FINAL JEE-MAIN EXAMINATION - JULY, 2022

Held On Wednesday 27 July,2022

TIME :9:00 AM to 12:00 NOON

SECTION-A

1. Let R_1 and R_2 be two relations defined on $\mathbb{R}$ by $a\, R_1\, b \Leftrightarrow ab \geq 0$ and $a\, R_2 b \Leftrightarrow a \geq b$, then

(A) R_1 is an equivalence relation but not R_2

(B) R_2 is an equivalence relation but not R_1

(C) both R_1 and R_2 are equivalence relations

(D) neither R_1 nor R_2 is an equivalence relation

Official Ans. by NTA (D)

Ans. (D)

Sol. $R_1 = \{xy \geq 0, x, y \in R\}$

For reflexive $x \times x \geq 0$ which is true.

For symmetric

If $xy \geq 0 \Rightarrow yx \geq 0$

If x = 2, y = 0 and z = -2

Then $x.y \geq 0$ & $y.z \geq 0$ but x.z ≥ 0 is not true

$\Rightarrow$ not transitive relation.

$\Rightarrow R_1$ is not equivalence

R_2 if $a \geq b$ it does not implies $b \geq a$

$\Rightarrow R_2$ is not equivalence relation

$\Rightarrow D$

2. Let $f, g : \mathbb{N} - \{1\} \to \mathbb{N}$ be functions defined by $f(a) = \alpha$, where α is the maximum of the powers of those primes p such that p^{α} divides a, and $g(a) = a + 1$, for all $a \in \mathbb{N} - \{1\}$. Then, the function f + g is

(A) one-one but not onto

(B) onto but not one-one

(C) both one-one and onto

(D) neither one-one nor onto

Official Ans. by NTA (D)

Ans. (D)

Sol. $f : N - \{1\} \to N \quad f(a) = \alpha$

Where α is max of powers of prime P such that p^{α} divides a. Also g(a) = a + 1

$\therefore$ f(2) = 1 g(2) = 3

f (3) = 1 g(3) = 4

f(4) = 2 g(4) = 5

f(5) = 1 g(5) = 6

$\Rightarrow$ f (2) + g (2) = 4

$(f(3) + g(3)) = 5$

f(4) + g (4) = 7

f (5) + g (5) = 7

$\therefore$ Many one f(x) + g(x) does not cotain 1

$\Rightarrow$ into function

$\therefore$ Ans. (D) [neither one-one nor onto]

3. Let the minimum value v_0 of $v = |z|^2 + |z-3|^2 + |z-6i|^2$, $z \in \mathbb{C}$ is attained at z = z_0. Then $\left|2z_0^2 - \bar{z}_0^3 + 3\right|^2 + v_0^2$ is equal to

(A) 1000 (B) 1024

(C) 1105 (D) 1196

Official Ans. by NTA (A)

Ans. (A)

Sol. $z_0 = \left(\frac{0+3+0}{3}, \frac{0+6+0}{3}\right) = (1,2)$

$v_0 = |1+2i|^2 + |1+2i-3|^2 + |1+2i-6i|^2 = 30$

Then $\left|2z_0^2 - \bar{z}_0^3 + 3\right|^2 + v_0^2$

$= \left|2(1+2i)^2 - (1-2i)^3 + 3\right|^2 + 900$

$= |2(1-4+4i) - (1-4-4i)(1-2i) + 3|^2 + 900$

$= |8+6i|^2 + 900 = 100 + 900 = 1000$

4. Let $A=\begin{pmatrix}1 & 2\\ -2 & -5\end{pmatrix}$. Let $\alpha,\beta\in\mathbb{R}$ be such that $\alpha A^2+\beta A=2I$. Then $\alpha+\beta$ is equal to -

(A) -10 (B) -6

(C) 6 (D) 10

Official Ans. by NTA (D)

Ans. (D)

