FINAL JEE(Advanced) EXAMINATION - 2021

(Held On Sunday 03rd OCTOBER, 2021)

PAPER-2 | **TEST PAPER WITH SOLUTION**

PART-3 : MATHEMATICS

SECTION-1 : (Maximum Marks : 24)

- This section contains **SIX (06)** questions.
- Each question has **FOUR** options (A), (B), (C) and (D). **ONE OR MORE THAN ONE** of these four option(s) is (are) correct answer(s).
- For each question, choose the option(s) corresponding to (all) the correct answer(s).
- Answer to each question will be evaluated according to the following marking scheme:

  Full Marks : +4 If only (all) the correct option(s) is(are) chosen;
  Partial Marks : +3 If all the four options are correct but ONLY three options are chosen;
  Partial Marks : +2 If three or more options are correct but ONLY two options are chosen, both of which are correct;
  Partial Marks : +1 If two or more options are correct but ONLY one option is chosen and it is a correct option;
  Zero Marks : 0 If unanswered;
  Negative Marks : –2 In all other cases.
- For example, in a question, if (A), (B) and (D) are the ONLY three options corresponding to correct answers, then
  choosing ONLY (A), (B) and (D) will get +4 marks;
  choosing ONLY (A) and (B) will get +2 marks;
  choosing ONLY (A) and (D) will get +2 marks;
  choosing ONLY (B) and (D) will get +2 marks;
  choosing ONLY (A) will get +1 mark;
  choosing ONLY (B) will get +1 mark;
  choosing ONLY (D) will get +1 mark;
  choosing no option(s) (i.e. the question is unanswered) will get 0 marks and
  choosing any other option(s) will get –2 marks.

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1. Let

$$S_1 = \{(i, j, k) : i, j, k \in \{1, 2, \ldots, 10\}\}$$
$$S_2 = \{(i, j) : 1 \le i < j + 2 \le 10,\ i, j \in \{1, 2, \ldots, 10\}\},$$
$$S_3 = \{(i, j, k, l) : 1 \le i < j < k < l,\ i, j, k, l \in \{1, 2, \ldots, 10\}\}.$$

and

$$S_4 = \{(i, j, k, l) : i, j, k \text{ and } l \text{ are distinct elements in } \{1, 2, \ldots, 10\}\}.$$

If the total number of elements in the set S_r is n_r, $r = 1, 2, 3, 4$, then which of the following statements is (are) **TRUE**?

(A) $n_1 = 1000$ (B) $n_2 = 44$ (C) $n_3 = 220$ (D) $\frac{n_4}{12} = 420$

Ans. (A,B,D)

Sol. (A) $n_1 = 10 \times 10 \times 10 = 1000$

(B) As per given condition $1 \le i < j + 2 \le 10 \Rightarrow\ j \le 8\ \&\ i \ge 1$

for $i = 1, 2,\quad j = 1, 2, 3, ..., 8 \rightarrow (8 + 8)$ possibilities

for $i = 3,\quad j = 2, 3, ..., 8 \rightarrow 7$ possibilities

$i = 4,\quad j = 3, ..., 8 \rightarrow 6$ possibilities

$i = 9,\quad j = 1\quad \rightarrow 1$ possibility

So $n_2 = (1 + 2 + 3 + ...+ 8) + 8 = 44$

(C) $n_3 = {}^{10}C_4$ (Choose any four)

$= 210$

(D) $n_4 = {}^{10}C_4 . 4! = (210)(24)$

$\Rightarrow \dfrac{n_4}{12} = 420$

So correct Ans. (A), (B), (D)

2. Consider a triangle PQR having sides of lengths p, q and r opposite to the angles P, Q and R, respectively. Then which of the following statements is (are) **TRUE**?

(A) $\cos P \ge 1 - \dfrac{p^2}{2qr}$

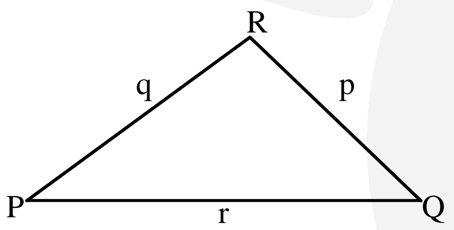
(B) $\cos R \ge \left(\dfrac{q-r}{p+q}\right)\cos P + \left(\dfrac{p-r}{p+q}\right)\cos Q$

(C) $\dfrac{q+r}{p} < 2\dfrac{\sqrt{\sin Q \sin R}}{\sin P}$

(D) If $p < q$ and $p < r$, then $\cos Q > \dfrac{p}{r}$ and $\cos R > \dfrac{p}{q}$

Ans. **(A,B)**

Sol.



