FINAL JEE–MAIN EXAMINATION – APRIL,2019

(Held On Monday 08th APRIL, 2019) TIME : 9 : 30 AM To 12 : 30 PM

MATHEMATICS

TEST PAPER WITH ANSWER & SOLUTION

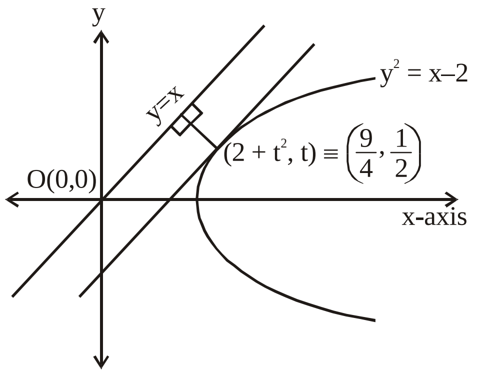
1. The shortest distance between the line y = x and the curve $y^2 = x - 2$ is :

(1) $\frac{7}{4\sqrt{2}}$ (2) $\frac{7}{8}$

(3) $\frac{11}{4\sqrt{2}}$ (4) 2

Official Ans. by NTA (1)

Sol.



we have, $2y.\frac{dy}{dx} = 1 \Rightarrow \left.\frac{dy}{dx}\right]_{P(2+t^2,t)} = \frac{1}{2t} = 1$

$\Rightarrow \quad t = \frac{1}{2}$

$\therefore \quad P\left(\frac{9}{4}, \frac{1}{2}\right)$

So, shortest distance

$= \frac{\left|\frac{9}{4} - \frac{2}{4}\right|}{\sqrt{2}} = \frac{7}{4\sqrt{2}}$

2. Let y = y(x) be the solution of the differential equation, $(x^2+1)^2\frac{dy}{dx} + 2x(x^2+1)y = 1$ such that y(0) = 0. If $\sqrt{a}y(1) = \frac{\pi}{32}$, then the value of 'a' is :

(1) $\frac{1}{2}$ (2) $\frac{1}{16}$

(3) $\frac{1}{4}$ (4) 1

Official Ans. by NTA (2)

Sol. $\frac{dy}{dx} + \left(\frac{2x}{x^2+1}\right)y = \frac{1}{(x^2+1)^2}$

(Linear differential equation)

$\therefore \quad$ I.F. $= e^{\ell n(x^2+1)} = (x^2+1)$

So, general solution is $y.(x^2 + 1) = \tan^{-1}x + c$

As y(0) = 0 $\Rightarrow$ c = 0

$\therefore \quad y(x) = \frac{\tan^{-1}x}{x^2+1}$

As, $\sqrt{a}.\,y(1) = \frac{\pi}{32}$

$\Rightarrow \quad \sqrt{a} = \frac{1}{4} \Rightarrow a = \frac{1}{16}$

3. A point on the straight line, 3x + 5y = 15 which is equidistant from the coordinate axes will lie only in :

(1) 1^{st} and 2^{nd} quadrants

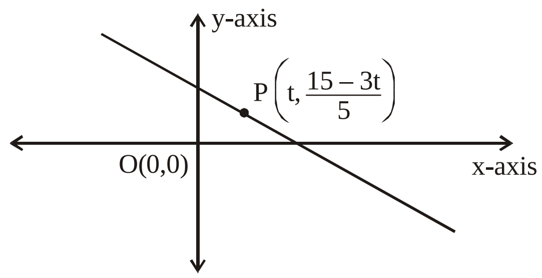
(2) 4^{th} quadrant

(3) 1^{st}, 2^{nd} and 4^{th} quadrant

(4) 1^{st} quadrant

Official Ans. by NTA (1)

Sol.



