarrow \mathbb{R}$ be a function defined by

$$f(x) = \frac{1}{1 - e^{-x}}, \text{ and}$$

$g(x) = (f(-x) - f(x))$. Consider two statements

- (I) g is an increasing function in $(0, 1)$
- (II) g is one-one in $(0, 1)$

Then,

- (1) Only (I) is true
- (2) Only (II) is true
- (3) Neither (I) nor (II) is true
- (4) Both (I) and (II) are true

Official Ans. by NTA (4)

Ans. (4)

Sol. $g(x) = f(-x) - f(x) = \frac{1 + e^x}{1 - e^x}$

$$\Rightarrow g'(x) = \frac{2e^x}{(1 - e^x)^2} > 0$$

$\Rightarrow g$ is increasing in $(0, 1)$

$\Rightarrow g$ is one-one in $(0, 1)$

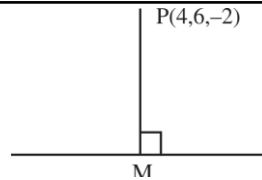
78. The distance of the point $P(4, 6, -2)$ from the line passing through the point $(-3, 2, 3)$ and parallel to a line with direction ratios $3, 3, -1$ is equal to :

- (1) 3
- (2) $\sqrt{6}$
- (3) $2\sqrt{3}$
- (4) $\sqrt{14}$

Official Ans. by NTA (4)

Ans. (4)

Sol.



$$\text{Equation of line is } \frac{x+3}{3} = \frac{y-2}{3} = \frac{z-3}{-1} = \lambda$$

$$M(3\lambda - 3, 3\lambda + 2, 3 - \lambda)$$

D.R of $PM(3\lambda - 7, 3\lambda - 4, 5 - \lambda)$

$$\text{Since } PM \text{ is perpendicular to line}$$

$$\Rightarrow 3(3\lambda - 7) + 3(3\lambda - 4) - 1(5 - \lambda) = 0$$

$$\Rightarrow \lambda = 2$$

$$\Rightarrow M(3, 8, 1) \Rightarrow PM = \sqrt{14}$$

79. Let $x, y, z > 1$ and

$$A = \begin{bmatrix} 1 & \log_x y & \log_x z \\ \log_y x & 2 & \log_y z \\ \log_z x & \log_z y & 3 \end{bmatrix}$$

Then $|\text{adj}(\text{adj } A^2)|$ is equal to

- (1) 6^4
- (2) 2^8
- (3) 4^8
- (4) 2^4

Official Ans. by NTA (2)

Ans. (2)

Sol. $|A| = \frac{1}{\log x \cdot \log y \cdot \log z} \begin{vmatrix} \log x & \log y & \log z \\ \log x & 2\log y & \log z \\ \log x & \log y & 3\log z \end{vmatrix} = 2$

$$\Rightarrow |\text{adj}(\text{adj } A^2)| = |A^2|^4 = 2^8$$

80. If a_r is the coefficient of x^{10-r} in the Binomial

expansion of $(1+x)^{10}$, then $\sum_{r=1}^{10} r^3 \left(\frac{a_r}{a_{r-1}} \right)^2$ is equal

to

- (1) 4895
- (2) 1210
- (3) 5445
- (4) 3025

Official Ans. by NTA (2)

Ans. (2)

Sol. $a_r = {}^{10}C_{10-r} = {}^{10}C_r$

$$\Rightarrow \sum_{r=1}^{10} r^3 \left(\frac{{}^{10}C_r}{{}^{10}C_{r-1}} \right)^2 = \sum_{r=1}^{10} r^3 \left(\frac{11-r}{r} \right)^2 = \sum_{r=1}^{10} r(11-r)^2$$

$$= \sum_{r=1}^{10} (121r + r^3 - 22r^2) = 1210$$

SECTION-B

81. Let $S = \{1, 2, 3, 5, 7, 10, 11\}$. The number of non-empty subsets of S that have the sum of all elements a multiple of 3, is _____.

Official Ans. by NTA (43)

Ans. (43)

Sol. Elements of the type $3k = 3$

Elements of the type $3k + 1 = 1, 7, 9$

Elements of the type $3k + 2 = 2, 5, 11$

Subsets containing one element $S_1 = 1$

Subsets containing two elements

$$S_2 = {}^3C_1 \times {}^3C_1 = 9$$

Subsets containing three elements

$$S_3 = {}^3C_1 \times {}^3C_1 + 1 + 1 = 11$$

Subsets containing four elements

$$S_4 = {}^3C_3 + {}^3C_3 + {}^3C_2 \times {}^3C_2 = 11$$

Subsets containing five elements

$$S_5 = {}^3C_2 \times {}^3C_2 \times 1 = 9$$

Subsets containing six elements $S_6 = 1$

Subsets containing seven elements $S_7 = 1$

$$\Rightarrow \text{sum} = 43$$

82. For some $a, b, c \in \mathbb{N}$, let $f(x) = ax - 3$ and

$$g(x) = x^b + c, x \in \mathbb{R}. \text{ If } (f \circ g)^{-1}(x) = \left(\frac{x-7}{2}\right)^{1/3},$$

then $(f \circ g)(ac) + (g \circ f)(b)$ is equal to _____.