(A) $\text{Cos}P = \dfrac{q^2 + r^2 - p^2}{2qr} = \dfrac{q^2 + r^2}{2qr} - \dfrac{p^2}{2qr} \ge 1 - \dfrac{p^2}{2qr}$

(as $p^2 + q^2 \ge 2qr$ (AM $\ge$ GM)), so (A) is correct

(B) $(p + q) \cos R \ge (q - r) \cos P + (p - r) \cos Q$

$\Rightarrow (p \cos R + r \cos P) + (q \cos R + r \cos Q) \ge q\cos P + p \cos Q$

$\Rightarrow q + p \ge r$

So (B) is correct

(C) $\dfrac{q+r}{p} = \dfrac{\sin Q + \sin R}{\sin P} \ge \dfrac{2\sqrt{\sin Q \times \sin R}}{\sin P}$ so (C) is incorrect

(D) $\cos Q > \dfrac{p}{r}$ $\quad\Rightarrow \sin R \cos Q > \sin P$

$\Rightarrow \sin P + \sin (R - Q) > 2 \sin P$

$\Rightarrow \sin (R - Q) > \sin P$

need not necessarily hold true if $R < Q$

Hence (A), (B)

3. Let $f:\left[-\frac{\pi}{2},\frac{\pi}{2}\right]\to\mathbb{R}$ be a continuous function such that

$$f(0)=1 \text{ and } \int_0^{\frac{\pi}{3}} f(t)\,dt=0$$

Then which of the following statements is (are) **TRUE**?

(A) The equation $f(x)-3\cos 3x=0$ has at least one solution in $\left(0,\frac{\pi}{3}\right)$

(B) The equation $f(x)-3\sin 3x=-\frac{6}{\pi}$ has at least one solution in $\left(0,\frac{\pi}{3}\right)$

(C) $\lim\limits_{x\to 0}\frac{x\int_0^x f(t)dt}{1-e^{x^2}}=-1$

(D) $\lim\limits_{x\to 0}\frac{\sin x\int_0^x f(t)dt}{x^2}=-1$

Ans. **(A,B,C)**

Sol. (A) Let $g(x)=f(x)-3\cos 3x$

Now $\int\limits_0^{\pi/3} g(x)dx=\int\limits_0^{\pi/3} f(x)dx-3\int\limits_0^{\pi/3}\cos 3x\,dx=0$

Hence $g(x)=0$ has a root in $\left(0,\frac{\pi}{3}\right)$

(B) Let $h(x)=f(x)-3\sin 3x+\frac{6}{\pi}$

Now $\int\limits_0^{\pi/3} h(x)dx=\int\limits_0^{\pi/3} f(x)dx-3\int\limits_0^{\pi/3}\sin 3x\,dx+\int\limits_0^{\pi/3}\frac{6}{\pi}dx$

$=0-2+2=0$

Hence $h(x)=0$ has a root in $\left(0,\frac{\pi}{3}\right)$

(C) $\lim\limits_{x\to 0}\frac{x\int\limits_0^x f(t)\,dt}{1-e^{x^2}}=\lim\limits_{x\to 0}\underbrace{\left(\frac{x^2}{1-e^{x^2}}\right)}_{-1}\underbrace{\frac{\int\limits_0^x f(t)\,dt}{x}}_{\text{Apply L' Hopital's Rule}}$

$=-1\lim\limits_{x\to 0}\frac{f(x)}{1}=-1$

(D) $\lim_{x\to 0} \frac{(\sin x)\int_0^x f(t)\,dt}{x^2}$

$$= \lim_{x\to 0} \underbrace{\left(\frac{\sin x}{x}\right)}_{1} \underbrace{\frac{\int_0^x f(t)\,dt}{x}}_{\text{Apply L' Hopitals Rule}}$$

$$= 1 \lim_{x\to 0} \frac{f(x)}{1} = 1$$

Ans. A,B,C

4. For any real numbers α and β, let $y_{\alpha,\beta}(x)$, $x \in \mathbb{R}$, be the solution of the differential equation

$$\frac{dy}{dx} + \alpha y = xe^{\beta x},\ y(1) = 1$$

Let $S = \{y_{\alpha,\beta}(x) : \alpha, \beta \in \mathbb{R}\}$. Then which of the following functions belong(s) to the set S?