Now, $\left|\frac{15-3t}{5}\right| = |t|$

$\Rightarrow \quad \frac{15-3t}{5} = t$ or $\frac{15-3t}{5} = -t$

$\therefore \quad t = \frac{15}{8}$ or $t = \frac{-15}{2}$

So, $P\left(\frac{15}{8},\frac{15}{8}\right) \in I^{st}$ quadrant

or $P\left(\frac{-15}{2},\frac{15}{2}\right) \in II^{nd}$ quadrant

4. If α and β be the roots of the equation $x^2 - 2x + 2 = 0$, then the least value of n for which $\left(\frac{\alpha}{\beta}\right)^n = 1$ is :

(1) 2 (2) 3

(3) 4 (4) 5

Official Ans. by NTA (3)

Sol. $(x-1)^2 + 1 = 0 \Rightarrow x = 1 + i,\ 1 - i$

$\therefore \left(\frac{\alpha}{\beta}\right)^n = 1 \Rightarrow (\pm i)^n = 1$

$\therefore$ n (least natural number) = 4

5. $\lim\limits_{x\to 0} \frac{\sin^2 x}{\sqrt{2} - \sqrt{1+\cos x}}$ equals :

(1) $2\sqrt{2}$ (2) $4\sqrt{2}$

(3) $\sqrt{2}$ (4) 4

Official Ans. by NTA (2)

Sol. $$\lim_{x\to 0} \frac{\left(\frac{\sin^2 x}{x^2}\right)\left(\sqrt{2}+\sqrt{1+\cos x}\right)}{\left(\frac{1-\cos x}{x^2}\right)}$$

$$= \frac{(1)^2.\left(2\sqrt{2}\right)}{\frac{1}{2}} = 4\sqrt{2}$$

6. The length of the perpendicular from the point $(2, -1, 4)$ on the straight line, $\frac{x+3}{10} = \frac{y-2}{-7} = \frac{z}{1}$ is :

(1) less than 2

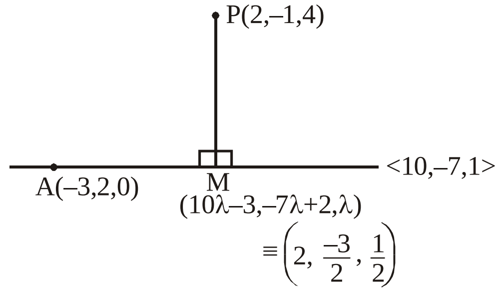
(2) greater than 3 but less than 4

(3) greater than 4

(4) greater than 2 but less than 3

Official Ans. by NTA (2)

Sol.



Now, $\overrightarrow{MP}.\left(10\hat{i} - 7\hat{j} + \hat{k}\right) = 0$

$\Rightarrow \lambda = \frac{1}{2}$

$\therefore$ Length of perpendicular

$$(= PM) = \sqrt{0 + \frac{1}{4} + \frac{49}{4}}$$

$$= \sqrt{\frac{50}{4}} = \sqrt{\frac{25}{2}} = \frac{5}{\sqrt{2}},$$

which is greater than 3 but less than 4.

7. The magnitude of the projection of the vector $2\hat{i} + 3\hat{j} + \hat{k}$ on the vector perpendicular to the plane containing the vectors $\hat{i} + \hat{j} + \hat{k}$ and $\hat{i} + 2\hat{j} + 3\hat{k}$, is :

(1) $\frac{\sqrt{3}}{2}$ (2) $\sqrt{\frac{3}{2}}$

(3) $\sqrt{6}$ (4) $3\sqrt{6}$

Official Ans. by NTA (2)

Sol. Vector perpendicular to plane containing the vectors $\hat{i} + \hat{j} + \hat{k}$ & $\hat{i} + 2\hat{j} + 3\hat{k}$ is parallel to vector

$$= \begin{vmatrix} \hat{i} & \hat{j} & \hat{k} \\ 1 & 1 & 1 \\ 1 & 2 & 3 \end{vmatrix} = \hat{i} - 2\hat{j} + \hat{k}$$

$\therefore$ Required magnitude of projection

$$= \frac{\left|(2\hat{i} + 3\hat{j} + \hat{k}).(\hat{i} - 2\hat{j} + \hat{k})\right|}{\left|\hat{i} - 2\hat{j} + \hat{k}\right|}$$

$$= \frac{|2 - 6 + 1|}{\left|\sqrt{6}\right|} = \frac{3}{\sqrt{6}} = \sqrt{\frac{3}{2}}$$

8. The contrapositive of the statement "If you are born in India, then you are a citizen of India", is :

(1) If you are born in India, then you are not a citizen of India.

(2) If you are not a citizen of India, then you are not born in India.

(3) If you are a citizen of India, then you are born in India.

(4) If you are not born in India, then you are not a citizen of India.

