FINAL JEE(Advanced) EXAMINATION - 2021

(Held On Sunday 03rd OCTOBER, 2021)

PAPER-1 | **TEST PAPER WITH SOLUTION**

PART-3 : MATHEMATICS

SECTION-1 : (Maximum Marks : 12)

- This section contains **FOUR (04)** questions.
- Each question has **FOUR** options (A), (B), (C) and (D). **ONLY ONE** of these four options is the correct answer.
- For each question, choose the option corresponding to the correct answer.
- Answer to each question will be evaluated according to the following marking scheme:

Full Marks : +3 If ONLY the correct option is chosen;
Zero Marks : 0 If none of the options is chosen (i.e. the question is unanswered);
Negative Marks : −1 In all other cases.

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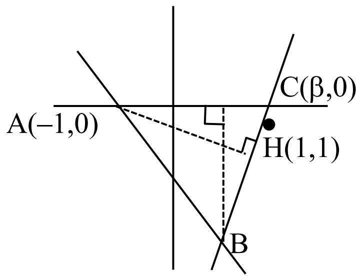
1. Consider a triangle Δ whose two sides lie on the x-axis and the line $x + y + 1 = 0$. If the orthocenter of Δ is (1, 1), then the equation of the circle passing through the vertices of the triangle Δ is

(A) $x^2 + y^2 - 3x + y = 0$ (B) $x^2 + y^2 + x + 3y = 0$
(C) $x^2 + y^2 + 2y - 1 = 0$ (D) $x^2 + y^2 + x + y = 0$

Ans. (B)

Sol.

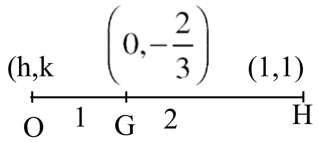


$(1,-2)=(\alpha,-\alpha-1)$
$\Rightarrow \alpha = 1$

one of the vectex is intersection of x-axis and $x + y + 1 = 0 \Rightarrow A(-1,0)$
Let vertex B be $(\alpha,-\alpha-1)$
Line AC $\perp$ BH $\Rightarrow \alpha = 1 \Rightarrow B(1,-2)$
Let vertex C be$(\beta,0)$
Line AH $\perp$ BC
$m_{AH}.m_{BC} = -1$
$\frac{1}{2}.\frac{2}{\beta-1} = -1 \Rightarrow \beta = 0$

Centroid of ΔABC is $\left(0,-\frac{2}{3}\right)$

Now G(centroid) divides line joining circum centre (O) and ortho centre (H) in the ratio 1: 2

$\Rightarrow$ 

$2h + 1 = 0 \qquad 2k + 1 = -z$

$h = -\frac{1}{2} \qquad k = -\frac{3}{2}$

$\Rightarrow$ circum centre is $\left(-\frac{1}{2}, -\frac{3}{2}\right)$

Equation of circum circle is (passing through C(0,0)) is

$x^2 + y^2 + x + 3y = 0$

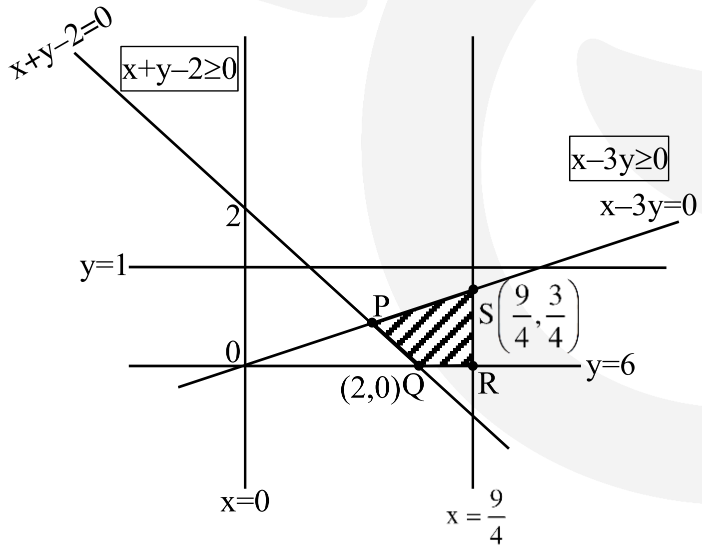
2. The area of the region $\left\{(x,y) : 0 \le x \le \frac{9}{4},\quad 0 \le y \le 1,\quad x \ge 3y,\quad x + y \ge 2\right\}$ is

(A) $\frac{11}{32}$ (B) $\frac{35}{96}$ (C) $\frac{37}{96}$ (D) $\frac{13}{32}$

Ans. (A)