(A) $f(x) = \frac{x^2}{2}e^{-x} + \left(e - \frac{1}{2}\right)e^{-x}$

(B) $f(x) = -\frac{x^2}{2}e^{-x} + \left(e + \frac{1}{2}\right)e^{-x}$

(C) $f(x) = \frac{e^x}{2}\left(x - \frac{1}{2}\right) + \left(e - \frac{e^2}{4}\right)e^{-x}$

(D) $f(x) = \frac{e^x}{2}\left(\frac{1}{2} - x\right) + \left(e + \frac{e^2}{4}\right)e^{-x}$

Ans. (A,C)

Sol. Integrating factor = $e^{\alpha x}$

So $ye^{\alpha x} = \int xe^{(\alpha+\beta)x}dx$

Case-I

If $\alpha + \beta = 0$ $\quad ye^{\alpha x} = \frac{x^2}{2} + c$

It passes through (1, 1) $\Rightarrow C = e^{\alpha} - \frac{1}{2}$

So $ye^{\alpha x} = \frac{x^2 - 1}{2} + e^{\alpha}$

for $\alpha = 1$

$$y = \frac{x^2}{2}e^{-x} + \left(e - \frac{1}{2}\right)e^{-x} \to (A)$$

Case-II

If $\alpha + \beta \neq 0$

$$ye^{\alpha x} = \frac{x.e^{(\alpha+\beta)x}}{\alpha+\beta} - \frac{1}{\alpha+\beta}e^{(\alpha+\beta)x}\,dx$$

$$\Rightarrow ye^{\alpha x} = \frac{xe^{(\alpha+\beta)x}}{\alpha+\beta} - \frac{e^{(\alpha+\beta)x}}{(\alpha+\beta)^2} + c$$

$$\Rightarrow \text{So } c = e^{\alpha} - \frac{e^{\alpha+\beta}}{\alpha+\beta} + \frac{e^{\alpha+\beta}}{(\alpha+\beta)^2}$$

$$y = \frac{e^{\beta x}}{(\alpha+\beta)^2}\left((\alpha+\beta)x - 1\right) + e^{-\alpha x}\left(e^{x} - \frac{e^{\alpha+\beta}}{\alpha+\beta} + \frac{e^{\alpha+\beta}}{(\alpha+\beta)^2}\right)$$

If $\alpha = \beta = 1$

$$y = \frac{e^{x}}{4}(2x-1) + e^{-x}\left(e - \frac{e^2}{2} + \frac{e^2}{4}\right)$$

$$y = \frac{e^{x}}{2}\left(x - \frac{1}{2}\right) + e^{-x}\left(e - \frac{e^2}{4}\right) \rightarrow (c)$$

Ans. (A) & (C)

5. Let O be the origin and $\overrightarrow{OA} = 2\hat{i} + 2\hat{j} + \hat{k}$, $\overrightarrow{OB} = \hat{i} - 2\hat{j} + 2\hat{k}$ and $\overrightarrow{OC} = \frac{1}{2}\left(\overrightarrow{OB} - \lambda\overrightarrow{OA}\right)$ for some $\lambda > 0$. If $\left|\overrightarrow{OB} \times \overrightarrow{OC}\right| = \frac{9}{2}$, then which of the following statements is (are) **TRUE**?

(A) Projection of $\overrightarrow{OC}$ on $\overrightarrow{OA}$ is $-\frac{3}{2}$

(B) Area of the triangle OAB is $\frac{9}{2}$

(C) Area of the triangle ABC is $\frac{9}{2}$

(D) The acute angle between the diagonals of the parallelogram with adjacent sides $\overrightarrow{OA}$ and $\overrightarrow{OC}$ is $\frac{\pi}{3}$

Ans. (A,B,C)

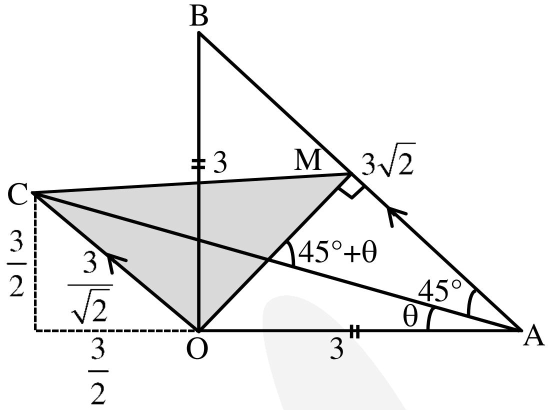
Sol. $\overrightarrow{OB} \times \overrightarrow{OC} = \frac{1}{2}\overrightarrow{OB} \times \left(\overrightarrow{OB} - \lambda\overrightarrow{OA}\right)$

$= \frac{\lambda}{2}\left(\overrightarrow{OA} \times \overrightarrow{OB}\right)$

$\left|\overrightarrow{OB}\right| \times \left|\overrightarrow{OC}\right| = \frac{|\lambda|}{2}\left|\overrightarrow{OA}\right| \times \left|\overrightarrow{OB}\right|$ (Note $\overrightarrow{OA}$ & $\overrightarrow{OB}$ are perpendicular)

$\Rightarrow \frac{9\lambda}{2} = \frac{9}{2} \Rightarrow \lambda = 1$ (given $\lambda > 0$)