Official Ans. by NTA (2)

Sol. The contrapositive of statement

$p \to q$ is $\sim q \to \sim p$

Here, p : you are born in India.

q : you are citizen of India.

So, contrapositive of above statement is

" If you are not a citizen of India, then you are not born in India".

9. The mean and variance of seven observations are 8 and 16, respectively. If 5 of the observations are 2, 4, 10, 12, 14, then the product of the remaining two observations is :

(1) 40 (2) 49

(3) 48 (4) 45

Official Ans. by NTA (3)

Sol. Let 7 observations be $x_1, x_2, x_3, x_4, x_5, x_6, x_7$

$$\bar{x} = 8 \Rightarrow \sum_{i=1}^{7} x_i = 56 \qquad ......(1)$$

Also $\sigma^2 = 16$

$$\Rightarrow \quad 16 = \frac{1}{7}\left(\sum_{i=1}^{7} x_i^2\right) - (\bar{x})^2$$

$$\Rightarrow \quad 16 = \frac{1}{7}\left(\sum_{i=1}^{7} x_i^2\right) - 64$$

$$\Rightarrow \quad \left(\sum_{i=1}^{7} x_i^2\right) = 560 \qquad .....(2)$$

Now, $x_1 = 2$, $x_2 = 4$, $x_3 = 10$, $x_4 = 12$, $x_5 = 14$

$\Rightarrow \quad x_6 + x_7 = 14$ (from (1))

& $\quad x_6^2 + x_7^2 = 100$ (from (2))

$$\therefore \quad x_6^2 + x_7^2 = (x_6 + x_7)^2 - 2x_6.x_7 \Rightarrow x_6.x_7 = 48$$

10. If $f(x) = \dfrac{2 - x\cos x}{2 + x\cos x}$ and $g(x) = \log_e x, (x > 0)$ then the value of integral $\int_{-\frac{\pi}{4}}^{\frac{\pi}{4}} g(f(x))\,dx$ is :

(1) $\log_e 3$ (2) $\log_e 2$

(3) $\log_e e$ (4) $\log_e 1$

Official Ans. by NTA (4)

Sol. $g(f(x)) = \ell n(f(x)) = \ell n\left(\dfrac{2 - x.\cos x}{2 + x.\cos x}\right)$

$$\therefore \quad I = \int_{-\pi/4}^{\pi/4} \ell n\left(\frac{2 - x.\cos x}{2 + x.\cos x}\right) dx$$

$$= \int_{0}^{\pi/4}\left(\ell n\left(\frac{2 - x.\cos x}{2 + x.\cos x}\right) + \ell n\left(\frac{2 + x.\cos x}{2 - x.\cos x}\right)\right) dx$$

$$= \int_{0}^{\pi/2} (0)dx = 0 = \log_e(1)$$

11. If the tangents on the ellipse $4x^2 + y^2 = 8$ at the points (1, 2) and (a, b) are perpendicular to each other, then a^2 is equal to :

(1) $\dfrac{64}{17}$ (2) $\dfrac{2}{17}$

(3) $\dfrac{128}{17}$ (4) $\dfrac{4}{17}$

Official Ans. by NTA (2)

Sol. $4a^2 + b^2 = 8 \qquad ....(1)$

also $\left.\dfrac{dy}{dx}\right)_{(1,2)} = -\dfrac{4x}{y} = -2$

$$\Rightarrow \quad -\frac{4a}{b} = \frac{1}{2}$$

$b = -8a$

$\Rightarrow \quad b^2 = 64a^2$

$68a^2 = 8$

$$a^2 = \frac{2}{17}$$

12. If $\alpha = \cos^{-1}\left(\frac{3}{5}\right), \beta = \tan^{-1}\left(\frac{1}{3}\right)$,

where $0 < \alpha, \beta < \frac{\pi}{2}$, then $\alpha - \beta$ is equal to :

(1) $\sin^{-1}\left(\frac{9}{5\sqrt{10}}\right)$ (2) $\tan^{-1}\left(\frac{9}{14}\right)$

(3) $\cos^{-1}\left(\frac{9}{5\sqrt{10}}\right)$ (4) $\tan^{-1}\left(\frac{9}{5\sqrt{10}}\right)$

Official Ans. by NTA (1)

