

FINAL JEE(Advanced) EXAMINATION - 2020**(Held On Sunday 27th SEPTEMBER, 2020)****PAPER-1****TEST PAPER WITH SOLUTION****PART-3 : MATHEMATICS****SECTION-1 : (Maximum Marks : 18)**

- This section contains **SIX (06)** questions.
- Each question has **FOUR** options. **ONLY ONE** of these four options is the correct answer.
- For each question, choose the option corresponding to the correct answer.
- Answer to each question will be evaluated according to the following marking scheme :

Full Marks : +3 If **ONLY** the correct option is chosen;

Zero Marks : 0 If none of the options is chosen (i.e. the question is unanswered);

Negative Marks : -1 In all other cases.

1. Suppose a, b denote the distinct real roots of the quadratic polynomial $x^2 + 20x - 2020$ and suppose c, d denote the distinct complex roots of the quadratic polynomial $x^2 - 20x + 2020$. Then the value of

$$ac(a - c) + ad(a - d) + bc(b - c) + bd(b - d)$$

is

(A) 0

(B) 8000

(C) 8080

(D) 16000

Ans. (D)

Sol. $x^2 + 20x - 2020 = 0$ has two roots $a, b \in \mathbb{R}$

$x^2 - 20x + 2020 = 0$ has two roots $c, d \in \text{complex}$

$$ac(a - c) + ad(a - d) + bc(b - c) + bd(b - d)$$

$$= a^2c - ac^2 + a^2d - ad^2 + b^2c - bc^2 + b^2d - bd^2$$

$$= a^2(c + d) + b^2(c + d) - c^2(a + b) - d^2(a + b)$$

$$= (c + d)(a^2 + b^2) - (a + b)(c^2 + d^2)$$

$$= (c + d)((a + b)^2 - 2ab) - (a + b)((c + d)^2 - 2cd)$$

$$= 20[(20)^2 + 4040] + 20[(20)^2 - 4040]$$

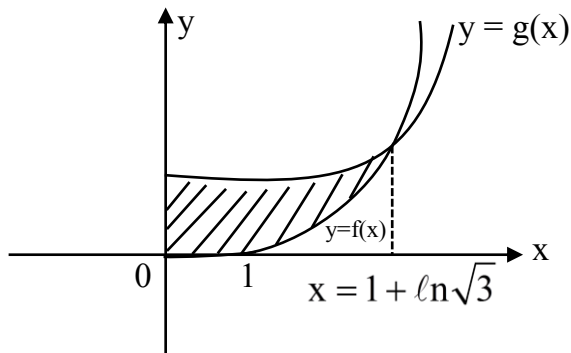
$$= 20[(20)^2 + 4040 + (20)^2 - 4040]$$

$$= 20 \times 800 = 16000$$

Sol. Here

$$f(x) = \begin{cases} 0 & x \leq 1 \\ e^{x-1} - e^{1-x} & x \geq 1 \end{cases}$$

$$\& g(x) = \frac{1}{2}(e^{x-1} + e^{1-x})$$



$$\text{solve } f(x) \& g(x) \Rightarrow x = 1 + \ln\sqrt{3}$$

$$\text{So bounded area} = \int_0^1 \frac{1}{2}(e^{x-1} + e^{1-x}) dx + \int_1^{1+\ln\sqrt{3}} \frac{1}{2}(e^{x-1} + e^{1-x}) - (e^{x-1} - e^{1-x}) dx$$

$$= \frac{1}{2} [e^{x-1} - e^{1-x}]_0^1 + \left[-\frac{1}{2}e^{x-1} - \frac{3}{2}e^{1-x} \right]_1^{1+\ln\sqrt{3}}$$

$$= \frac{1}{2} \left[e - \frac{1}{e} \right] + \left[\left(-\frac{\sqrt{3}}{2} - \frac{\sqrt{3}}{2} \right) + 2 \right] = 2 - \sqrt{3} + \frac{1}{2} \left(e - \frac{1}{e} \right)$$