Sol. $x + y - 2 = 0$

$P\left(\frac{3}{2}, \frac{1}{2}\right)$; Q(2, 0); $R\left(\frac{9}{4}, 0\right)$; $S\left(\frac{9}{4}, \frac{3}{4}\right)$



$$\text{Area} = \frac{1}{2}\left|\begin{vmatrix} \frac{3}{2} & \frac{1}{2} \\ 2 & 0 \end{vmatrix} + \begin{vmatrix} 2 & 0 \\ \frac{9}{4} & 0 \end{vmatrix} + \begin{vmatrix} \frac{9}{4} & 0 \\ \frac{9}{4} & \frac{3}{4} \end{vmatrix} + \begin{vmatrix} \frac{9}{4} & \frac{3}{4} \\ \frac{3}{2} & \frac{1}{2} \end{vmatrix}\right|$$

$$= \frac{1}{2}\left|(0-1) + (0-0) + \left(\frac{27}{16} - 0\right) + \left(\frac{9}{8} - \frac{9}{8}\right)\right| = \frac{11}{32}$$

3. Consider three sets $E_1 = \{1, 2, 3\}$, $F_1 = \{1, 3, 4\}$ and $G_1 = \{2, 3, 4, 5\}$. Two elements are chosen at random, without replacement, from the set E_1, and let S_1 denote the set of these chosen elements.

Let $E_2 = E_1 - S_1$ and $F_2 = F_1 \cup S_1$. Now two elements are chosen at random, without replacement, from the set F_2 and let S_2 denote the set of these chosen elements.

Let $G_2 = G_1 \cup S_2$. Finally, two elements are chosen at random, without replacement, from the set G_2 and let S_3 denote the set of these chosen elements.

Let $E_3 = E_2 \cup S_3$. Given that $E_1 = E_3$, let p be the conditional probability of the event $S_1 = \{1, 2\}$. Then the value of p is

(A) $\frac{1}{5}$ (B) $\frac{3}{5}$ (C) $\frac{1}{2}$ (D) $\frac{2}{5}$

Ans. (A)

Sol. $P = \dfrac{P(S_1 \cap (E_1 = E_3))}{P(E_1 = E_3)} = \dfrac{P(B_{1,2})}{P(B)}$

$P(B) = P(B_{1,2}) + P(B_{1,3}) + P(B_{2,3})$

↑ If 1,2 chosen at start; ↑ If 1,3 chosen at start; ↑ If 2,3 chosen at start

$$P(B_{1,2}) = \frac{1}{3} \times \underbrace{\frac{1 \times {}^3C_1}{{}^4C_2}}_{\text{1 is definitely chosen from } F_2} \times \underbrace{\frac{1}{{}^5C_2}}_{\text{1,2 chosen from } G_2}$$

$$P(B_{1,3}) = \frac{1}{3} \times \underbrace{\frac{1 \times {}^2C_1}{{}^3C_2}}_{\text{1 is definitely chosen from } F_2} \times \underbrace{\frac{1}{{}^5C_2}}_{\text{1,2 chosen from } G_2}$$

$$P(B_{2,3}) = \frac{1}{3} \times \left[\underbrace{\frac{{}^3C_2 \times 1}{{}^4C_2}}_{\text{If 1 is not chosen from } F_2} \times \frac{1}{{}^4C_2} + \underbrace{\frac{1 \times {}^3C_1}{{}^4C_2}}_{\text{If 1 is chosen from } F_2} \times \frac{1}{{}^5C_2} \right]$$

$$\frac{P(B_{1,2})}{P(B)} = \frac{1}{5}$$

4. Let $\theta_1, \theta_2, \ldots, \theta_{10}$ be positive valued angles (in radian) such that $\theta_1 + \theta_2 + \ldots + \theta_{10} = 2\pi$. Define the complex numbers $z_1 = e^{i\theta_1}$, $z_k = z_{k-1}e^{i\theta_k}$ for k = 2, 3, ..., 10, where $i = \sqrt{-1}$. Consider the statements P and Q given below :

$P : |z_2 - z_1| + |z_3 - z_2| + \ldots + |z_{10} - z_9| + |z_1 - z_{10}| \le 2\pi$

$Q : |z_2^2 - z_1^2| + |z_3^2 - z_2^2| + \ldots + |z_{10}^2 - z_9^2| + |z_1^2 - z_{10}^2| \le 4\pi$

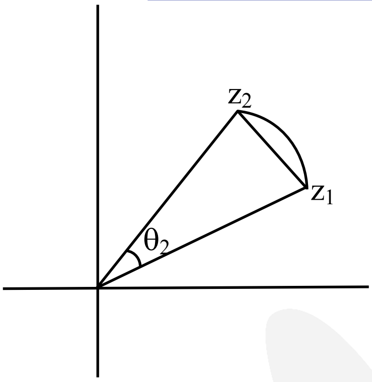
Then,

(A) P is **TRUE** and Q is **FALSE** (B) Q is **TRUE** and P is **FALSE**

(C) both P and Q are **TRUE** (D) both P and Q are **FALSE**

Ans. (C)

Sol.