Sol. $\cos\alpha = \frac{3}{5}, \tan\beta = \frac{1}{3}$

$\Rightarrow \tan\alpha = \frac{4}{3}$

$\Rightarrow \tan(\alpha-\beta) = \frac{\frac{4}{3}-\frac{1}{3}}{1+\frac{4}{3}\cdot\frac{1}{3}} = \frac{9}{13}$

$\Rightarrow \sin(\alpha-\beta) = \frac{9}{5\sqrt{10}}$

$\Rightarrow \alpha-\beta = \sin^{-1}\left(\frac{9}{5\sqrt{10}}\right)$

13. If S_1 and S_2 are respectively the sets of local minimum and local maximum points of the function, $f(x) = 9x^4 + 12x^3 - 36x^2 + 25$, $x \in R$, then :

(1) $S_1 = \{-2, 1\}; S_2 = \{0\}$

(2) $S_1 = \{-2, 0\}; S_2 = \{1\}$

(3) $S_1 = \{-2\}; S_2 = \{0, 1\}$

(4) $S_1 = \{-1\}; S_2 = \{0, 2\}$

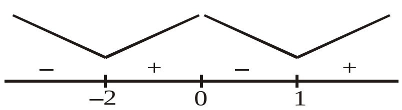
Official Ans. by NTA (1)

Sol. $f(x) = 9x^4 + 12x^3 - 36x^2 + 25$

$f'(x) = 36x^3 + 36x^2 - 72x$

$= 36x(x^2 + x - 2)$

$= 36x(x - 1)(x + 2)$



Points of minima = $\{-2, 1\} = S_1$

Point of maxima = $\{0\} = S_2$

14. Let O(0, 0) and A(0, 1) be two fixed points. Then the locus of a point P such that the perimeter of ΔAOP is 4, is :

(1) $8x^2 - 9y^2 + 9y = 18$

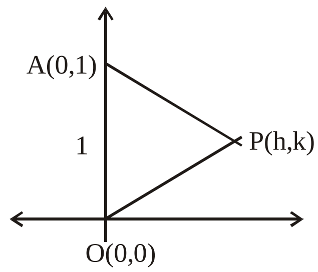
(2) $9x^2 + 8y^2 - 8y = 16$

(3) $8x^2 + 9y^2 - 9y = 18$

(4) $9x^2 - 8y^2 + 8y = 16$

Official Ans. by NTA (2)

Sol.



AP + OP + AO = 4

$\sqrt{h^2+(k-1)^2} + \sqrt{h^2+k^2} + 1 = 4$

$\sqrt{h^2+(k-1)^2} + \sqrt{h^2+k^2} = 3$

$h^2+(k-1)^2 = 9 + h^2 + k^2 - 6\sqrt{h^2+k^2}$

$-2k - 8 = -6\sqrt{h^2+k^2}$

$k + 4 = 3\sqrt{h^2+k^2}$

$k^2 + 16 + 8k = 9(h^2 + k^2)$

$9h^2 + 8k^2 - 8k - 16 = 0$

Locus of P is $9x^2 + 8y^2 - 8y - 16 = 0$

15. Let $A = \begin{pmatrix} \cos\alpha & -\sin\alpha \\ \sin\alpha & \cos\alpha \end{pmatrix}, (\alpha \in R)$ such that

$A^{32} = \begin{pmatrix} 0 & -1 \\ 1 & 0 \end{pmatrix}$. Then a value of α is

(1) $\frac{\pi}{16}$ (2) 0

(3) $\frac{\pi}{32}$ (4) $\frac{\pi}{64}$

Official Ans. by NTA (4)

Sol. $A = \begin{bmatrix} \cos\alpha & -\sin\alpha \\ \sin\alpha & \cos\alpha \end{bmatrix}$

$A^2 = \begin{bmatrix} \cos\alpha & -\sin\alpha \\ \sin\alpha & \cos\alpha \end{bmatrix}\begin{bmatrix} \cos\alpha & -\sin\alpha \\ \sin\alpha & \cos\alpha \end{bmatrix}$

$$=\begin{bmatrix}\cos 2\alpha & -\sin 2\alpha\\ \sin 2\alpha & \cos 2\alpha\end{bmatrix}$$

$$A^3=\begin{bmatrix}\cos 2\alpha & -\sin 2\alpha\\ \sin 2\alpha & \cos 2\alpha\end{bmatrix}\begin{bmatrix}\cos \alpha & -\sin \alpha\\ \sin \alpha & \cos \alpha\end{bmatrix}$$

$$=\begin{bmatrix}\cos 3\alpha & -\sin 3\alpha\\ \sin 3\alpha & \cos 3\alpha\end{bmatrix}$$