Sol. Characteristic equation of matric A

$$|A-\lambda I|=0$$

$$\begin{vmatrix}1-\lambda & 2\\ -2 & -5-\lambda\end{vmatrix}=0$$

$$\Rightarrow \lambda^2+4\lambda=1$$

$$\Rightarrow A^2+4A=I$$

$$\Rightarrow 2A^2+8A=2I \quad \ldots\ldots\ldots\ldots\ldots (1)$$

Given that $\alpha A^2+\beta A=2I \quad \ldots\ldots\ldots (2)$

Comparing equation (1) & (2) we get

$\alpha=2,\quad \beta=8$

$\therefore \alpha+\beta=10$

Ans. (D) (10)

5. The remainder when $(2021)^{2022}+(2022)^{2021}$ is divided by 7 is

(A) 0 (B) 1

(C) 2 (D) 6

Official Ans. by NTA (A)

Ans. (A)

Sol. $(2021)^{2022}+(2022)^{2021}$

$=(2023-2)^{2022}+(2023-1)^{2021}$

$=7n_1+2^{2022}+7n_2-1$

$=7(n_1+n_2)+8^{674}-1$

$=7(n_1+n_2)+(7-1)^{674}-1$

$=7(n_1+n_2)+7n_3+1-1$

$=7(n_1+n_2+n_3)$

$\therefore$ Given number is divisible by 7 hence remainder is zero

6. Suppose $a_1, a_2, \ldots, a_n, \ldots$ be an arithmetic progression of natural numbers. If the ratio of the sum of the first five terms of the sum of first nine terms of the progression is 5 : 17 and $110 < a_{15} < 120$, then the sum of the first ten terms of the progression is equal to -

(A) 290 (B) 380

(C) 460 (D) 510

Official Ans. by NTA (B)

Ans. (B)

Sol. $$\frac{S_5}{S_9}=\frac{5}{17}\Rightarrow\frac{\frac{5}{2}(2a+4d)}{\frac{9}{2}(2a+8d)}=\frac{5}{17}$$

$\Rightarrow d=4a$

$a_{15}=a+14d=57a$

Now, $110<a_{15}<120$

$\Rightarrow 110<57a<120$

$\Rightarrow a=2 \therefore d=8$

$S_{10}=\frac{10}{2}(2\times2+9\times8)=380$

7. Let $f:\mathbb{R}\to\mathbb{R}$ be a function defined as

$f(x)=a\sin\left(\frac{\pi[x]}{2}\right)+[2-x],\ a\in\mathbb{R}$, where [t] is the greatest integer less than or equal to t. If $\lim_{x\to-1} f(x)$ exists, then the value of $\int_0^4 f(x)dx$ is equal to :

(A) -1 (B) -2

(C) 1 (D) 2

Official Ans. by NTA (B)

Ans. (B)

Sol. $\lim\limits_{x\to-1^+} a\sin\left(\pi\frac{[x]}{2}\right)+[2-x]=-a+2$

$\lim\limits_{x\to-1^-} a\sin\left(\pi\frac{[x]}{2}\right)+[2-x]=0+3=3$

$\lim\limits_{x\to-1} f(x)$ exist when $a=-1$

Now,

$\int_0^4 f(x)dx=\int_0^1 f(x)dx+\int_1^2 f(x)dx+\int_2^3 f(x)dx+\int_3^4 f(x)dx$

$=\int_0^1(0+1)dx+\int_1^2(-1+0)dx+\int_2^3(0-1)dx+\int_3^4(1-2)dx$

$=1-1-1-1=-2$

8. $I=\int_{\pi/4}^{\pi/3}\left(\frac{8\sin x-\sin 2x}{x}\right)dx$. Then

(A) $\frac{\pi}{2}<I<\frac{3\pi}{4}$

(B) $\frac{\pi}{5}<I<\frac{5\pi}{12}$

(C) $\frac{5\pi}{12}<I<\frac{\sqrt{2}}{3}\pi$

(D) $\frac{3\pi}{4}<I<\pi$

Official Ans. by NTA (C)

Ans. (C)