So $\overrightarrow{OC} = \frac{\overrightarrow{OB} - \overrightarrow{OA}}{2} = \frac{\overrightarrow{AB}}{2}$



M is mid point of AB

Note projection of $\overrightarrow{OC}$ on $\overrightarrow{OA} = -\frac{3}{2}$

$\tan\theta = \frac{1}{3}$

Area of $\Delta ABC = \frac{9}{2}$

Acute angle between diagonals is

$$\tan^{-1}\left(\frac{1+\frac{1}{3}}{1-\frac{1}{3}}\right) = \tan^{-1} 2$$

6. Let E denote the parabola $y^2 = 8x$. Let $P = (-2, 4)$, and let Q and Q' be two distinct points on E such that the lines PQ and PQ' are tangents to E. Let F be the focus of E. Then which of the following statements is (are) **TRUE**?

(A) The triangle PFQ is a right-angled triangle

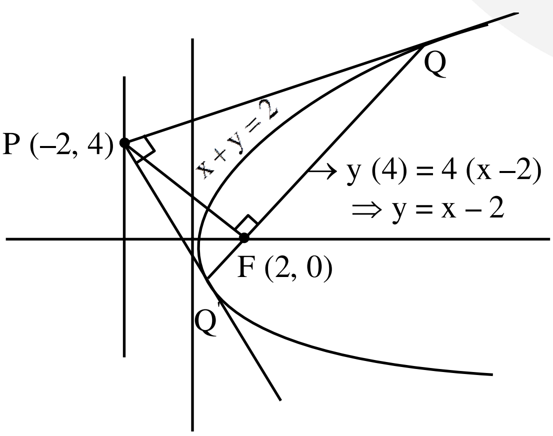
(B) The triangle QPQ' is a right-angled triangle

(C) The distance between P and F is $5\sqrt{2}$

(D) F lies on the line joining Q and Q'

Ans. (A,B,D)

Sol.



Note that P lies on directrix so triangle PQQ' is right angled, hence QQ' passes through focus F.
PF = $4\sqrt{2}$
Equation of QF is y = x – 2 & PF is x + y = 2
Hence A, B, D.

SECTION-2 : (Maximum Marks : 12)

- This section contains **THREE (03)** question stems.
- There are **TWO (02)** questions corresponding to each question stem.
- The answer to each question is a **NUMERICAL VALUE**.
- For each question, enter the correct numerical value corresponding to the answer in the designated place using the mouse and the on-screen virtual numeric keypad.
- If the numerical value has more than two decimal places, **truncate/round-off** the value to **TWO** decimal places.
- Answer to each question will be evaluated according to the following marking scheme:

  Full Marks : +2 If ONLY the correct numerical value is entered at the designated place;

  Zero Marks : 0 In all other cases.

Question Stem for Questions Nos. 7 and 8

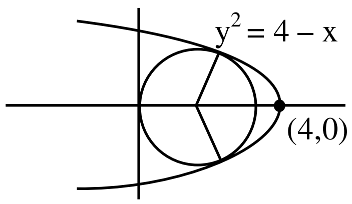
Question Stem

Consider the region R = $\{(x, y) \in \mathbb{R} \times \mathbb{R} : x \geq 0 \text{ and } y^2 \leq 4 - x\}$. Let *F* be the family of all circles that are contained in R and have centers on the x-axis. Let C be the circle that has largest radius among the circles in *F*. Let (α, β) be a point where the circle C meets the curve $y^2 = 4 - x$.

7. The radius of the circle C is _____.

Ans. (1.50)

Sol.



Let the circle be
$x^2 + y^2 + \lambda x = 0$
For point of intersection of circle & parabola $y^2 = 4 - x$.
$x^2 + 4 - x + \lambda x = 0 \Rightarrow x^2 + x(\lambda - 1) + 4 = 0$
For tangency : $\Delta = 0 \Rightarrow (\lambda - 1)^2 - 16 = 0 \Rightarrow \lambda = 5$ (rejected) or $\lambda = -3$
Circle : $x^2 + y^2 - 3x = 0$
Radius = $\frac{3}{2}$ = 1.5

8. The value of α is _____.

Ans. **(2.00)**

Sol. For point of intersection :