$|z_1| = |z_2| = \ldots |z_{10}| = 1$

$\text{angle} = \frac{\text{arc}}{\text{rad}}$

$\theta_2 = \text{arc}(z_1 z_2) > (z_2 > z_1)$

$P : |z_2 - z_1| + \ldots + |z_1 - z_{10}| \le \theta_1 + \theta_2 + \ldots + \theta_{10}$

$\Rightarrow |z_2 - z_1| + \ldots + |z_1 - z_{10}| \le 2\pi$ P is true

$z_1^2 = e^{i2\theta_1}$, $z_k^2 = z_{k-1}^2 . e^{i2\theta_k}$

Let $2\theta_k = \alpha k$

$z_1^2 = e^{i\alpha_1}$, $z_k^2 = z_{k-1}^2 . e^{i\alpha_k}$

$\alpha_1 + \alpha_2 + \ldots + \alpha_k = 4\pi$

one similar sense

$|z_1{}^2 - z_2{}^2| + \ldots |z_1{}^2 - z_{10}{}^2| \le 4\pi$

Q is also true

SECTION-2 : (Maximum Marks : 12)

- This section contains **THREE (03)** question stems.
- There are **TWO (02)** questions corresponding to each question stem.
- The answer to each question is a **NUMERICAL VALUE**.
- For each question, enter the correct numerical value corresponding to the answer in the designated place using the mouse and the on-screen virtual numeric keypad.
- If the numerical value has more than two decimal places, **truncate/round-off** the value to **TWO** decimal places.
- Answer to each question will be evaluated according to the following marking scheme:

  Full Marks : +2 If ONLY the correct numerical value is entered at the designated place;

  Zero Marks : 0 In all other cases.

Question Stem for Question Nos. 5 and 6

Question Stem

Three numbers are chosen at random, one after another with replacement, from the set $S = \{1, 2, 3, ..., 100\}$. Let p_1 be the probability that the maximum of chosen numbers is at least 81 and p_2 be the probability that the minimum of chosen numbers is at most 40.

5. The value of $\frac{625}{4} p_1$ is __________.

Ans. (76.25)

Sol. p_1 = probability that maximum of chosen numbers is at least 81

$p_1 = 1$ – probability that maximum of chosen number is at most 80

$$p_1 = 1 - \frac{80 \times 80 \times 80}{100 \times 100 \times 100} = 1 - \frac{64}{125}$$

$$p_1 = \frac{61}{125}$$

$$\frac{625 p_1}{4} = \frac{625}{4} \times \frac{61}{125} = \frac{305}{4} = 76.25$$

the value of $\frac{625 p_1}{4}$ is 76.25

6. The value of $\frac{125}{4} p_2$ is __________.

Ans. (24.50)

Sol. p_2 = probability that minimum of chosen numbers is at most 40

= 1 – probability that minimum of chosen numbers is at least 41

$$= 1 - \left(\frac{600}{100}\right)^3$$

$$= 1 - \frac{27}{125} = \frac{98}{125}$$

$$\therefore \frac{125}{4} p_2 = \frac{125}{4} \times \frac{98}{125} = 24.50$$

Question Stem for Question Nos. 7 and 8

Question Stem

Let α, β and γ be real numbers such that the system of linear equations

$$x + 2y + 3z = \alpha$$

$$4x + 5y + 6z = \beta$$

$$7x + 8y + 9z = \gamma - 1$$

is consistent. Let $|M|$ represent the determinant of the matrix

$$M = \begin{bmatrix} \alpha & 2 & \gamma \\ \beta & 1 & 0 \\ -1 & 0 & 1 \end{bmatrix}$$

Let P be the plane containing all those (α, β, γ) for which the above system of linear equations is consistent, and D be the **square** of the distance of the point (0, 1, 0) from the plane P.

7. The value of $|M|$ is ___________.

Ans. (1.00)

8. The value of D is ___________.

Ans. (1.50)

Solutions 7 & 8

Sol. $7x + 8y + 9z - (\gamma - 1) = A(4x + 5y + 6z - \beta) + B(x + 2y + 3z - \alpha)$

$x : 7 = 4A + B$

$y : 8 = 5A + 2B$

$A = 2, B = -1$

const. term : $-(\gamma - 1) = -A\beta - \alpha B \Rightarrow -(\gamma - 1) \equiv 2\beta + \alpha$

$\alpha - 2\beta + \gamma = 1$

$$M = \begin{pmatrix} \alpha & 2 & \gamma \\ \beta & 1 & 0 \\ -1 & 0 & 1 \end{pmatrix} = \alpha - 2\beta + \gamma = 1$$

Plane P : $x - 2y + z = 1$

Perpendicular distance $= \left|\dfrac{3}{\sqrt{6}}\right| = P \Rightarrow D = P^2 = \dfrac{9}{6} = 1.5$

Question Stem for Question Nos. 9 and 10

Question Stem

Consider the lines L_1 and L_2 defined by

$L_1 : x\sqrt{2}+y-1=0$ and $L_2 : x\sqrt{2}-y+1=0$

For a fixed constant λ, let C be the locus of a point P such that the product of the distance of P from L_1 and the distance of P from L_2 is λ^2. The line y = 2x + 1 meets C at two points R and S, where the distance between R and S is $\sqrt{270}$.