Ans. (A)

4. Let a, b and λ be positive real numbers. Suppose P is an end point of the latus rectum of the parabola $y^2 = 4\lambda x$, and suppose the ellipse $\frac{x^2}{a^2} + \frac{y^2}{b^2} = 1$ passes through the point P . If the tangents to the parabola and the ellipse at the point P are perpendicular to each other, then the eccentricity of the ellipse is

- (A) $\frac{1}{\sqrt{2}}$ (B) $\frac{1}{2}$ (C) $\frac{1}{3}$ (D) $\frac{2}{5}$

Ans. (A)

Sol. $y^2 = 4\lambda x, P(\lambda, 2\lambda)$

Slope of the tangent to the parabola at point P

$$\frac{dy}{dx} = \frac{4\lambda}{2y} = \frac{4\lambda}{2 \times 2\lambda} = 1$$

Slope of the tangent to the ellipse at P

$$\frac{2x}{a^2} + \frac{2yy'}{b^2} = 0$$

As tangents are perpendicular $y' = -1$

$$\Rightarrow \frac{2\lambda}{a^2} - \frac{4\lambda}{b^2} = 0 \Rightarrow \frac{a^2}{b^2} = \frac{1}{2}$$

$$\Rightarrow e = \sqrt{1 - \frac{1}{2}} = \frac{1}{\sqrt{2}}$$

Ans. A

5. Let C_1 and C_2 be two biased coins such that the probabilities of getting head in a single toss are $\frac{2}{3}$ and $\frac{1}{3}$, respectively. Suppose α is the number of heads that appear when C_1 is tossed twice, independently, and suppose β is the number of heads that appear when C_2 is tossed twice, independently, Then probability that the roots of the quadratic polynomial $x^2 - \alpha x + \beta$ are real and equal, is

- (A) $\frac{40}{81}$ (B) $\frac{20}{81}$ (C) $\frac{1}{2}$ (D) $\frac{1}{4}$

Ans. (B)

Sol. $P(H) = \frac{2}{3}$ for C_1

$P(H) = \frac{1}{3}$ for C_2

for C_1

No. of Heads (α)	0	1	2
Probability	$\frac{1}{9}$	$\frac{4}{9}$	$\frac{4}{9}$

for C_2

No. of Heads (β)	0	1	2
Probability	$\frac{4}{9}$	$\frac{4}{9}$	$\frac{1}{9}$

for real and equal roots

$$\alpha^2 = 4\beta$$

$(\alpha, \beta) = (0, 0), (2, 1)$

So, probability = $\frac{1}{9} \times \frac{4}{9} + \frac{4}{9} \times \frac{4}{9} = \frac{20}{81}$

6. Consider all rectangles lying in the region

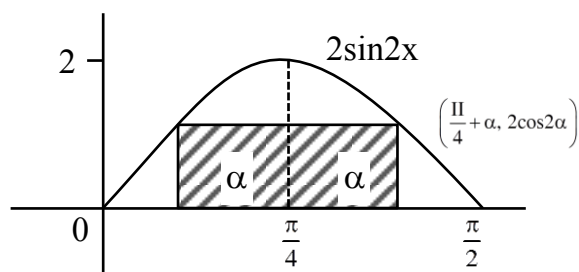
$$\left\{ (x, y) \in \mathbb{R} \times \mathbb{R} : 0 \leq x \leq \frac{\pi}{2} \text{ and } 0 \leq y \leq 2\sin(2x) \right\}$$

and having one side on the x-axis. The area of the rectangle which has the maximum perimeter among all such rectangles, is

- (A) $\frac{3\pi}{2}$ (B) π (C) $\frac{\pi}{2\sqrt{3}}$ (D) $\frac{\pi\sqrt{3}}{2}$

Ans. (C)

Sol.