$x^2 - 4x + 4 = 0 \Rightarrow x = 2$ so $\alpha = 2$

Question Stem for Questions Nos. 9 and 10

Question Stem

Let $f_1 : (0, \infty) \to \mathbb{R}$ and $f_2 : (0, \infty) \to \mathbb{R}$ be defined by

$$f_1(x) = \int_0^x \prod_{j=1}^{21} (t - j)^j \, dt, \quad x > 0$$

and

$$f_2(x) = 98(x-1)^{50} - 600(x-1)^{49} + 2450, \quad x > 0,$$

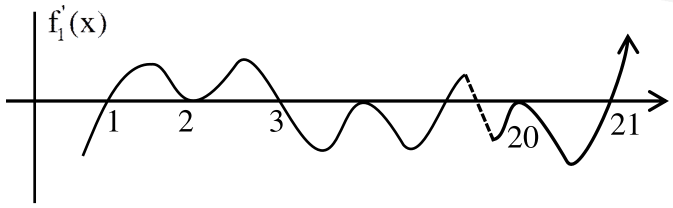
where, for any positive integer n and real numbers $a_1, a_2, \ldots, a_n$, $\prod_{i=1}^{n} a_i$ denotes the product of $a_1, a_2, \ldots, a_n$. Let m_i and n_i, respectively, denote the number of points of local minima and the number of points of local maxima of function f_i, $i = 1, 2$, in the interval $(0, \infty)$

9. The value of $2m_1 + 3n_1 + m_1 n_1$ is _____.

Ans. **(57.00)**

Sol. $f_1(x) = \int_0^x \prod_{j=1}^{21} (t - j)^j dt$

$$f_1'(x) = \prod_{j=1}^{21} (x - j)^j = (x-1)(x-2)^2(x-3)^3 \ldots\ldots(x-21)^{21}$$



So points of minima one $4m + 1$ where $m = 0, 1, \ldots\ldots 5 \Rightarrow m_1 = 6$

Points of maxima are $4m - 1$ where $m = 1, 2, \ldots\ldots 5 \Rightarrow n_1 = 5$

$\Rightarrow 2m_1 + 3n_1 + m_1 n_1 = 57$

10. The value of $6m_2 + 4n_2 + 8m_2n_2$ is _____.

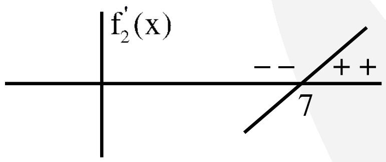
Ans. (6.00)

Sol. $f'_2(x) = 98(x-1)^{50} - 600(x-1)^{49} + 2450$

$\Rightarrow f'_2(x) = 2 \times 49 \times 50(x-1)^{49} - 50 \times 12 \times 49(x-1)^{48}$

$= 50 \times 49 \times 2(x-1)^{48}(x-1-6)$

$= 50 \times 49 \times 2\,(x-1)^{48}\,(x-7)$



Point of minima = 7

$\Rightarrow m_2 = 1$

No point of maxima

$\Rightarrow n_2 = 0$

$6m_2 + 4n_2 + 8m_2n_2 = 6$

Question Stem for Questions Nos. 11 and 12

Question Stem

Let $g_i : \left[\frac{\pi}{8}, \frac{3\pi}{8}\right] \to \mathbb{R}$, $i = 1, 2$, and $f : \left[\frac{\pi}{8}, \frac{3\pi}{8}\right] \to \mathbb{R}$ be functions such that

$g_1(x) = 1$, $g_2(x) = |4x - \pi|$ and $f(x) = \sin^2 x$, for all $x \in \left[\frac{\pi}{8}, \frac{3\pi}{8}\right]$

Define $S_i = \int_{\frac{\pi}{8}}^{\frac{3\pi}{8}} f(x) \cdot g_i(x)\,dx$, $i = 1, 2$

11. The value of $\frac{16S_1}{\pi}$ is _____.

Ans. (2.00)

Sol. $S_1 = \int_{\pi/8}^{3\pi} f(x)\,dx = \int_{\pi/8}^{3\pi/8} \sin^2 x\,dx = \int_{\pi/8}^{3\pi/8} \sin^2\left(\frac{\pi}{8} + \frac{3\pi}{8} - x\right)dx = \int_{\pi/8}^{3\pi/8} \cos^2 x\,dx$

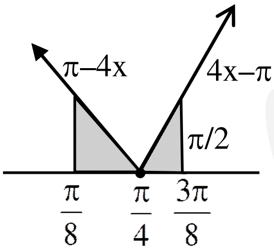
$2S_1 = \int_{\pi/8}^{3\pi/8} (\sin^2 x + \cos^2 x)\,dx = \frac{3\pi}{8} - \frac{\pi}{8} = \frac{\pi}{4}$

$\Rightarrow \frac{16S_1}{\pi} = 2$

12. The value of $\frac{48S_2}{\pi^2}$ is ______.

Ans. (1.50)

Sol.