Let the perpendicular bisector of RS meet C at two distinct points R' and S'. Let D be the **square** of the distance between R' and S'.

9. The value of λ^2 is __________.

Ans. **(9.00)**

Sol. $P(x,y)$ $\left|\frac{\sqrt{2}x+y-1}{\sqrt{3}}\right|\left|\frac{\sqrt{2}x-y+1}{\sqrt{3}}\right|=\lambda^2$

$\left|\frac{2x^2-(y-1)^2}{3}\right|=\lambda^2$, $C:\left|2x^2-(y-1)^2\right|=3\lambda^2$

line y = 2x + 1, $RS=\sqrt{(x_1-x_2)^2+(y_1-y_2)^2}$, $R(x_1,y_1)$ and $S(x_2,y_2)$

$y_1 = 2x_1 + 1$ and $y_2 = 2x_2 + 1 \Rightarrow (y_1 - y_2) = 2(x_1 - x_2)$

$RS=\sqrt{5(x_1-x_2)^2}=\sqrt{5}|x_1-x_2|$

solve curve C and line y = 2x+1 we get

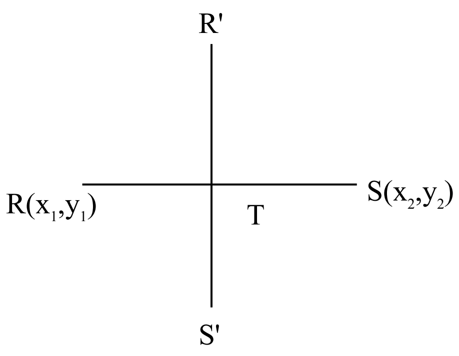
$\left|2x^2-(2x)^2\right|=3\lambda^2 \Rightarrow x^2=\frac{3\lambda^2}{2}$

$RS=\sqrt{5}\left|\frac{2\sqrt{3}\lambda}{\sqrt{2}}\right|=\sqrt{30}\lambda=\sqrt{270} \Rightarrow 30\lambda^2=270 \Rightarrow \lambda^2=9$

10. The value of D is __________.

Ans. **(77.14)**

Sol.



$\perp$ bisector of RS

$$T \equiv \left(\frac{x_1+x_2}{2}, \frac{y_1+y_2}{2}\right)$$

Here $x_1 + x_2 = 0$

$T = (0,1)$

Equation of

$$R'S' : (y-1) = -\frac{1}{2}(x-0) \Rightarrow x+2y=2$$

$R'(a_1,b_1)$ $S'(a_2,b_2)$

$D = (a_1 - a_2)^2 + (b_1 - b_2)^2 = 5(b_1 - b_2)^2$

solve $x+2y=2$ and $\left|2x^2-(y-1)^2\right| = 3\lambda^2$

$$\left|8(y-1)^2-(y-1)^2\right| = 3\lambda^2 \Rightarrow (y-1)^2 = \left(\frac{\sqrt{3}\lambda}{\sqrt{7}}\right)^2$$

$$y-1 = \pm\frac{\sqrt{3}\lambda}{\sqrt{7}} \Rightarrow b_1 = 1+\frac{\sqrt{3}\lambda}{\sqrt{7}}, b_2 = 1-\frac{\sqrt{3}\lambda}{\sqrt{17}}$$

$$D = 5\left(\frac{2\sqrt{3}\lambda}{\sqrt{7}}\right)^2 = \frac{5\times4\times3\lambda^2}{7} = \frac{5\times4\times27}{7} = 77.14$$

SECTION-3 : (Maximum Marks : 24)

- This section contains **SIX (06)** questions.
- Each question has **FOUR** options (A), (B), (C) and (D). **ONE OR MORE THAN ONE** of these four option(s) is (are) correct answer(s).
- For each question, choose the option(s) corresponding to (all) the correct answer(s).
- Answer to each question will be evaluated according to the following marking scheme:

Full Marks	: +4	If only (all) the correct option(s) is(are) chosen;
Partial Marks	: +3	If all the four options are correct but ONLY three options are chosen;
Partial Marks	: +2	If three or more options are correct but ONLY two options are chosen, both of which are correct;
Partial Marks	: +1	If two or more options are correct but ONLY one option is chosen and it is a correct option;
Zero Marks	: 0	If unanswered;
Negative Marks	: −2	In all other cases.