$$\text{Perimeter} = 2(2\alpha + 2 \cos 2\alpha)$$

$$P = 4(\alpha + \cos 2\alpha)$$

$$\frac{dP}{d\alpha} = 4(1 - 2 \sin 2\alpha) = 0$$

$$\sin 2\alpha = \frac{1}{2}$$

$$2\alpha = \frac{\pi}{6}, \frac{5\pi}{6}$$

$$\frac{d^2P}{d\alpha^2} = -4 \cos 2\alpha$$

$$\text{for maximum } \alpha = \frac{\pi}{12}$$

$$\text{Area} = (2\alpha) (2 \cos 2\alpha)$$

$$= \frac{\pi}{6} \times 2 \times \frac{\sqrt{3}}{2} = \frac{\pi}{2\sqrt{3}}$$

SECTION-2 : (Maximum Marks : 24)

- This section contains **SIX (06)** questions.
- Each question has **FOUR** options. **ONE OR MORE THAN ONE** of these four option(s) is(are) correct answer(s).
- For each question, choose the option(s) corresponding to (all) the correct answer(s).
- Answer to each question will be evaluated according to the following marking scheme :

<i>Full Marks</i>	:	+4	If only (all) the correct option(s) is(are) chosen;
<i>Partial Marks</i>	:	+3	If all the four options are correct but ONLY three options are chosen;
<i>Partial Marks</i>	:	+2	If three or more options are correct but ONLY two options are chosen, both of which are correct;
<i>Partial Marks</i>	:	+1	If two or more options are correct but ONLY one option is chosen and it is a correct option;
<i>Zero Marks</i>	:	0	If none of the options is chose (i.e. the question is unanswered);
<i>Negative Marks</i>	:	-2	In all other cases

7. Let the function $f : \mathbb{R} \rightarrow \mathbb{R}$ be defined by $f(x) = x^3 - x^2 + (x - 1) \sin x$ and let $g : \mathbb{R} \rightarrow \mathbb{R}$ be an arbitrary function. Let $fg : \mathbb{R} \rightarrow \mathbb{R}$ be the product function defined by $(fg)(x) = f(x)g(x)$. Then which of the following statements is/are TRUE ?
- (A) If g is continuous at $x = 1$, then fg is differentiable at $x = 1$
 (B) If fg is differentiable at $x = 1$, then g is continuous at $x = 1$
 (C) If g is differentiable at $x = 1$, then fg is differentiable at $x = 1$
 (D) If fg is differentiable at $x = 1$, then g is differentiable at $x = 1$

Ans. (A,C)

Sol. $f : \mathbb{R} \rightarrow \mathbb{R}$ $f(x) = (x^2 + \sin x)(x-1)$ $f(1^+) = f(1^-) = f(1) = 0$

$fg(x) : \mathbb{R} \rightarrow \mathbb{R}$ $fg : \mathbb{R} \rightarrow \mathbb{R}$

let $fg(x) = h(x) = f(x).g(x)$ $h : \mathbb{R} \rightarrow \mathbb{R}$

option (c) $h'(x) = f'(x)g(x) + f(x)g'(x)$

$h'(1) = f'(1)g(1) + 0,$

(as $f(1) = 0, g'(x)$ exists}

\Rightarrow if $g(x)$ is differentiable then $h(x)$ is also differentiable (true)

option (A) If $g(x)$ is continuous at $x = 1$ then $g(1^+) = g(1^-) = g(1)$

$$h'(1^+) = \lim_{h \rightarrow 0^+} \frac{h(1+h) - h(1)}{h}$$

$$h'(1^+) = \lim_{h \rightarrow 0^+} \frac{f(1+h)g(1+h) - 0}{h} = f'(1)g(1)$$

$$h'(1^-) = \lim_{h \rightarrow 0^+} \frac{f(1-h)g(1-h) - 0}{-h} = f'(1)g(1)$$

So $h(x) = f(x).g(x)$ is differentiable

at $x = 1$ (True)

option (B) (D)
$$h'(1^+) = \lim_{h \rightarrow 0^+} \frac{h(1+h) - h(1)}{-h}$$

$$h'(1^+) = \lim_{h \rightarrow 0^+} \frac{f(1+h)g(1+h)}{h} = f'(1)g(1^+)$$

$$h'(1^-) = \lim_{h \rightarrow 0^+} \frac{f(1-h)g(1-h)}{-h} = f'(1).g(1^-)$$

$$\Rightarrow g(1^+) = g(1^-)$$

So we cannot comment on the continuity and differentiability of the function.