$$S_2 = \int_{\pi/8}^{3\pi/8} f(x)g_2(x)dx = \int_{\pi/8}^{3\pi/8} \sin^2 x\,|4x-\pi|\,dx$$

$$= \int_{\pi/8}^{3\pi/8} \sin^2\left(\frac{\pi}{2}-x\right)\left|4\left(\frac{\pi}{2}-x\right)-\pi\right|dx$$

$$= \int_{\pi/8}^{3\pi/8} (\cos^2 x)|\pi-4x|\,dx$$

$$\Rightarrow 2S_2 = \int_{\pi/8}^{3\pi/8} |4x-\pi|\ (\sin^2 x+\cos^2 x)dx = \int_{\pi/8}^{3\pi/8} |4x-\pi|\,dx$$

$$= 2\times\frac{1}{2}\times\frac{\pi}{8}\times\frac{\pi}{2} = \frac{\pi^2}{16}$$

$$\Rightarrow \frac{48S_2}{\pi^2} = \frac{3}{2} = 1.5$$

SECTION-3 : (Maximum Marks : 12)

- This section contains **TWO (02) paragraphs**. Based on each paragraph, there are **TWO (02)** questions.
- Each question has **FOUR** options (A), (B), (C) and (D). **ONLY ONE** of these four options is the correct answer.
- For each question, choose the option corresponding to the correct answer.
- Answer to each question will be evaluated according to the following marking scheme:
  Full Marks : +3 If ONLY the correct option is chosen;
  Zero Marks : 0 If none of the options is chosen (i.e. the question is unanswered);
  Negative Marks : –1 In all other cases.

Paragraph

Let

$$M = \{(x, y) \in \mathbb{R} \times \mathbb{R} : x^2 + y^2 \le r^2\},$$

where r > 0. Consider the geometric progression $a_n = \frac{1}{2^{n-1}}$, n = 1, 2, 3, ... Let $S_0 = 0$ and, for $n \ge 1$, let S_n denote the sum of the first n terms of this progression. For $n \ge 1$, let C_n denote the circle with center $(S_{n-1}, 0)$ and radius a_n, and D_n denote the circle with center (S_{n-1}, S_{n-1}) and radius a_n.

13. Consider M with $r = \frac{1025}{513}$. Let k be the number of all those circles C_n that are inside M. Let l be the maximum possible number of circles among these k circles such that no two circles intersect. Then

(A) $k + 2l = 22$ (B) $2k + l = 26$ (C) $2k + 3l = 34$ (D) $3k + 2l = 40$

Ans. (D)

Sol. $S_n = 1 + \frac{1}{2} + \frac{1}{2^2} + ... + \frac{1}{2^{n-1}}$

$= 2\left(1 - \frac{1}{2^n}\right) = 2 - \frac{1}{2^{n-1}}$

Centre of C_n is $\left(2 - \frac{1}{2^{n-2}}, 0\right)$

and radius of C_n is $\frac{1}{2^{n-1}}$

when $r = \frac{1025}{S13} < 2$

C_n will lie inside m

when $2 - \frac{1}{2^{n-2}} + \frac{1}{2^{n-1}} < \frac{1025}{S13}$

$\Rightarrow k = 10$

Also $\ell = 5$

$3k + 2\ell = 30 + 10 = 40$

Ans. (D)

14. Consider M with $r = \frac{(2^{199}-1)\sqrt{2}}{2^{198}}$. The number of all those circles D_n that are inside M is

(A) 198 (B) 199 (C) 200 (D) 201

Ans. (B)

Sol. Center of D_n is (S_{n-1}, S_{n-1})

$$r = \frac{1}{2^{n-1}}$$

D_n will lie inside

when $\sqrt{2}\left(S_{n-1}\right) < \frac{2^{199}-1}{2^{198}}\sqrt{2}$

$$\Rightarrow \frac{\sqrt{2}}{2^{n-2}} > \frac{\sqrt{2}}{2^{198}} + \frac{1}{2^{n-1}}$$

$\Rightarrow n = 199$

Paragraph

Let $\psi_1 : [0, \infty) \to \mathbb{R}$, $\psi_2 : [0, \infty) \to \mathbb{R}$, $f : [0, \infty) \to \mathbb{R}$ and $g : [0, \infty) \to \mathbb{R}$ be functions such that

$f(0) = g(0) = 0$,

$\psi_1(x) = e^{-x} + x, \quad x \geq 0$,

$\psi_2(x) = x^2 - 2x - 2e^{-x} + 2, \quad x \geq 0$,

$f(x) = \int_{-x}^{x} \left(|t| - t^2\right)e^{-t^2}\, dt, \quad x > 0$

and

$g(x) = \int_0^{x^2} \sqrt{t}\, e^{-t}\, dt, \quad x > 0$

15. Which of the following statements is **TRUE** ?

(A) $f\left(\sqrt{\ln 3}\right) + g\left(\sqrt{\ln 3}\right) = \frac{1}{3}$

(B) For every $x > 1$, there exists an $\alpha \in (1, x)$ such that $\psi_1(x) = 1 + \alpha x$