Sol. Consider

$f(x) = 8\sin x - \sin 2x$

$f'(x) = 8\sin x - 2\cos 2x$

$f''(x) = -8\sin x + 4\sin 2x$

$= -8\sin x(1-\cos x)$

$\therefore f''(x)<0 \quad x\in\left(\frac{\pi}{4},\frac{\pi}{3}\right)$

$\therefore f'(x)$ is $\downarrow$ function

$f'\left(\frac{\pi}{3}\right)<f'(x)<f'\left(\frac{\pi}{4}\right)$

$5<f'(x)<\frac{8}{\sqrt{2}}$

$5<f'(x)<4\sqrt{2}$

$5x<f(x)<4\sqrt{2}x$

$5<\frac{f(x)}{x}<4\sqrt{2}$

$\int_{\pi/4}^{\pi/3}5<\int\frac{f(x)}{x}<\int_{\pi/4}^{\pi/3}4\sqrt{2}$

$\int_{\pi/4}^{\pi/3}5<\int\frac{8\sin x-\sin 2x}{x}<\int_{\pi/4}^{\pi/3}4\sqrt{2}$

$\frac{5\pi}{12}<I<\frac{\sqrt{2}\pi}{3}$

9. The area of the smaller region enclosed by the curves $y^2 = 8x + 4$ and $x^2+y^2+4\sqrt{3}x-4=0$ is equal to

(A) $\frac{1}{3}\left(2-12\sqrt{3}+8\pi\right)$

(B) $\frac{1}{3}\left(2-12\sqrt{3}+6\pi\right)$

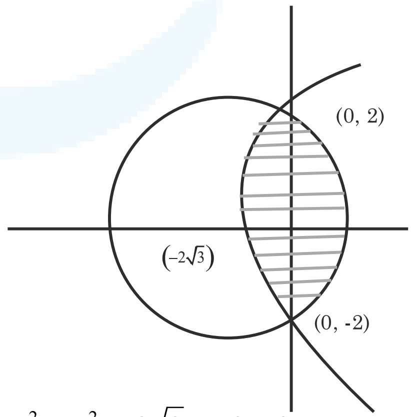
(C) $\frac{1}{3}\left(4-12\sqrt{3}+8\pi\right)$

(D) $\frac{1}{3}\left(4-12\sqrt{3}+6\pi\right)$

Official Ans. by NTA (C)

Ans. (C)

Sol.



$x^2+y^2+4\sqrt{3}x-4=0$

$y^2 = 8x + 4$

Point of intersections are (0, 2) & (0, -2)

Both are symmetric about x-axis

Area $=2\int_0^2\left(\left(\sqrt{16-y^2}-2\sqrt{3}\right)-\left(\frac{y^2-4}{8}\right)\right)dy$

On solving $Area=\frac{1}{3}\left[8\pi+4-12\sqrt{3}\right]$

10. Let $y = y_1(x)$ and $y = y_2(x)$ be two distinct solutions of the differential equation $\frac{dy}{dx} = x + y$, with $y_1(0) = 0$ and $y_2(0) = 1$ respectively. Then, the number of points of intersection of $y = y_1(x)$ and $y = y_2(x)$ is

(A) 0 (B) 1

(C) 2 (D) 3

Official Ans. by NTA (A)

Ans. (A)

Sol. $\frac{dy}{dx} = x + y \Rightarrow \frac{dy}{dx} - y = x$

$\text{If} = e^{-x}$

$\therefore$ solution is $ye^{-x} = \int xe^{-x}dx$

$\Rightarrow ye^{-x} = -xe^{-x} - e^{-x} + c$

$\Rightarrow y = -x - 1 + ce^{x}$

$y_1(0) = 0 \Rightarrow c = 1$

$\therefore y_1 = -x - 1 + e^{x}$ ....(1)

$y_2(0) = 1 \Rightarrow c = 2$

$\therefore y_2 = -x - 1 + 2e^{x}$ .....(2)

Now $y_2 - y_1 = e^{x} > 0 \therefore y_2 \neq y_1$

$\therefore$ Number of points of intersection of y_1 & y_2 is zero.