Official Ans. by NTA (2039)

Ans. (2039)

Sol. Let $f \circ g(x) = h(x)$

$$\Rightarrow h^{-1}(x) = \left(\frac{x-7}{2}\right)^{1/3}$$

$$\Rightarrow h(x) = f \circ g(x) = 2x^3 + 7$$

$$f \circ g(x) = a(x^b + c) - 3$$

$$\Rightarrow a = 2, b = 3, c = 7$$

$$\Rightarrow f \circ g(ac) = f \circ g(10) = 2007$$

$$g(f(x)) = (2x - 3)^3 + 7$$

$$\Rightarrow g \circ f(b) = g \circ f(3) = 32$$

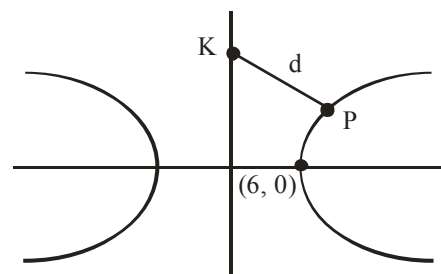
$$\Rightarrow \text{sum} = 2039$$

83. The vertices of a hyperbola H are $(\pm 6, 0)$ and its eccentricity is $\frac{\sqrt{5}}{2}$. Let N be the normal to H at a point in the first quadrant and parallel to the line $\sqrt{2}x + y = 2\sqrt{2}$. If d is the length of the line segment of N between H and the y -axis then d^2 is equal to _____.

Official Ans. by NTA (216)

Ans. (216)

Sol.



$$H: \frac{x^2}{36} - \frac{y^2}{9} = 1$$

$$\text{equation of normal is } 6x \cos\theta + 3y \cot\theta = 45$$

$$\text{slope} = -2 \sin\theta = -\sqrt{2}$$

$$\Rightarrow \theta = \frac{\pi}{4}$$

$$\text{Equation of normal is } \sqrt{2}x + y = 15$$

$$P: (a \sec\theta, b \tan\theta)$$

$$\Rightarrow P(6\sqrt{2}, 3) \text{ and } K(0, 15)$$

$$d^2 = 216$$

84. Let

$$S = \left\{ \alpha : \log_2(9^{2\alpha-4} + 13) - \log_2\left(\frac{5}{2} \cdot 3^{2\alpha-4} + 1\right) = 2 \right\}.$$

Then the maximum value of β for which the

$$\text{equation } x^2 - 2\left(\sum_{\alpha \in S} \alpha\right)^2 x + \sum_{\alpha \in S} (\alpha + 1)^2 \beta = 0 \text{ has}$$

real roots, is _____ .

Official Ans. by NTA (25)

Ans. (25)

Sol. $\log_2(9^{2\alpha-4} + 13) - \log_2\left(\frac{5}{2} \cdot 3^{2\alpha-4} + 1\right) = 2$

$$\Rightarrow \frac{9^{2\alpha-4} + 13}{\frac{5}{2} \cdot 3^{2\alpha-4} + 1} = 4$$

$$\Rightarrow \alpha = 2 \quad \text{or} \quad 3$$

$$\sum_{\alpha \in S} \alpha = 5 \quad \text{and} \quad \sum_{\alpha \in S} (\alpha + 1)^2 = 25$$

$$\Rightarrow x^2 - 50x + 25\beta = 0 \text{ has real roots}$$

$$\Rightarrow \beta \leq 25$$

$$\Rightarrow \beta_{\max} = 25$$

85. The constant term in the expansion of

$$\left(2x + \frac{1}{x^7} + 3x^2\right)^5 \text{ is } \underline{\hspace{2cm}}.$$

Official Ans. by NTA (1080)

Ans. (1080)

Sol. General term is $\sum \frac{5!(2x)^{n_1}(x^{-7})^{n_2}(3x^2)^{n_3}}{n_1! n_2! n_3!}$

For constant term,

$$n_1 + 2n_3 = 7n_2$$

$$\& n_1 + n_2 + n_3 = 5$$

Only possibility $n_1 = 1, n_2 = 1, n_3 = 3$

$$\Rightarrow \text{constant term} = 1080$$

86. Let A_1, A_2, A_3 be the three A.P. with the same common difference d and having their first terms as $A, A + 1, A + 2$, respectively. Let a, b, c be the 7th, 9th, 17th terms of A_1, A_2, A_3 , respectively such

$$\text{that } \begin{vmatrix} a & 7 & 1 \\ 2b & 17 & 1 \\ c & 17 & 1 \end{vmatrix} + 70 = 0$$

If $a = 29$, then the sum of first 20 terms of an AP whose first term is $c - a - b$ and common difference is $\frac{d}{12}$, is equal to _____ .