(C) For every $x > 0$, there exists a $\beta \in (0, x)$ such that $\psi_2(x) = 2x(\psi_1(\beta) - 1)$

(D) f is an increasing function on the interval $\left[0, \frac{3}{2}\right]$

Ans. (C)

Sol. $f'(x) = (|x| - x^2)e^{-x^2} + (|x| - x^2)e^{-x^2},\ x \ge 0$

$f' = 2\,(x - x^2)\ e^{-x^2}$

$$\begin{array}{ccccc} & & +++ & & --- \\ \hline & 0 & & 1 & \end{array}$$

hence option (D) is wrong

$g'(x) = xe^{-x^2}\,2x$

$f'(x) + g'(x) = 2 \times e^{-x^2}$

$f(x) + g(x) = -e^{-x^2} + c$

$f(x) + g(x) = -e^{-x^2} + 1$

$F(\ell n3) + g\left(\sqrt{\ell n3}\right) = 1 - \frac{1}{3} = \frac{2}{3}$ (option (A) is wrong)

$H(x) = \psi_1(x) - 1 - \alpha x = e^{-x} + x - 1 - \alpha\, x,$ $\qquad x \ge 1\ \&\ \alpha \in (1, x)$

$H(1) = e^{-1} + 1 - 1 - \alpha < 0$

$H'(x) = -e^{-x} + 1 - \alpha > 0 \quad \Rightarrow H(x)$ is $\downarrow \Rightarrow$ option (B) is wrong

(C) $\psi_2(x) = 2\,(\psi_1(\beta) - 1)$

Applying L.M.V.T to $\psi_2(x)$ in $[0, x]$

$\psi_2'(\beta) = \dfrac{\psi_2(x) - \psi_2(0)}{x}$

$2\beta - 2 + 2e^{-\beta} = \dfrac{\psi_2(x) - 0}{x}$

$\Rightarrow \psi_2(x) = 2x(\psi_1(\beta - 1))$ has one solution

option (C) is correct.

16. Which of the following statements is **TRUE** ?

(A) $\psi_1(x) \le 1$, for all $x > 0$

(B) $\psi_2(x) \le 0$, for all $x > 0$

(C) $f(x) \ge 1 - e^{-x^2} - \frac{2}{3}x^3 + \frac{2}{5}x^5$, for all $x \in \left(0, \frac{1}{2}\right)$

(D) $g(x) \le \frac{2}{3}x^3 - \frac{2}{5}x^5 + \frac{1}{7}x^7$, for all $x \in \left(0, \frac{1}{2}\right)$

Ans. (D)

Sol. (A) $\psi_1(x) = e^{-x} + x,\quad x \ge, 0$

$\psi_1'(x) = 1 - e^{-x} > 0 \quad \Rightarrow \psi_1(x)$ is $\uparrow$

$\psi_1(x) \ge \psi_1(0) \quad \forall x \ge 0 \Rightarrow \psi_1(x) \ge 1$

(B) $\psi_2(x) = x^2 - 2x + 2 - 2e^{-x} \quad x \ge 0$

$\psi_2'(x) = 2x - 2 + 2e^{-x} = 2\psi_1(x) - 2 \ge 0 \quad \forall x \ge 0$

$\Rightarrow \psi_2(x)$ is $\uparrow \Rightarrow \psi_2(x) \ge \psi_2(0) \Rightarrow \psi_2(x) \ge 0$

(C) $f(x)=2\int_0^x (t-t^2)e^{-t^2}\,dt \quad \& \quad x\in\left(0,\frac{1}{2}\right)$

$=\int_0^x 2te^{-t^2}\,dt-\int_0^x 2t^2e^{-t^2}dl$

$=-e^{-x^2}\Big|_0^x -$

Let $H(x) = f(x) -1 + e^{-x^2} + \frac{2}{3}x^3 - \frac{2}{5}x^5,\ \ x\in\left(0,\frac{1}{2}\right)$

$H(0) = 0$

$H'(x) = 2(x-x^2)\,e^{-x^2} - 2xe^{-x^2} + 2x^2 - 2x^4$

$=-2x^2e^{-x^2}+2x^2-2x^4$

$=2x^2\left(1-x^2-e^{-x^2}\right)$

$\because\ e^{-x}\geq 1-x\ \ \forall\ x\geq 0$

$\Rightarrow H'(x)\leq 0$

$\Rightarrow H(x)$ is $\downarrow\ \Rightarrow\ 1{-}1(x)<0\ \forall x\in\left(0,\frac{1}{2}\right)$