11. Let P (*a*, b) be a point on the parabola $y^2 = 8x$ such that the tangent at P passes through the centre of the circle $x^2 + y^2 - 10x - 14y + 65 = 0$. Let A be the product of all possible values of *a* and *B* be the product of all possible values of *b*. Then the value of A + B is equal to :

(A) 0 (B) 25

(C) 40 (D) 65

Official Ans. by NTA (D)

Ans. (D)

Sol. P(a, b) is point on $y^2 = 8x$, such that tangent at P pass through centre of $x^2 + y^2 - 10x - 14y + 65 = 0$ i.e. (5, 7)

Tangent at $P(at^2, 2at)$ is $ty = x + at^2$

A = 2 & it pass through (5, 7)

$7t = 5 + 2t^2$

$\Rightarrow t = 1, t = \frac{5}{2}$

$\therefore P(at^2, 2at) \Rightarrow (2, 4)$ when t = 1

& $\left(\frac{25}{2}, 10\right)$ when $t = \frac{5}{2}$

$\therefore A = 2 \times \frac{25}{2} = 25$

$B = 4 \times 10 = 40$ $\therefore A + B = 65$

12. Let $\vec{a} = \alpha\hat{i} + \hat{j} + \beta\hat{k}$ and $\vec{b} = 3\hat{i} - 5\hat{j} + 4\hat{k}$ be two vectors, such that $\vec{a} \times \vec{b} = -\hat{i} + 9\hat{i} + 12\hat{k}$. Then the projection of $\vec{b} - 2\vec{a}$ on $\vec{b} + \vec{a}$ is equal to

(A) 2 (B) $\frac{39}{5}$

(C) 9 (D) $\frac{46}{5}$

Official Ans. by NTA (D)

Ans. (Bonus)

Sol. Let $\vec{a} = \alpha\hat{i} + \hat{j} + \beta\hat{k}, \vec{b} = 3\hat{i} - 5\hat{j} + 4\hat{k}$

$\vec{a} \times \vec{b} = -\hat{i} + 9\hat{j} + 12\hat{k}$

$$\Rightarrow \begin{vmatrix} \hat{i} & \hat{j} & \hat{k} \\ \alpha & 1 & \beta \\ 3 & -5 & 4 \end{vmatrix}$$

$\Rightarrow (4 + 5\beta)\hat{i} + (3\beta - 4\alpha)\hat{j} + (-5\alpha - 3)\hat{k}$

$= -\hat{i} + 9\hat{j} + 12\hat{k}$

$\therefore 4 + 5\beta = -1, 3\beta - 4\alpha = 9, -5\alpha - 3 = 12$

$\beta = -1, \quad \alpha = -3$

$\therefore \vec{a} = -3\hat{i} + \hat{j} - \hat{k}, \vec{b} = 3\hat{i} - 5\hat{j} + 4\hat{k}$

$\therefore \vec{a} + \vec{b} = -4\hat{j} + 3\hat{k}$

$|\vec{a}|^2 = 11, |\vec{b}|^2 = 50$

$\vec{a}.\vec{b} = -9 + (-5) - 4 = -18$

$\therefore$ Projectile of $\left(\vec{b}-2\vec{a}\right)$ on $\vec{a}+\vec{b}$ is

$$\frac{\left(\vec{b}-2\vec{a}\right).\left(\vec{a}+\vec{b}\right)}{\left|\vec{a}+\vec{b}\right|}$$

$$=\frac{\left|\vec{b}\right|^2-2\left|\vec{a}\right|^2-\left(\vec{a}.\vec{b}\right)}{\left|\vec{a}+\vec{b}\right|}=\frac{50-22-(-18)}{5}=\frac{46}{5}$$

Ans. $\left(\frac{46}{5}\right)$

13. Let $\vec{a}=2\hat{i}-\hat{j}+5\hat{k}$ and $\vec{b}=\alpha\hat{i}+\beta\hat{j}+2\hat{k}$. If $\left(\left(\vec{a}\times\vec{b}\right)\times\hat{i}\right).\hat{k}=\frac{23}{2}$, then $\left|\vec{b}\times 2\hat{j}\right|$ is equal to

(A) 4 (B) 5

(C) $\sqrt{21}$ (D) $\sqrt{17}$

Official Ans. by NTA (B)

Ans. (B)

Sol. $\vec{a}=2\hat{i}-\hat{j}+5\hat{k},\ \vec{b}=\alpha\hat{i}+\beta\hat{j}+2\hat{k}$