Official Ans. by NTA (495)

Ans. (495)

Sol. $\begin{vmatrix} A + 6d & 7 & 1 \\ 2(A + 1 + 8d) & 17 & 1 \\ A + 2 + 16d & 17 & 1 \end{vmatrix} + 70 = 0$

$$\Rightarrow A = -7 \text{ and } d = 6$$

$$\therefore c - a - b = 20$$

$$S_{20} = 495$$

87. If the sum of all the solutions of

$$\tan^{-1}\left(\frac{2x}{1-x^2}\right) + \cot^{-1}\left(\frac{1-x^2}{2x}\right) = \frac{\pi}{3},$$

$-1 < x < 1, x \neq 0$, is $\alpha - \frac{4}{\sqrt{3}}$, then α is equal to _____ .

Official Ans. by NTA (2)

Ans. (2)

Sol. Case I : $x > 0$

$$\tan^{-1} \frac{2x}{1-x^2} + \tan^{-1} \frac{2x}{1-x^2} = \frac{\pi}{3}$$

$$x = 2 - \sqrt{3}$$

Case II : $x < 0$

$$\tan^{-1} \frac{2x}{1-x^2} + \tan^{-1} \frac{2x}{1-x^2} + \pi = \frac{\pi}{3}$$

$$x = \frac{-1}{\sqrt{3}} \Rightarrow \alpha = 2$$

88. Let the equation of the plane passing through the line $x - 2y - z - 5 = 0 = x + y + 3z - 5$ and parallel to the line $x + y + 2z - 7 = 0 = 2x + 3y + z - 2$ be $ax + by + cz = 65$. Then the distance of the point (a, b, c) from the plane $2x + 2y - z + 16 = 0$ is _____.

Official Ans. by NTA (9)

Ans. (9)

Sol. Equation of plane is $(x - 2y - z - 5) + b(x + y + 3z - 5) = 0$

$$\begin{vmatrix} 1+b & -2+b & -1+3b \\ 1 & 1 & 2 \\ 2 & 3 & 1 \end{vmatrix} = 0$$

$$\Rightarrow b = 12$$

$$\therefore \text{plane is } 13x + 10y + 35z = 65$$

$$\text{Distance from given point to plane} = 9$$

89. Let x and y be distinct integers where $1 \leq x \leq 25$ and $1 \leq y \leq 25$. Then, the number of ways of choosing x and y , such that $x + y$ is divisible by 5, is _____.

Official Ans. by NTA (120)

Ans. (120)

Sol. $x + y = 5\lambda$

Cases :

x	y	Number of ways
5λ	5λ	20
$5\lambda + 1$	$5\lambda + 4$	25
$5\lambda + 2$	$5\lambda + 3$	25
$5\lambda + 3$	$5\lambda + 2$	25
$5\lambda + 4$	$5\lambda + 1$	25

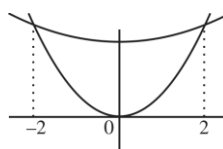
$$\text{Total} = 120$$

90. It the area enclosed by the parabolas $P_1 : 2y = 5x^2$ and $P_2 : x^2 - y + 6 = 0$ is equal to the area enclosed by P_1 and $y = \alpha x$, $\alpha > 0$, then α^3 is equal to _____.

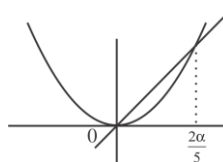
Official Ans. by NTA (600)

Ans. (600)

Sol.



Abscissa of point of intersection of $2y = 5x^2$ and $y = x^2 + 6$ is ± 2



$$\text{Area} = 2 \int_0^2 \left(x^2 + 6 - \frac{5x^2}{2} \right) dx = \int_0^{\frac{2\alpha}{5}} \left(\alpha x - \frac{5x^2}{2} \right) dx$$

$$\Rightarrow \int_0^{\frac{2\alpha}{5}} \left(\alpha x - \frac{5x^2}{2} \right) dx = 16$$

$$\Rightarrow \alpha^3 = 600$$