Similarly $A^{32}=\begin{bmatrix}\cos 32\alpha & -\sin 32\alpha\\ \sin 32\alpha & \cos 32\alpha\end{bmatrix}=\begin{bmatrix}0 & -1\\ 1 & 0\end{bmatrix}$

$\Rightarrow$ $\cos 32\alpha = 0$ & $\sin 32\alpha = 1$

$\Rightarrow$ $32\alpha=(4n+1)\frac{\pi}{2}, n\in I$

$\alpha=(4n+1)\frac{\pi}{64}, n\in I$

$\alpha=\frac{\pi}{64}$ for $n=0$

16. If $f(x)=\log_e\left(\frac{1-x}{1+x}\right), |x|<1$, then $f\left(\frac{2x}{1+x^2}\right)$ is equal to :

(1) $2f(x)$ (2) $2f(x^2)$

(3) $(f(x))^2$ (4) $-2f(x)$

Official Ans. by NTA (1)

Sol. $f(x)=\log_e\left(\frac{1-x}{1+x}\right), |x|<1$

$$f\left(\frac{2x}{1+x^2}\right)=\ell n\left(\frac{1-\frac{2x}{1+2x^2}}{1+\frac{2x}{1+x^2}}\right)$$

$$=\ell n\left(\frac{(x-1)^2}{(x+1)^2}\right)=2\ell n\left|\frac{1-x}{1+x}\right|=2f(x)$$

17. The equation of a plane containing the line of intersection of the planes $2x - y - 4 = 0$ and $y + 2z - 4 = 0$ and passing through the point (1, 1, 0) is :

(1) $x + 3y + z = 4$ (2) $x - y - z = 0$

(3) $x - 3y - 2z = -2$ (4) $2x - z = 2$

Official Ans. by NTA (2)

Sol. The required plane is

$(2x - y - 4) + \lambda(y + 2z - 4) = 0$

it passes through (1, 1, 0)

$\Rightarrow$ $(2 - 1 - 4) + \lambda(1 - 4) = 0$

$\Rightarrow$ $-3 - 3\lambda = 0$ $\Rightarrow$ $\lambda = -1$

$\Rightarrow$ $x - y - z = 0$

18. The sum of all natural numbers 'n' such that $100 < n < 200$ and H.C.F. $(91, n) > 1$ is :

(1) 3221 (2) 3121

(3) 3203 (4) 3303

Official Ans. by NTA (2)

Sol. S_A = sum of numbers between 100 & 200 which are divisible by 7.

$\Rightarrow$ $S_A = 105 + 112 + ... + 196$

$$S_A=\frac{14}{2}[105+196]=2107$$

S_B = Sum of numbers between 100 & 200 which are divisible by 13.

$$S_B=104+117+....+195=\frac{8}{2}[104+195]=1196$$

S_C = Sum of numbers between 100 & 200 which are divisible by both 7 & 13.

$S_C = 182$

$\Rightarrow$ H.C.F. $(91, n) > 1$ $= S_A + S_B - S_C = 3121$

19. The sum of the series

$2.{}^{20}C_0 + 5.{}^{20}C_1 + 8.{}^{20}C_2 + 11.{}^{20}C_3 + ... + 62.{}^{20}C_{20}$ is equal to :

(1) 2^{24} (2) 2^{25}

(3) 2^{26} (4) 2^{23}

Official Ans. by NTA (2)

Sol. $2.{}^{20}C_0 + 5.{}^{20}C_1 + 8.{}^{20}C_2 + 11.{}^{20}C_3 + ... + 62.{}^{20}C_{20}$

$$=\sum_{r=0}^{20}(3r+2)\,{}^{20}C_r$$

$$=3\sum_{r=0}^{20}r.\,{}^{20}C_r+2\sum_{r=0}^{20}{}^{20}C_r$$

$$=3\sum_{r=0}^{20}r\left(\frac{20}{r}\right){}^{19}C_{r-1}+2.2^{20}$$

$= 60.2^{19} + 2.2^{20} = 2^{25}$

20. The sum of the solutions of the equation $\left|\sqrt{x}-2\right|+\sqrt{x}\left(\sqrt{x}-4\right)+2=0,\ (x>0)$ is equal to :