$\left(\left(\vec{a}\times\vec{b}\right)\times\hat{i}\right).\hat{k}=\frac{23}{2}$, then $\left|\vec{b}\times 2\hat{j}\right|$ is

$$\left(\left(\vec{a}.\hat{i}\right)\vec{b}-\left(\vec{b}.\hat{i}\right)\vec{a}\right).\hat{k}=\frac{23}{2}$$

$$\left(\vec{a}.\hat{i}\right)\left(\vec{b}.\hat{i}\right)-\left(\vec{b}.\hat{i}\right)\left(\vec{a}.\hat{k}\right)=\frac{23}{2}$$

$$2\times2-\alpha\times5=\frac{23}{2}\Rightarrow5\alpha=4-\frac{23}{2}\Rightarrow\alpha=\frac{-3}{2}$$

$$\vec{b}\times2\hat{j}=\begin{vmatrix}\hat{i}&\hat{j}&\hat{k}\\ \alpha&\beta&2\\ 0&2&0\end{vmatrix}=-4\hat{i}+2\alpha\hat{k}$$

$$\therefore\left|\vec{b}\times2\hat{j}\right|=\sqrt{16+4\alpha^2}=\sqrt{16+4\times\frac{9}{4}}=5$$

14. Let S be the sample space of all five digit numbers. If p is the probability that a randomly selected number from S, is a multiple of 7 but not divisible by 5, then $9p$ is equal to

(A) 1.0146 (B) 1.2085

(C) 1.0285 (D) 1.1521

Official Ans. by NTA (C)

Ans. (C)

Sol. n(S) = all 5 digit nos $= 9\times10^4$

A : no is multiple of 7 but not divisible by 5

Smallest 5 digit divisible by 7 is 10003

Largest 5 digit divisible by 7 is 99995

$\therefore$ 99995 = 10003 + (n – 1) 7 n = 12857

Numbers divisible by 35

99995 = 10010 + (P-1) 35 $\Rightarrow$ P = 2572

$\therefore$ Numbers divisible by 7 but not by 35 are

12857 – 2572 = 10285

$\therefore P=\frac{10285}{90000}$ $\therefore$ 9P = 1.0285

Ans. (C) [1.0285]

15. Let a vertical tower AB of height $2h$ stands on a horizontal ground. Let from a point P on the ground a man can see upto height h of the tower with an angle of elevation 2α. When from P, he moves a distance d in the direction of $\overrightarrow{AP}$, he can see the top B of the tower with an angle of elevation α. If $d=\sqrt{7}h$, then tan α is equal to

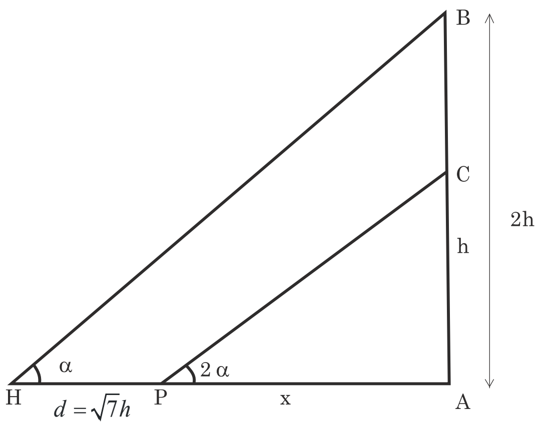
(A) $\sqrt{5}-2$ (B) $\sqrt{3}-1$

(C) $\sqrt{7}-2$ (D) $\sqrt{7}-\sqrt{3}$

Official Ans. by NTA (C)

Ans. (C)

Sol.



$$\tan2\alpha=\frac{h}{x}$$

and $\tan\alpha = \dfrac{2h}{x+\sqrt{7}h}$

$\tan\alpha = \dfrac{2h}{h\cot 2\alpha + \sqrt{7}h}$

$\tan\alpha = \dfrac{2}{\dfrac{(1-\tan^2\alpha)}{2\tan\alpha}+\sqrt{7}}$

Put $\tan\alpha = t$ & simplify

$\Rightarrow \tan\alpha = \sqrt{7}-2$

16. $(p \wedge r) \Leftrightarrow (p \wedge (\sim q))$ is equivalent to $(\sim p)$ when r is

(A) p (B) $\sim p$

(C) q (D) $\sim q$

Official Ans. by NTA (C)

Ans. (C)