11. For any 3 × 3 matrix M, let |M| denote the determinant of M. Let

$$E = \begin{bmatrix} 1 & 2 & 3 \\ 2 & 3 & 4 \\ 8 & 13 & 18 \end{bmatrix}, P = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 0 & 1 \\ 0 & 1 & 0 \end{bmatrix} \text{ and } F = \begin{bmatrix} 1 & 3 & 2 \\ 8 & 18 & 13 \\ 2 & 4 & 3 \end{bmatrix}$$

If Q is a nonsingular matrix of order 3 × 3, then which of the following statements is (are) **TRUE** ?

(A) $F = PEP$ and $P^2 = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}$

(B) $|EQ + PFQ^{-1}| = |EQ| + |PFQ^{-1}|$

(C) $\left|(EF)^3\right| > |EF|^2$

(D) Sum of the diagonal entries of $P^{-1}EP + F$ is equal to the sum of diagonal entries of $E + P^{-1}FP$

Ans. (A,B,D)

$$PEP = \begin{pmatrix} 1 & 0 & 0 \\ 0 & 0 & 1 \\ 0 & 1 & 0 \end{pmatrix}\begin{pmatrix} 1 & 2 & 3 \\ 2 & 3 & 4 \\ 8 & 13 & 18 \end{pmatrix}\begin{pmatrix} 1 & 0 & 0 \\ 0 & 0 & 1 \\ 0 & 1 & 0 \end{pmatrix}$$

$$\begin{pmatrix} 1 & 2 & 3 \\ 8 & 13 & 18 \\ 2 & 3 & 4 \end{pmatrix}\begin{pmatrix} 1 & 0 & 0 \\ 0 & 0 & 1 \\ 0 & 1 & 0 \end{pmatrix} = \begin{pmatrix} 1 & 3 & 2 \\ 8 & 18 & 13 \\ 2 & 4 & 3 \end{pmatrix}$$

$$P^2 = \begin{pmatrix} 1 & 0 & 0 \\ 0 & 0 & 1 \\ 0 & 1 & 0 \end{pmatrix}\begin{pmatrix} 1 & 0 & 0 \\ 0 & 0 & 1 \\ 0 & 1 & 0 \end{pmatrix} = \begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix}$$

(B) $\left|EQ + PFQ^{-1}\right| = |EQ| + \left|PFQ^{-1}\right|$

$|E| = 0$ and $|F| = 0$ and $|Q| \neq 0$

$|EQ| = |E||Q| = 0$, $\left|PFQ^{-1}\right| = \dfrac{|P||F|}{|Q|} = 0$

$T = EQ + PFQ^{-1}$

$TQ = EQ^2 + PF = EQ^2 + P^2EP = EQ^2 + EP = E(Q^2 + P)$

$|TQ| = \left|E\left(Q^2 + P\right)\right| \Rightarrow |T||Q| = |E|\left|Q^2 + P\right| = 0 \Rightarrow |T| = 0$ (as $|Q| \neq 0$)

(C) $\left|(EF)^3\right| > |EF|^2$

Here 0 > 0 (false)

(D) as $P^2 = I \Rightarrow P^{-1} = P$ so $P^{-1}FP = PFP = PPEPP = E$

so $E + P^{-1}FP = E + E = 2E$

$P^{-1}EP + F \Rightarrow PEP + F = 2PEP$

Tr(2PEP) = 2Tr(PEP) = 2Tr(EPP) = 2Tr(E)

12. Let $f:\mathbb{R}\to\mathbb{R}$ be defined by

$$f(x)=\frac{x^2-3x-6}{x^2+2x+4}.$$

Then which of the following statements is (are) **TRUE** ?

(A) f is decreasing in the interval (–2,–1)

(B) f is increasing in the interval (1,2)

(C) f is onto

(D) Range of f is $\left[-\frac{3}{2},2\right]$

Ans. (A,B)

Sol. $f(x)=\dfrac{x^2-3x-6}{x^2+2x+4}$

$$f'(x)=\frac{\left(x^2+2x+4\right)(2x-3)-\left(x^2-3x-6\right)(2x+2)}{\left(x^2+2x+4\right)^2}$$

$$f'(x)=\frac{5x(x+4)}{\left(x^2+2x+4\right)^2}$$

f '(x) : + (–∞, –4), – (–4, 0), + (0, ∞)

$$f(-4)=\frac{11}{6},\qquad f(0)=-\frac{3}{2},\qquad \lim_{x\to\pm\infty} f(x)=1$$

Range : $\left[-\frac{3}{2},\frac{11}{6}\right]$, clearly f(x) is into

13. Let E,F and G be three events having probabilities

$P(E)=\frac{1}{8},P(F)=\frac{1}{6}$ and $P(G)=\frac{1}{4}$, and let $P(E\cap F\cap G)=\frac{1}{10}$.