8. Let M be a 3×3 invertible matrix with real entries and let I denote the 3×3 identity matrix. If $M^{-1} = \text{adj}(\text{adj } M)$, then which of the following statement is/are ALWAYS TRUE ?

- (A) $M = I$ (B) $\det M = 1$ (C) $M^2 = I$ (D) $(\text{adj } M)^2 = I$

Ans. (B,C,D)

Sol. $\det(M) \neq 0$

$$M^{-1} = \text{adj}(\text{adj } M)$$

$$M^{-1} = \det(M).M$$

$$M^{-1}M = \det(M).M^2$$

$$I = \det(M).M^2 \quad \dots (i)$$

$$\det(I) = (\det(M))^5$$

$$1 = \det(M) \quad \dots (ii)$$

From (i) $I = M^2$

$$(\text{adj } M)^2 = \text{adj}(M^2) = \text{adj } I = I$$

9. Let S be the set of all complex numbers z satisfying $|z^2 + z + 1| = 1$. Then which of the following statements is/are TRUE ?

(A) $\left|z + \frac{1}{2}\right| \leq \frac{1}{2}$ for all $z \in S$

(B) $|z| \leq 2$ for all $z \in S$

(C) $\left|z + \frac{1}{2}\right| \geq \frac{1}{2}$ for all $z \in S$

(D) The set S has exactly four elements

Ans. (B,C)

Sol. $|z^2 + z + 1| = 1$

$$\Rightarrow \left| \left(z + \frac{1}{2} \right)^2 + \frac{3}{4} \right| = 1$$

$$\Rightarrow \left| \left(z + \frac{1}{2} \right)^2 + \frac{3}{4} \right| \leq \left| z + \frac{1}{2} \right|^2 + \frac{3}{4}$$

$$\Rightarrow 1 \leq \left| z + \frac{1}{2} \right|^2 + \frac{3}{4} \Rightarrow \left| \left(z + \frac{1}{2} \right)^2 \right| \geq \frac{1}{4}$$

$$\Rightarrow \left| z + \frac{1}{2} \right| \geq \frac{1}{2}$$

also $|(z^2 + z) + 1| = 1 \geq ||z^2 + z| - 1|$

$$\Rightarrow |z^2 + z| - 1 \leq 1$$

$$\Rightarrow |z^2 + z| \leq 2$$

$$\Rightarrow ||z^2| - |z|| \leq |z^2 + z| \leq 2$$

$$\Rightarrow |r^2 - r| \leq 2$$

$$\Rightarrow r = |z| \leq 2 ; \forall z \in S$$

Also we can always find root of the equation $z^2 + z + 1 = e^{i\theta} ; \forall \theta \in \mathbb{R}$

Hence set 'S' is infinite

10. Let x , y and z be positive real numbers. Suppose x , y and z are lengths of the sides of a triangle opposite to its angles X , Y and Z , respectively. If

$$\tan \frac{X}{2} + \tan \frac{Z}{2} = \frac{2y}{x+y+z},$$

then which of the following statements is/are TRUE?