Sol. Given $(p \wedge r) \Leftrightarrow (p \wedge (\sim q)) \equiv (\sim p)$

Taking r = q

p	q	~p	~q	p^q	P^~q	$(p\wedge r)\Leftrightarrow(p\wedge(\sim q))$
T	T	F	F	T	F	F
T	F	F	T	F	T	F
F	T	T	F	F	F	T
F	F	T	T	F	F	T

So, clear $(p \wedge r) \Leftrightarrow (p \wedge (\sim q)) \equiv (\sim p)$

17. If the plane P passes through the intersection of two mutually perpendicular planes 2x + ky – 5z = 1 and 3kx – ky + z = 5, k < 3 and intercepts a unit length on positive x-axis, then the intercept made by the plane P on the y-axis is

(A) $\dfrac{1}{11}$ (B) $\dfrac{5}{11}$

(C) 6 (D) 7

Official Ans. by NTA (D)

Ans. (D)

Sol. Two given planes mutually perpendicular

2(3k) + k (-k) + (-5) 1 = 0

k = 1, 5

but k < 3 So k = 1

Plane passing through these planes is

$2x + y - 5z - 1 + \lambda(3x - y + z - 5) = 0$

$$\frac{x}{\frac{5\lambda+1}{2+3\lambda}} + \frac{y}{\frac{5\lambda+1}{1-\lambda}} + \frac{z}{\frac{5\lambda+1}{\lambda-5}} = 1$$

Given $\dfrac{5\lambda+1}{2+3\lambda} = 1 \Rightarrow \lambda = \dfrac{1}{2}$

So intercept on y - axis = $\dfrac{5\lambda+1}{1-\lambda} = 7$

18. Let A(1, 1), B(-4, 3) C(-2, -5) be vertices of a triangle ABC, P be a point on side BC, and Δ_1 and Δ_2 be the areas of triangle APB and ABC. Respectively.

If $\Delta_1 : \Delta_2 = 4:7$, then the area enclosed by the lines AP, AC and the x-axis is

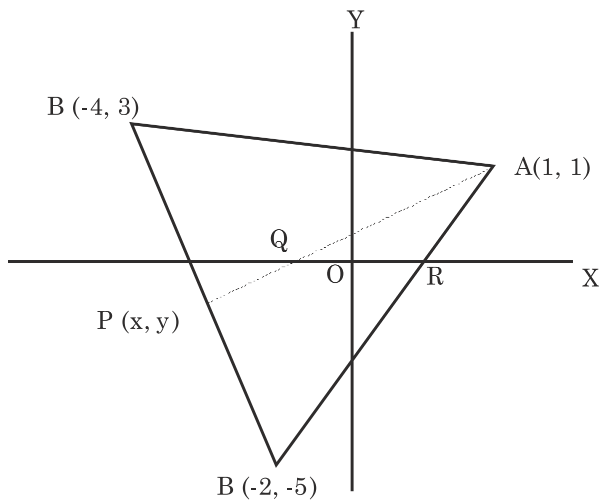
(A) $\dfrac{1}{4}$ (B) $\dfrac{3}{4}$

(C) $\dfrac{1}{2}$ (D) 1

Official Ans. by NTA (C)

Ans. (C)

Sol.



Given $\Delta_1 = \frac{1}{2}\begin{vmatrix} x & y & 1 \\ 1 & 1 & 1 \\ -4 & 3 & 1 \end{vmatrix}$

& $\Delta_2 = \frac{1}{2}\begin{vmatrix} 1 & 1 & 1 \\ -4 & 3 & 1 \\ -2 & -5 & 1 \end{vmatrix}$

Given $\frac{\Delta_1}{\Delta_2} = \frac{4}{7} \Rightarrow \frac{-2x-5y+7}{36} = \frac{4}{7}$

$\Rightarrow 14x + 35y = -95$ ....(1)

Equation of BC is 4x + y = -13 ....(2)

Solve equation (1) & (2)

Point $P\left(\frac{-20}{7}, \frac{-11}{7}\right)$

Here point $Q\left(\frac{-1}{2}, 0\right)$ & $R\left(\frac{1}{2}, 0\right)$

So Area of triangle AQR $= \frac{1}{2} \times 1 \times 1 = \frac{1}{2}$

19. If the circle $x^2 + y^2 - 2gx + 6y - 19c = 0$, $g, c \in \mathbb{R}$ passes through the point (6, 1) and its centre lies on the line x – 2cy = 8, then the length of intercept made by the circle on x-axis is