For any event H, if H^C denotes its complement, then which of the following statements is(are) **TRUE ?**

(A) $P\left(E\cap F\cap G^C\right)\le\frac{1}{40}$

(B) $P\left(E^C\cap F\cap G\right)\le\frac{1}{15}$

(C) $P\left(E\cup F\cup G\right)\le\frac{13}{24}$

(D) $P\left(E^C\cap F^C\cap G^C\right)\le\frac{5}{12}$

Ans. (A,B,C)

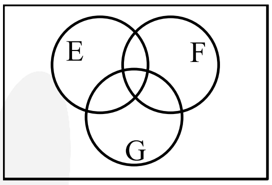
Sol. $P(E)=\frac{1}{8}$; $P(F)=\frac{1}{6}$; $P(G)=\frac{1}{4}$; $P(E\cap F\cap G)=\frac{1}{10}$

(C) $P(E\cup F\cup G) = P(E) + P(F) + P(G) - P(E\cap F) - P(F\cap G) - P(G\cap E) + P(E\cap F\cap G)$

$$=\frac{1}{8}+\frac{1}{6}+\frac{1}{4}-\sum P(E\cap F)+\frac{1}{10}$$

$= \frac{3+4+6}{24} + \frac{1}{10} - \sum P(E \cap F) = \frac{13}{24} + \frac{1}{10} - \sum P(E \cap F)$

$\Rightarrow P(E \cup F \cup G) \le \frac{13}{24}$ [(C) is Correct]



(D) $P(E^C \cap F^C \cap G^C) = 1 - P(E \cup F \cup G) \ge 1 - \frac{13}{24}$

$\Rightarrow P(E^C \cap F^C \cap G^C) \ge \frac{11}{24}$ [(D) is Incorrect]

(A) $P(E) = \frac{1}{8} \ge P(E \cap F \cap G^C) + P(E \cap F \cap G)$

$\Rightarrow \frac{1}{8} \ge P(E \cap F \cap G^c) + \frac{1}{10} \Rightarrow \frac{1}{8} - \frac{1}{10} \ge P(E \cap F \cap G^c)$

$\Rightarrow \frac{1}{40} \ge P(E \cap F \cap G^c)$ [(A) is Correct]

(B) $P(F) = \frac{1}{6} \ge P(E^C \cap F \cap G) + P(E \cap F \cap G)$

$\Rightarrow \frac{1}{6} - \frac{1}{10} \ge P(E^C \cap F \cap G)$

$\Rightarrow \frac{4}{60} \ge P(E^C \cap F \cap G)$

$\Rightarrow \frac{1}{15} \ge P(E^C \cap F \cap G)$ [(B) is Correct]

14. For any 3×3 matrix M, let |M| denote the determinant of M. Let I be the 3×3 identity matrix. Let E and F be two 3×3 matrices such that $(I - EF)$ is invertible. If $G = (I - EF)^{-1}$, then which of the following statements is (are) **TRUE ?**

(A) $|FE| = |I - FE||FGE|$

(B) $|I - FE|(I + FGE) = I$

(C) $EFG = GEF$

(D) $(I - FE)(I - FGE) = I$

Ans. (A,B,C)

Sol. $|I - EF| \ne 0$; $G = (I - EF)^{-1} \Rightarrow G^{-1} = I - EF$

Now, $G.G^{-1} = I = G^{-1} G$

$\Rightarrow G (I - EF) = I = (I - EF) G$

$\Rightarrow G - GEF = I = G - EFG$

$\Rightarrow GEF = EFG$ [C is Correct]

$(I - FE)(I + FGE) = I + FGE - FE - FEFGE$
$= I + FGE - FE - F(G - I)E$
$= I + FGE - FE - FGE + FE$
$= I$ [(B) is Correct]

(So 'D' is Incorrect)

We have

$(I - FE)(I + FGE) = I$ …..(I)

Now

FE(I + FGE)
= FE + FEFGE
= FE + F(G – I)E
= FE + FGE – FE
= FGE
$\Rightarrow$ |FE| |I + FGE| = |FGE|
$\Rightarrow |FE| \times \dfrac{1}{|I - FE|} = |FGE|$ (from (1))
$\Rightarrow$ |FE| = |I–FE| |FGE|
(option (A) is correct)

15. For any positive integer n, let $S_n : (0, \infty) \to \mathbb{R}$ be defined by

$$S_n(x) = \sum_{k=1}^{n} \cot^{-1}\left(\frac{1 + k(k+1)x^2}{x}\right),$$

where for any $x \in \mathbb{R}$, $\cot^{-1}x \in (0,\pi)$ and $\tan^{-1}(x) \in \left(-\frac{\pi}{2}, \frac{\pi}{2}\right)$. Then which of the following statements is (are) **TRUE** ?