(A) $2Y = X + Z$

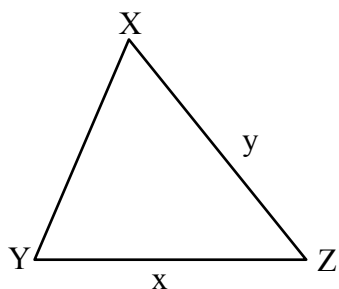
(B) $Y = X + Z$

(C) $\tan \frac{X}{2} = \frac{x}{y+z}$

(D) $x^2 + z^2 - y^2 = xz$

Ans. (B,C)

Sol.



$$\tan \frac{x}{2} + \tan \frac{z}{2} = \frac{2y}{x+y+z}$$

$$\frac{\Delta}{S(S-x)} + \frac{\Delta}{S(S-z)} = \frac{2y}{2S}$$

$$\frac{\Delta}{S} \left(\frac{2S-(x+z)}{(S-x)(S-z)} \right) = \frac{y}{S}$$

$$\Rightarrow \frac{\Delta y}{S(S-x)(S-z)} = \frac{y}{S}$$

$$\Rightarrow \Delta^2 = (S-x)^2 (S-z)^2$$

$$\Rightarrow S(S-y) = (S-x)(S-z)$$

$$\Rightarrow (x+y+z)(x+z-y) = (y+z-x)(x+y-z)$$

$$\Rightarrow (x+z)^2 - y^2 = y^2 - (z-x)^2$$

$$\Rightarrow (x+z)^2 + (x-z)^2 = 2y^2$$

$$\Rightarrow x^2 + z^2 = y^2 \Rightarrow \angle Y = \frac{\pi}{2}$$

$$\Rightarrow \angle Y = \angle X + \angle Z$$

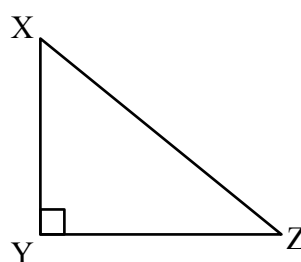
$$\tan \frac{x}{2} = \frac{\Delta}{S(S-x)}$$

$$\tan \frac{x}{2} = \frac{\frac{1}{2}xz}{\frac{(y+z)^2 - x^2}{4}}$$

$$\tan \frac{x}{2} = \frac{2xz}{y^2 + z^2 + 2yz - x^2}$$

$$\tan \frac{x}{2} = \frac{2xz}{2z^2 + 2yz} \quad (\text{using } y^2 = x^2 + z^2)$$

$$\tan \frac{x}{2} = \frac{x}{y+z}$$



11. Let L_1 and L_2 be the following straight line.

$$L_1 : \frac{x-1}{1} = \frac{y}{-1} = \frac{z-1}{3} \quad \text{and} \quad L_2 : \frac{x-1}{-3} = \frac{y}{-1} = \frac{z-1}{1}$$

Suppose the straight line

$$L : \frac{x-\alpha}{l} = \frac{y-1}{m} = \frac{z-\gamma}{-2}$$

lies in the plane containing L_1 and L_2 , and passes through the point of intersection of L_1 and L_2 . If the line L bisects the acute angle between the lines L_1 and L_2 , then which of the following statements is/are TRUE?

- (A) $\alpha - \gamma = 3$ (B) $l + m = 2$ (C) $\alpha - \gamma = 1$ (D) $l + m = 0$

Ans. (A,B)

Sol. Point of intersection of L_1 & L_2 is $(1, 0, 1)$

Line L passes through $(1, 0, 1)$

$$\frac{1-\alpha}{l} = -\frac{1}{m} = \frac{1-\gamma}{-2} \quad \dots(1)$$

acute angle bisector of L_1 & L_2

$$\vec{r} = \hat{i} + \hat{k} + \lambda \left(\frac{\hat{i} - \hat{j} + 3\hat{k} - 3\hat{i} - \hat{j} + \hat{k}}{\sqrt{11}} \right)$$