(A) $\sqrt{11}$ (B) 4

(C) 3 (D) $2\sqrt{23}$

Official Ans. by NTA (D)

Ans. (D)

Sol. Given circle $x^2 + y^2 - 2gx + 6y - 19c = 0$

Passes through (6, 1)

12g + 19 c = 43 ....(1)

Centre (g, -3) lies on given line

So, g + 6c = 8 ....(2)

Solve equation (1) & (2)

c = 1 & g = 2

equation of circle $x^2 + y^2 - 4x + 6y - 19 = 0$

Length of intercept on x-axis

$= 2\sqrt{g^2 - c} = 2\sqrt{23}$

20. Let a function $f : \mathbb{R} \to \mathbb{R}$ be defined as :

$$f(x) = \begin{cases} \int_0^x (5 - |t-3|)\,dt, & x > 4 \\ x^2 + bx, & x \le 4 \end{cases}$$

where $b \in \mathbb{R}$. If f is continuous at x = 4, then which of the following statements is NOT true ?

(A) f is not differentiable at x = 4

(B) $f'(3) + f'(5) = \frac{35}{4}$

(C) f is increasing in $\left(-\infty, \frac{1}{8}\right) \cup (8, \infty)$

(D) f has a local minima at $x = \frac{1}{8}$

Official Ans. by NTA (C)

Ans. (C)

Sol. Given $f(x) \begin{cases} \int_0^x (5 - |t-3|)\,dt, & x > 4 \\ x^2 + bx, & x \le 4 \end{cases}$

f(x) is continuous at x = 4

So $\lim_{x \to 4^-} f(x) = \lim_{x \to 4^+} f(x) = f(4)$

So 16 + 4b $= \int_0^3 (2 - t)\,dt + \int_3^4 (8 - t)\,dt$

$\Rightarrow 16 + 4b = 15$

So $b = \frac{-1}{4}$

At x = 4

LHD = 2x + b = $\frac{31}{4}$

RHD = 5 - |x – 3| = 4

LHD $\neq$ RHD

Option (A) is true

and f'(3) + f'(5) = $\frac{23}{4} + 3 = \frac{35}{4}$

Option (B) is true

$\because f(x) = x^2 - \frac{x}{4}$ at $x \le 4$

$f'(x)=2x-\frac{1}{4}$

This function is not increasing.

In the interval in $x\in\left(-\infty,\frac{1}{8}\right)$

Option (C) is NOT TRUE.

This function f(x) is also local minima at $x=\frac{1}{8}$

SECTION-B

1. For $k\in\mathbb{R}$, let the solutions of the equation $\cos\left(\sin^{-1}\left(x\cot\left(\tan^{-1}\left(\cos\left(\sin^{-1}x\right)\right)\right)\right)\right)=k, 0<|x|<\frac{1}{\sqrt{2}}$ be α and β, where the inverse trigonometric functions take only principal values. If the solutions of the equation $x^2 - bx - 5 = 0$ are $\frac{1}{\alpha^2}+\frac{1}{\beta^2}$ and $\frac{\alpha}{\beta}$, then $\frac{b}{k^2}$ is equal to ______.

Official Ans. by NTA (12)

Ans. (12)

Sol. $\cos\left(\sin^{-1}x\right)=\cos\left(\cos^{-1}\sqrt{1-x^2}\right)=\sqrt{1-x^2}$

$\cot\left(\tan^{-1}\sqrt{1-x^2}\right)=\cot\ \cot^{-1}\left(\sqrt{\frac{1}{\sqrt{1-x^2}}}\right)=\frac{1}{\sqrt{1-x^2}}$

$\Rightarrow\cos\left(\sin^{-1}\left(\frac{x}{\sqrt{1-x^2}}\right)\right)=\frac{\sqrt{1-2x^2}}{\sqrt{1-x^2}}$