(1) 4 (2) 9

(3) 10 (4) 12

Official Ans. by NTA (3)

Sol. $\left|\sqrt{x}-2\right|+\sqrt{x}\left(\sqrt{x}-4\right)+2=0$

$\left|\sqrt{x}-2\right|+\left(\sqrt{x}\right)^2-4\sqrt{x}+2=0$

$\left|\sqrt{x}-2\right|^2+\left|\sqrt{x}-2\right|-2=0$

$\left|\sqrt{x}-2\right|=-2$ (not possible) or $\left|\sqrt{x}-2\right|=1$

$\sqrt{x}-2=1,-1$

$\sqrt{x}=3,1$

x = 9, 1

Sum = 10

21. Let A and B be two non-null events such that $A\subset B$. Then, which of the following statements is always correct ?

(1) P(A|B) = 1

(2) P(A|B) = P(B) – P(A)

(3) P(A|B) $\le$ P(A)

(4) P(A|B) $\ge$ P(A)

Official Ans. by NTA (4)

Sol. $P(A\,|\,B)=\dfrac{P(A\cap B)}{P(B)}=\dfrac{P(A)}{P(B)}$

$\left(\text{as } A\subset B\Rightarrow P(A\cap B)=P(A)\right)$

$\Rightarrow\quad P(A\,|\,B)\ge P(A)$

22. The sum of the co-efficients of all even degree terms in x in the expansion of $\left(x+\sqrt{x^3-1}\right)^6+\left(x-\sqrt{x^3-1}\right)^6,(x>1)$ is equal to :

(1) 32 (2) 26

(3) 29 (4) 24

Official Ans. by NTA (4)

Sol. $\left(x+\sqrt{x^3-1}\right)^6+\left(x-\sqrt{x^3-1}\right)^6$

$= 2[{}^6C_0x^6 + {}^6C_2x^4(x^3-1) + {}^6C_4x^2(x^3-1)^2 + {}^6C_6(x^3-1)^3]$

$= 2[{}^6C_0x^6 + {}^6C_2x^7 - {}^6C_2x^4 + {}^6C_4x^8 + {}^6C_4x^2 - 2\,{}^6C_4x^5 + (x^9 - 1 - 3x^6 + 3x^3)]$

$\Rightarrow$ Sum of coefficient of even powers of x

= 2[1 – 15 + 15 + 15 – 1 – 3] = 24

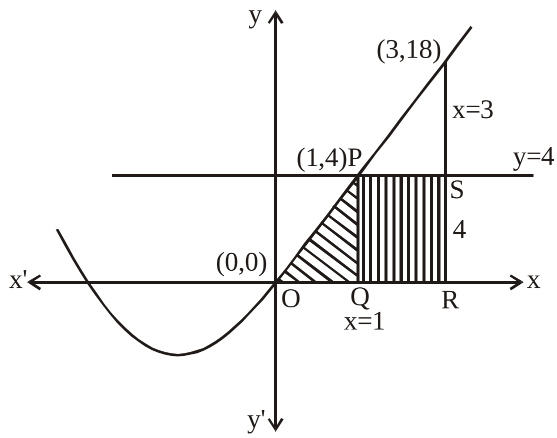
23. The area (in sq. units) of the region $A = \{(x, y) \in R \times R | 0 \le x \le 3,\ 0 \le y \le 4,\ y \le x^2 + 3x\}$ is :

(1) $\frac{53}{6}$ (2) $\frac{59}{6}$

(3) 8 (4) $\frac{26}{3}$

Official Ans. by NTA (2)

Sol.