$$\vec{r} = \hat{i} + \hat{k} + t(\hat{i} + \hat{j} - 2\hat{k})$$

$$\Rightarrow \frac{l}{1} = \frac{m}{1} = \frac{-2}{-2} \Rightarrow l = m = 1$$

$$\text{From (1)} \quad \frac{1-\alpha}{1} = -1 \Rightarrow \alpha = 2$$

$$\& \frac{1-\gamma}{-2} = -1 \Rightarrow \gamma = -1$$

12. Which of the following inequalities is/are TRUE?

(A) $\int_0^1 x \cos x dx \geq \frac{3}{8}$

(B) $\int_0^1 x \sin x dx \geq \frac{3}{10}$

(C) $\int_0^1 x^2 \cos x dx \geq \frac{1}{2}$

(D) $\int_0^1 x^2 \sin x dx \geq \frac{2}{9}$

Ans. (A,B,D)

Sol. (A) $\cos x = 1 - \frac{x^2}{2!} + \frac{x^4}{4!} - \dots$

$$\sin x = x - \frac{x^3}{3!} + \frac{x^5}{5!} - \dots$$

$$\cos x \geq 1 - \frac{x^2}{2}$$

$$\int_0^1 x \cos x \geq \int_0^1 x \left(1 - \frac{x^2}{2}\right) = \frac{1}{2} - \frac{1}{8}$$

$$\int_0^1 x \cos x \geq \frac{3}{8} \quad (\text{True})$$

(B) $\sin x \geq x - \frac{x^3}{6}$

$$\int_0^1 x \sin x \geq \int_0^1 x \left(x - \frac{x^3}{6}\right) dx$$

$$\int_0^1 x \sin x \geq \frac{1}{3} - \frac{1}{30} \Rightarrow \int_0^1 x \sin x \geq \frac{3}{8} \quad (\text{True})$$

(D) $\int_0^1 x^2 \sin x \geq \int_0^1 x^2 \left(x - \frac{x^3}{6}\right) dx$

$$\int_0^1 x^2 \sin x \geq \frac{1}{4} - \frac{1}{36}$$

$$\int_0^1 x^2 \sin x \geq \frac{2}{9} \quad (\text{True})$$

(C) $\cos x < 1$

$$x^2 \cos x < x^2$$

$$\int_0^1 x^2 \cos x \, dx < \int_0^1 x^2 \, dx$$

$$\int_0^1 x^2 \cos x \, dx < \frac{1}{3}$$

So option 'C' is incorrect.

SECTION-3 : (Maximum Marks : 24)

- This section contains **SIX (06)** questions. The answer to each question is a **NUMERICAL VALUE**.
- For each question, enter the correct numerical value of the answer using the mouse and the on-screen virtual numeric keypad in the place designated to enter the answer. If the numerical value has more than two decimal places, **truncate/round-off** the value to **TWO** decimal places.
- Answer to each question will be evaluated according to the following marking scheme :

<i>Full Marks</i>	: +4	If ONLY the correct numerical value is entered;
<i>Zero Marks</i>	: 0	In all other cases.

13. Let m be the minimum possible value of $\log_3(3^{y_1} + 3^{y_2} + 3^{y_3})$, where y_1, y_2, y_3 are real numbers for which $y_1 + y_2 + y_3 = 9$. Let M be the maximum possible value of $(\log_3 x_1 + \log_3 x_2 + \log_3 x_3)$, where x_1, x_2, x_3 are positive real numbers for which $x_1 + x_2 + x_3 = 9$. Then the value of $\log_2(m^3) + \log_3(M^2)$ is _____.

Ans. (8.00)

Sol.
$$\frac{3^{y_1} + 3^{y_2} + 3^{y_3}}{3} \geq \left[3^{(y_1+y_2+y_3)} \right]^{\frac{1}{3}}$$

$$\Rightarrow 3^{y_1} + 3^{y_2} + 3^{y_3} \geq 3^4$$

$$\Rightarrow \log_3(3^{y_1} + 3^{y_2} + 3^{y_3}) \geq 4$$

$$\Rightarrow m = 4$$

Also,
$$\frac{x_1 + x_2 + x_3}{3} \geq \sqrt[3]{x_1 x_2 x_3}$$

$$\Rightarrow x_1 x_2 x_3 \leq 27$$

$$\Rightarrow \log_3 x_1 + \log_3 x_2 + \log_3 x_3 \leq 3$$

$$\Rightarrow M = 3$$

$$\begin{aligned} \text{Thus, } \log_2(m^3) + \log_3(M^2) &= 6 + 2 \\ &= 8 \end{aligned}$$