(A) $S_{10}(x) = \frac{\pi}{2} - \tan^{-1}\left(\frac{1+11x^2}{10x}\right)$, for all $x > 0$

(B) $\lim_{n\to\infty} \cot(S_n(x)) = x$, for all $x > 0$

(C) The equation $S_3(x) = \frac{\pi}{4}$ has a root in $(0,\infty)$

(D) $\tan(S_n(x)) \le \frac{1}{2}$, for all $n \ge 1$ and $x > 0$

Ans. (A,B)

Sol. $S_n(x) = \sum_{k=1}^{n} \tan^{-1}\left(\frac{x}{1 + kx(kx + x)}\right)$

$= \sum_{k=1}^{n} \tan^{-1}\left(\frac{(kx+x)-(kx)}{1+(kx+x)(kx)}\right)$

$S_n(x) = \tan^{-1}(nx + x) - \tan^{-1}x = \tan^{-1}\left(\frac{nx}{1+(n+1)x^2}\right)$

(A) $S_{10}(x) = \tan^{-1}\frac{10x}{1+11x^2} = \frac{\pi}{2} - \tan^{-1}\left(\frac{1+11x^2}{10x}\right)$ $(x > 0)$

(B) $\lim_{n\to\infty} \cot(S_n(x)) = \lim_{n\to\infty} \frac{\frac{1}{n} + \left(1+\frac{1}{n}\right)x^2}{x} = x \ (x > 0)$

(C) $S_3(x) = \tan^{-1}\frac{3x}{1+4x^2} = \frac{\pi}{4} \Rightarrow 4x^2 - 3x + 1 = 0 \Rightarrow x \notin \mathbb{R}$

(D) $\tan(S_n(x)) = \frac{nx}{1+(n+1)x^2}$; $\forall\ n \geq 1$; $x > 0$

We need to check the validity of $\frac{nx}{1+(n+1)x^2} \leq \frac{1}{2}$ $\forall\ n \geq 1$; $x > 0$; $n \in \mathbb{N}$

$\Rightarrow 2nx \leq (n+1)x^2 + 1$

$\Rightarrow (n+1)x^2 - 2nx + 1 \geq 0\ \forall\ n \geq 1$; $x > 0$; $n \in \mathbb{N}$

Discriminant of $y = (n+1)x^2 - 2nx + 1$ is

$D = 4n^2 - 4(n+1)$ and $n \in \mathbb{N}$

$D < 0$ for $n = 1$; true for $x > 0$

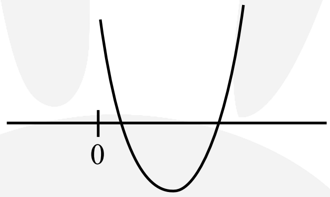
$D > 0$ for $n \geq 2 \Rightarrow \exists$ some $x > 0$

for which $y < 0$ as both roots of

$y = 0$ will be positive.

$y = (n+1)x^2 - 2nx + 1, n \geq 2$

So, $y \geq 0\ \forall\ n \geq 1$; $\forall\ x > 0$; $n \in N$ is false.



16. For any complex number $w = c + id$, let $\arg(w) \in (-\pi, \pi]$, where $i = \sqrt{-1}$. Let α and β be real numbers such that for all complex numbers $z = x + iy$ satisfying $\arg\left(\frac{z+\alpha}{z+\beta}\right) = \frac{\pi}{4}$, the ordered pair (x,y) lies on the circle

$$x^2 + y^2 + 5x - 3y + 4 = 0.$$

Then which of the following statements is (are) **TRUE** ?

(A) $\alpha = -1$ (B) $\alpha\beta = 4$ (C) $\alpha\beta = -4$ (D) $\beta = 4$

Ans. **(B,D)**

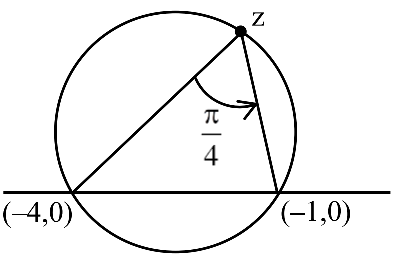
Sol. $\arg\left(\frac{z+\alpha}{z+\beta}\right) = \frac{\pi}{4}$ implies z is

on arc and $(-\alpha, 0)$ & $(-\beta, 0)$ subtend $\frac{\pi}{4}$ on z.

And z lies on $x^2 + y^2 + 5x - 3y + 4 = 0$

So put $y = 0$;

$x^2 + 5x + 4 = 0 \Rightarrow x = -1$; $x = -4$



Now, $\arg\left(\frac{z+\alpha}{z+\beta}\right)=\frac{\pi}{4} \quad \Rightarrow z+\alpha=(z+\beta).r.e^{i\frac{\pi}{4}}$

So, $z+\beta=z+4 \Rightarrow \beta=4$ & $z+\alpha=z+1 \Rightarrow \alpha=1$

SECTION-4 : (Maximum Marks : 12)

- This section contains **THREE (03)** questions.
- The answer to each question is a **NON-NEGATIVE INTEGER.**
- For each question, enter the correct integer corresponding to the answer using the mouse and the on-screen virtual numeric keypad in the place designated to enter the answer.
- Answer to each question will be evaluated according to the following marking scheme:

Full Marks : +4 If ONLY the correct integer is entered;

Zero Marks : 0 In all other cases.