Required Area

$=\int_0^1 (x^2+3x)dx + \text{Area of rectangle PQRS}$

$=\dfrac{11}{6}+8=\dfrac{59}{6}$

24. Let f : [0, 2] $\to$ R be a twice differentiable function such that $f''(x) > 0$, for all x $\in$ (0, 2). If $\phi(x) = f(x) + f(2-x)$, then ϕ is :

(1) decreasing on (0, 2)

(2) decreasing on (0, 1) and increasing on (1, 2)

(3) increasing on (0, 2)

(4) increasing on (0, 1) and decreasing on (1, 2)

Official Ans. by NTA (2)

Sol. $\phi(x) = f(x) + f(2 - x)$

$\phi'(x) = f'(x) - f'(2 - x)$ ......(1)

Since $f''(x) > 0$

$\Rightarrow f'(x)$ is increasing $\forall$ $x \in (0, 2)$

Case-I : When $x > 2 - x \Rightarrow x > 1$

$\Rightarrow \phi'(x) > 0 \ \forall \ x \in (1, 2)$

$\therefore \phi(x)$ is increasing on (1, 2)

Case-II : When $x < 2 - x \Rightarrow x < 1$

$\Rightarrow \phi'(x) < 0 \ \forall \ x \in (0,1)$

$\therefore \phi(x)$ is decreasing on (0, 1)

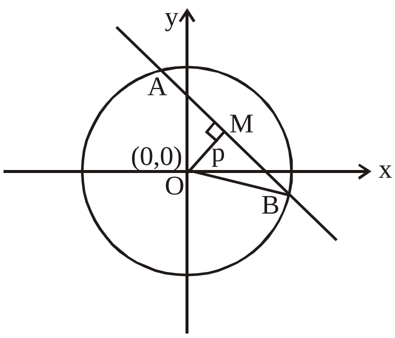
25. The sum of the squares of the lengths of the chords intercepted on the circle, $x^2 + y^2 = 16$, by the lines, $x + y = n$, $n \in N$, where N is the set of all natural numbers, is :

(1) 320 (2) 160

(3) 105 (4) 210

Official Ans. by NTA (4)

Sol.



$p = \frac{n}{\sqrt{2}}$, but $\frac{n}{\sqrt{2}} < 4 \Rightarrow n = 1, 2, 3, 4, 5.$

Length of chord AB = $2\sqrt{16 - \frac{n^2}{2}}$

$= \sqrt{64 - 2n^2} = \ell(\text{say})$

For $n = 1, \ell^2 = 62$

$n = 2, \ell^2 = 56$

$n = 3, \ell^2 = 46$

$n = 4, \ell^2 = 32$

$n = 5, \ell^2 = 14$

$\therefore$ Required sum = 62 + 56 + 46 + 32 + 14 = 210

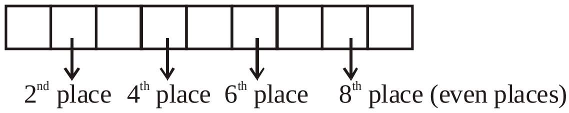
26. All possible numbers are formed using the digits 1, 1, 2, 2, 2, 2, 3, 4, 4 taken all at a time. The number of such numbers in which the odd digits occupy even places is :

(1) 175 (2) 162

(3) 160 (4) 180

Official Ans. by NTA (4)

Sol.



Number of such numbers $= {}^4C_3 \times \frac{3!}{2!} \times \frac{6!}{2!4!} = 180$

27. $\int \frac{\sin\frac{5x}{2}}{\sin\frac{x}{2}} dx$ is equal to :

(where c is a constant of integration)

(1) 2x + sinx + 2sin2x + c

(2) x + 2sinx + 2sin2x + c

(3) x + 2sinx + sin2x + c

(4) 2x + sinx + sin2x + c

Official Ans. by NTA (3)

Sol. $\int \frac{\sin\frac{5x}{2}}{\sin\frac{x}{2}} dx = \int \frac{2\sin\frac{5x}{2}\cos\frac{x}{2}}{2\sin\frac{x}{2}\cos\frac{x}{2}} dx$

$= \int \frac{\sin 3x + \sin 2x}{\sin x} dx$

$= \int \frac{3\sin x - 4\sin^3 x - 2\sin x \cos x}{\sin x} dx$

$= \int (3 - 4\sin^2 x + 2\cos x) dx$

$= \int \left(3 - 2(1 - \cos 2x) + 2\cos x\right) dx$

$= \int \left(1 + 2\cos 2x + 2\cos x\right) dx$

= x + sin2x + 2sinx + c

28. If $2y = \left(\cot^{-1}\left(\frac{\sqrt{3}\cos x + \sin x}{\cos x - \sqrt{3}\sin x}\right)\right)^2, x \in \left(0, \frac{\pi}{2}\right)$,

then $\frac{dy}{dx}$ is equal to :