14. Let a_1, a_2, a_3, \dots be a sequence of positive integers in arithmetic progression with common difference 2. Also, let b_1, b_2, b_3, \dots be a sequence of positive integers in geometric progression with common ratio 2. If $a_1 = b_1 = c$, then the number of all possible values of c , for which the equality

$$2(a_1 + a_2 + \dots + a_n) = b_1 + b_2 + \dots + b_n$$

holds for some positive integer n , is _____

Ans. (1.00)

Sol. Given $2(a_1 + a_2 + \dots + a_n) = b_1 + b_2 + \dots + b_n$

$$\Rightarrow 2 \times \frac{n}{2} (2c + (n-2)x_2) = c \left(\frac{2^n - 1}{2 - 1} \right)$$

$$\Rightarrow 2n^2 - 2n = c(2^n - 1 - 2n)$$

$$\Rightarrow c = \frac{2n^2 - 2n}{2^n - 1 - 2n} \in \mathbb{N}$$

So, $2n^2 - 2n \geq 2^n - 1 - 2n$

$$\Rightarrow 2n^2 + 1 \geq 2^n \Rightarrow n < 7$$

$$\Rightarrow n \text{ can be } 1, 2, 3, \dots,$$

Checking c against these values of n

we get $c = 12$ (when $n = 3$)

Hence number of such $c = 1$

15. Let $f : [0, 2] \rightarrow \mathbb{R}$ be the function defined by

$$f(x) = (3 - \sin(2\pi x)) \sin\left(\pi x - \frac{\pi}{4}\right) - \sin\left(3\pi x + \frac{\pi}{4}\right)$$

If $\alpha, \beta \in [0, 2]$ are such that $\{x \in [0, 2] : f(x) \geq 0\} = [\alpha, \beta]$, then the value of $\beta - \alpha$ is _____

Ans. (1.00)

Sol. Let $\pi x - \frac{\pi}{4} = \theta \in \left[\frac{-\pi}{4}, \frac{7\pi}{4}\right]$

$$\text{So, } \left(3 - \sin\left(\frac{\pi}{2} + 2\theta\right)\right) \sin \theta \geq \sin(\pi + 3\theta)$$

$$\Rightarrow (3 - \cos 2\theta) \sin \theta \geq -\sin 3\theta$$

$$\Rightarrow \sin \theta [3 - 4\sin^2 \theta + 3 - \cos 2\theta] \geq 0$$

$$\Rightarrow \sin \theta (6 - 2(1 - \cos 2\theta) - \cos 2\theta) \geq 0$$

$$\Rightarrow \sin \theta (4 + \cos 2\theta) \geq 0$$

$$\Rightarrow \sin \theta \geq 0$$

$$\Rightarrow \theta \in [0, \pi] \Rightarrow 0 \leq \pi x - \frac{\pi}{4} \leq \pi$$

$$\Rightarrow x \in \left[\frac{1}{4}, \frac{5}{4}\right]$$

$$\Rightarrow \beta - \alpha = 1$$

16. In a triangle PQR, let $\vec{a} = \overrightarrow{QR}$, $\vec{b} = \overrightarrow{RP}$ and $\vec{c} = \overrightarrow{PQ}$. If

$$|\vec{a}| = 3, |\vec{b}| = 4 \text{ and } \frac{\vec{a} \cdot (\vec{c} - \vec{b})}{\vec{c} \cdot (\vec{a} - \vec{b})} = \frac{|\vec{a}|}{|\vec{a}| + |\vec{b}|}, \text{ then the value of } |\vec{a} \times \vec{b}|^2 \text{ is } \underline{\hspace{2cm}}$$