Let $P(x) = g(x) -\frac{2}{3}x^3+\frac{2}{5}x^5-\frac{1}{7}x^7 x\in\left(0,\frac{1}{2}\right)$

$P'(x) = 2x^2\,e^{-x^2} - 2x^2 + 2x^4 - x^6$

$=2x^2\left(1-\frac{x^2}{1}+\frac{x^4}{2}-\frac{x^6}{3}+\ldots\right)-2x^2+2x^4-x^6$

$=-\frac{x^8}{3}+\frac{x^{10}}{12}\ldots\ldots\ldots$

$\Rightarrow P'(x)\leq 0$

$\Rightarrow P(x)$ is $\downarrow$

$\Rightarrow P(x)\leq 0$

option (D) is correct

SECTION-4 : (Maximum Marks : 12)

- This section contains **THREE (03)** questions.
- The answer to each question is a **NON-NEGATIVE INTEGER.**
- For each question, enter the correct integer corresponding to the answer using the mouse and the on-screen virtual numeric keypad in the place designated to enter the answer.
- Answer to each question will be evaluated according to the following marking scheme:

Full Marks : +4 If ONLY the correct integer is entered;

Zero Marks : 0 In all other cases.

17. A number is chosen at random from the set {1, 2, 3, ... , 2000}. Let p be the probability that the chosen number is a multiple of 3 or a multiple of 7. Then the value of 500p is ____.

Ans. (214)

Sol. A = set of numbers divisible by 3
A = {3, 6, 9, 12, ... 1998}
$\therefore$ n (A) = 666
B = set of numbers divisible by 7
B = {7, 14, 21, ...1995}
$\therefore$ n(B) = 285
$A \cap B = \{21, 42, ...1995\}$
$\therefore\ n(A \cup B) = 606 + 285 - 95 = 856$

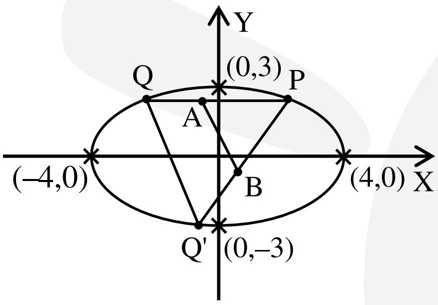
required probability = $\dfrac{856}{2000} = P$

so, 500 P = $\dfrac{856}{2000} \times 500 = 214$

18. Let E be the ellipse $\dfrac{x^2}{16} + \dfrac{y^2}{9} = 1$. For any three distinct points P, Q and Q' on E, let M (P, Q) be the mid-point of the line segment joining P and Q, and M (P, Q') be the mid-point of the line segment joining P and Q'. Then the maximum possible value of the distance between M(P, Q) and M(P, Q'), as P, Q and Q' vary on E, is _____.

Ans. **(4)**

Sol.



A and B be midpoints of segment PQ and PQ' respectively

AB = distance between M(P, Q) and M(P, Q') = $\dfrac{1}{2}$.QQ'

Since, Q, Q' must be on E, so,
maximum of QQ' = 8

$\therefore$ Maximum of AB = $\dfrac{8}{2}$ = 4

19. For any real number x, let [x] denote the largest integer less than or equal to x. If

$$I = \int_0^{10} \left[\sqrt{\frac{10x}{x+1}}\right] dx,$$

then the value of 9I is _____.

Ans. **(182)**

Sol. Let $f(x) = \left(\dfrac{10x}{x+1}\right)$

So, $f'(x) = 10\left(\dfrac{(x+1)-x}{(x+1)^2}\right) = \dfrac{10}{(x+1)^2} > 0\ \forall\ x \in [0, 10]$,

So, $f(x)$ is an increasing function

So, range of $f(\mathrm{x})$ is $\left[0, \sqrt{\frac{100}{11}}\right]$

$$I=\int_0^{1/9}\left[\sqrt{\frac{10x}{x+1}}\right]dx+\int_{2/3}^{9}\left[\sqrt{\frac{10x}{x+1}}\right]dx+\int_{1/9}^{2/3}\left[\sqrt{\frac{10x}{x+1}}\right]dx+\int_{2/3}^{9}\left[\sqrt{\frac{10x}{x+1}}\right]dx+\int_{9}^{10}\left[\sqrt{\frac{10x}{x+1}}\right]dx$$

$$=0+\int_{1/9}^{2/3}dx+2\int_{2/3}^{9}dx+3\int_{9}^{10}dx$$

$$=\frac{2}{3}-\frac{1}{9}+2\left(9-\frac{2}{3}\right)+3(10-9)$$

$$=\frac{6-1}{9}+2\times\frac{25}{3}+3=\frac{5}{9}+\frac{50}{3}+3$$

$$=\frac{5+150+27}{9}=\frac{182}{9}=182$$