(1) $2x - \frac{\pi}{3}$ (2) $\frac{\pi}{3} - x$

(3) $\frac{\pi}{6} - x$ (4) $x - \frac{\pi}{6}$

Official Ans. by NTA (4)

Sol. Consider $\cot^{-1}\left(\dfrac{\frac{\sqrt{3}}{2}\cos x+\frac{1}{2}\sin x}{\frac{1}{2}\sin x-\frac{\sqrt{3}}{2}\sin x}\right)$

$$=\cot^{-1}\left(\frac{\sin\left(x+\frac{\pi}{3}\right)}{\cos\left(x+\frac{\pi}{3}\right)}\right)$$

$$=\cot^{-1}\left(\tan\left(x+\frac{\pi}{3}\right)\right)=\frac{\pi}{2}-\tan^{-1}\left(\tan\left(x+\frac{\pi}{3}\right)\right)$$

$$\begin{cases}\frac{\pi}{2}-\left(x+\frac{\pi}{3}\right)=\left(\frac{\pi}{6}-x\right);\ 0<x<\frac{\pi}{6}\\ \frac{\pi}{2}-\left(\left(x-\frac{\pi}{3}\right)-\pi\right)=\left(\frac{7\pi}{6}-x\right);\ \frac{\pi}{6}<x<\frac{\pi}{2}\end{cases}$$

$$\therefore\quad 2y=\begin{cases}\left(\frac{\pi}{6}-x\right)^2;\quad 0<x<\frac{\pi}{6}\\ \left(\frac{7\pi}{6}-x\right)^2;\quad \frac{\pi}{6}<x<\frac{\pi}{2}\end{cases}$$

$$\therefore\quad 2\frac{dy}{dx}=\begin{cases}2\left(\frac{\pi}{6}-x\right).(-1);\quad 0<x<\frac{\pi}{6}\\ 2\left(\frac{7\pi}{6}-x\right).(-1);\quad \frac{\pi}{6}<x<\frac{\pi}{2}\end{cases}$$

29. The greatest value of $c \in R$ for which the system of linear equations

$x - cy - cz = 0$

$cx - y + cz = 0$

$cx + cy - z = 0$

has a non-trivial solution, is :

(1) $\frac{1}{2}$ (2) -1

(3) 0 (4) 2

Official Ans. by NTA (1)

Sol. For non-trivial solution

$D = 0$

$$\begin{vmatrix}1 & -c & -c\\ c & -1 & c\\ c & c & -1\end{vmatrix}=0\Rightarrow 2c^3-3c^2-1=0$$

$\Rightarrow\quad (c + 1)^2 (2c - 1) = 0$

$\therefore$ Greatest value of c is $\frac{1}{2}$

30. If $\cos(\alpha+\beta)=\frac{3}{5}, \sin(\alpha-\beta)=\frac{5}{13}$ and $0<\alpha,\beta<\frac{\pi}{4}$, then $\tan(2\alpha)$ is equal to :

(1) $\frac{21}{16}$ (2) $\frac{63}{52}$

(3) $\frac{33}{52}$ (4) $\frac{63}{16}$

Official Ans. by NTA (4)

Sol. $0<\alpha+\beta=\frac{\pi}{2}$ and $\frac{-\pi}{4}<\alpha-\beta<\frac{\pi}{4}$

if $\cos(\alpha+\beta)=\frac{3}{5}$ then $\tan(\alpha+\beta)=\frac{4}{3}$

and if $\sin(\alpha-\beta)=\frac{5}{13}$ then $\tan(\alpha-\beta)=\frac{5}{12}$

(since $\alpha - \beta$ here lies in the first quadrant)

Now $\tan(2\alpha)=\tan\{(\alpha+\beta)+(\alpha-\beta)\}$

$$=\frac{\tan(\alpha+\beta)+\tan(\alpha-\beta)}{1-\tan(\alpha+\beta).\tan(\alpha-\beta)}=\frac{\frac{4}{3}+\frac{5}{12}}{1-\frac{4}{3}.\frac{5}{12}}=\frac{63}{16}$$