Ans. (108.00)

Sol. We have $\vec{a} + \vec{b} + \vec{c} = \vec{0}$

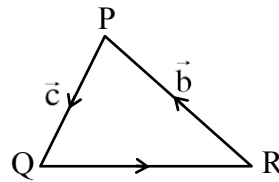
$$\Rightarrow \vec{c} = -\vec{a} - \vec{b}$$

$$\text{Now, } \frac{\vec{a} \cdot (-\vec{a} - 2\vec{b})}{(-\vec{a} - \vec{b}) \cdot (\vec{a} - \vec{b})} = \frac{3}{7}$$

$$\Rightarrow \frac{9 + 2\vec{a} \cdot \vec{b}}{9 - 16} = \frac{3}{7}$$

$$\Rightarrow \vec{a} \cdot \vec{b} = -6$$

$$\Rightarrow |\vec{a} \times \vec{b}|^2 = a^2 b^2 - (\vec{a} \cdot \vec{b})^2 = 9 \times 16 - 36 = 108$$



17. For a polynomial $g(x)$ with real coefficient, let m_g denote the number of distinct real roots of $g(x)$. Suppose S is the set of polynomials with real coefficient defined by

$$S = \{(x^2 - 1)^2 (a_0 + a_1 x + a_2 x^2 + a_3 x^3) : a_0, a_1, a_2, a_3 \in \mathbb{R}\}.$$

For a polynomial f , let f' and f'' denote its first and second order derivatives, respectively. Then the minimum possible value of $(m_{f'} + m_{f''})$, where $f \in S$, is _____

Ans. (5.00)

Sol. $f(x) = (x^2 - 1)^2 h(x)$; $h(x) = a_0 + a_1 x + a_2 x^2 + a_3 x^3$

$$\text{Now, } f(1) = f(-1) = 0$$

$$\Rightarrow f'(\alpha) = 0, \alpha \in (-1, 1) \quad [\text{Rolle's Theorem}]$$

Also, $f'(1) = f'(-1) = 0 \Rightarrow f'(x) = 0$ has atleast 3 root, $-1, \alpha, 1$ with $-1 < \alpha < 1$

$\Rightarrow f''(x) = 0$ will have at least 2 root, say β, γ such that

$$-1 < \beta < \alpha < \gamma < 1 \quad [\text{Rolle's Theorem}]$$

So, $\min(m_{f''}) = 2$

and we find $(m_{f'} + m_{f''}) = 5$ for $f(x) = (x^2 - 1)^2$.

Thus, Ans. 5

18. Let e denote the base of the natural logarithm. The value of the real number a for which the right hand limit

$$\lim_{x \rightarrow 0^+} \frac{(1-x)^{\frac{1}{x}} - e^{-1}}{x^a}$$

is equal to a nonzero real number, is _____.

Ans. (1.00)

$$\begin{aligned}
 \text{Sol. } \lim_{x \rightarrow 0^+} \frac{e^{\left(\frac{\ln(1-x)}{x}\right)} - 1}{x^a} & \\
 &= \lim_{x \rightarrow 0^+} \frac{1}{e} \frac{e^{\left(1 + \frac{\ln(1-x)}{x}\right)} - 1}{x^a} \\
 &= \frac{1}{e} \lim_{x \rightarrow 0^+} \frac{1 + \frac{\ln(1-x)}{x}}{x^a} \\
 &= \frac{1}{e} \lim_{x \rightarrow 0^+} \frac{\ln(1-x) + x}{x^{(a+1)}} \\
 &= \frac{1}{e} \lim_{x \rightarrow 0^+} \frac{\left(-x - \frac{x^2}{2} - \frac{x^3}{3} - \dots\right) + x}{x^{a+1}}
 \end{aligned}$$

Thus, $a = 1$