17. For $x \in \mathbb{R}$, then number of real roots of the equation $3x^2-4|x^2-1|+x-1=0$ is ____.

Ans. (4)

Sol. $3x^2+x-1=4|x^2-1|$

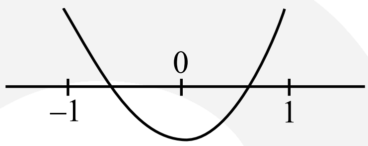
If $x \in [-1, 1]$,

$3x^2+x-1=-4x^2+4 \Rightarrow 7x^2+x-5=0$

say $f(x)=7x^2+x-5$

$f(1)=3;\ f(-1)=1;\ f(0)=-1$

[Two Roots]



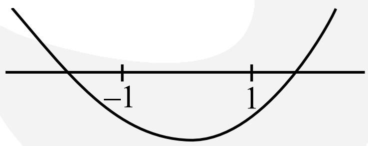
If $x \in (-\infty,-1] \cup [1,\infty)$

$3x^2+x-1=4x^2-4 \Rightarrow x^2-x-3=0$

Say $g(x)=x^2-x-3$

$g(-1)=-1;\ g(1)=-3$

[Two Roots]

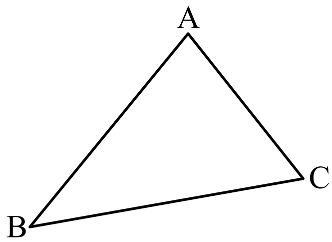


So total 4 roots.

18. In a triangle ABC, let $AB=\sqrt{23}$, $BC=3$ and CA= 4. Then the value of $\frac{\cot A+\cot C}{\cot B}$ is_____.

Ans. (2)

Sol.



Given $c=\sqrt{23}$; $a=3$; $b=4$

$$\cot A=\frac{\cos A}{\sin A}=\frac{b^2+c^2-a^2}{2bc\sin A}$$

$$=\frac{b^2+c^2-a^2}{2.2\Delta} \quad \left\{\Delta=\frac{1}{2}bc\sin A\right\}$$

$$\cot A=\frac{b^2+c^2-a^2}{4\Delta}$$

Similarly, $\cot B = \dfrac{a^2+c^2-b^2}{4\Delta}$ & $\cot C = \dfrac{a^2+b^2-c^2}{4\Delta}$

$\therefore \dfrac{\cot A + \cot C}{\cot B} = \dfrac{b^2+c^2-a^2+a^2+b^2-c^2}{a^2+c^2-b^2} = \dfrac{2b^2}{a^2+c^2-b^2} = \dfrac{32}{16} = 2$

19. Let $\vec{u}, \vec{v}$ and $\vec{w}$ be vectors in three-dimensional space, where $\vec{u}$ and $\vec{v}$ are unit vectors which are not perpendicular to each other and $\vec{u}.\vec{w} = 1$, $\vec{v}.\vec{w} = 1$, $\vec{w}.\vec{w} = 4$

If the volume of the parallelopiped, whose adjacent sides are represented by the vectors $\vec{u}, \vec{v}$ and $\vec{w}$, is $\sqrt{2}$, then the value of $|3\vec{u} + 5\vec{v}|$ is____.

Ans. **(7)**

Sol. Given, $|\vec{u}| = 1$; $|\vec{v}| = 1$; $\vec{u}.\vec{v} \neq 0$; $\vec{u}.\vec{w} = 1$; $\vec{v}.\vec{w} = 1$;

$\vec{w}.\vec{w} = |\vec{w}|^2 = 4 \Rightarrow |\vec{w}| = 2$; $[\vec{u} \;\; \vec{v} \;\; \vec{w}] = \sqrt{2}$

and $[\vec{u} \;\; \vec{v} \;\; \vec{w}]^2 = \begin{vmatrix} \vec{u}.\vec{u} & \vec{u}.\vec{v} & \vec{u}.\vec{w} \\ \vec{v}.\vec{u} & \vec{v}.\vec{v} & \vec{v}.\vec{w} \\ \vec{w}.\vec{u} & \vec{w}.\vec{v} & \vec{w}.\vec{w} \end{vmatrix} = 2$

$\Rightarrow \begin{vmatrix} 1 & \vec{u}.\vec{v} & 1 \\ \vec{u}.\vec{v} & 1 & 1 \\ 1 & 1 & 4 \end{vmatrix} = 2$

$\Rightarrow \vec{u}.\vec{v} = \dfrac{1}{2}$

So, $|3\vec{u} + 5\vec{v}| = \sqrt{9|\vec{u}|^2 + 25|\vec{v}|^2 + 2.3.5\vec{u}.\vec{v}}$

$= \sqrt{9 + 25 + 30\left(\dfrac{1}{2}\right)} = \sqrt{49} = 7$