$\Rightarrow\frac{\sqrt{1-2x^2}}{\sqrt{1-x^2}}=k$

$\Rightarrow 1-2x^2=k^2\left(1-x^2\right)$

$\Rightarrow\left(k^2-2\right)x^2=k^2-1$

$x^2=\frac{k^2-1}{k^2-2}$

$\alpha=\sqrt{\frac{k^2-1}{k^2-2}}\Rightarrow\alpha^2=\frac{k^2-1}{k^2-2}$

$\beta=\sqrt{\frac{k^2-1}{k^2-2}}\Rightarrow\beta^2=\frac{k^2-1}{k^2-2}$

$\frac{1}{\alpha^2}+\frac{1}{\beta^2}=2\left(\frac{k^2-2}{k^2-1}\right)\ \&\ \frac{\alpha}{\beta}=-1$

Sum of roots $=\frac{1}{\alpha^2}+\frac{1}{\beta^2}+\frac{\alpha}{\beta}=b$

$\Rightarrow\frac{2\left(k^2-2\right)}{k^2-1}-1=b$ .....(1)

Product of roots $=\left(\frac{1}{\alpha^2}+\frac{1}{\beta^2}\right)\frac{\alpha}{\beta}=-5$

$\Rightarrow\frac{2\left(k^2-2\right)}{k^2-1}(-1)=-5$

$\Rightarrow 2k^2-4=5k^2-5$

$\Rightarrow 3k^2=1\Rightarrow k^2=\frac{1}{3}$ .... Put in (1)

$\Rightarrow b=\frac{2\left(k^2-2\right)}{k^2-1}-1=5-1=4$

$\frac{b}{k^2}=\frac{4}{\frac{1}{3}}=12$

2. The mean and variance of 10 observations were calculated as 15 and 15 respectively by a student who took by mistake 25 instead of 15 for one observation. Then, the correct standard deviation is ___________.

Official Ans. by NTA (2)

Ans. (2)

Sol. n = 10, $\bar{x}=\frac{\sum x_i}{10}=15$

$6^2=\frac{\sum x_i^2}{10}-\left(\bar{x}\right)^2=15$

$\Rightarrow\sum_{i=1}^{10}x_i=150$

$\Rightarrow \sum_{i=1}^{9} x_i + 25 = 150$

$\Rightarrow \sum_{i=1}^{9} x_i = 125$

$\Rightarrow \sum_{i=1}^{9} x_i + 15 = 140$

Actual mean $= \frac{140}{10} = 14 = \bar{x}_{new}$

$\sum_{i=1}^{9} \frac{x_i^2 + 25^2 - 15^2}{10} = 15$

$\Rightarrow \sum_{i=1}^{9} x_i^2 + 625 = 2400$

$\sum_{i=1}^{9} x_i^2 = 1775$

$\sum_{i=1}^{9} x_i^2 + 15^2 = 2000 = \left(\sum x_i^2\right)_{actual}$

$$6^2_{actual} = \frac{\left(\sum x_i^2\right)_{actual} - (\bar{x}_{new})^2}{10}$$

$= \frac{2000}{10} - 14^2$

= 200-196 = 4

$(S.D)_{actual} = 6 = 2$

3. Let the line $\frac{x-3}{7} = \frac{y-2}{-1} = \frac{z-3}{-4}$ intersect the plane containing the lines $\frac{x-4}{1} = \frac{y+1}{-2} = \frac{z}{1}$ and $4ax - y + 5z - 7a = 0$ = 2x – 5y – z – 3, $a \in \mathbb{R}$ at the point $P(\alpha, \beta, \gamma)$. Then the value of $\alpha + \beta + \gamma$ equals ________.

Official Ans. by NTA (12)

Ans. (12)

Sol. Equation of plane

$4ax - y + 5z - 7a + \lambda$ (2x – 5y – z – 3)= 0

this satisfy (4, -1, 0)

$16a + 1 - 7a + \lambda(8 + 5 - 3) = 0$

$9a + 1 + 10\lambda = 0$ .....(1)

Normal vector of the plane A is $(4a + 2\lambda, -1 - 5\lambda, 5 - \lambda)$ vector along the line which contained the plane A is

i – 2j + k

$\therefore 4a + 2\lambda + 2 + 10\lambda + 5 - \lambda = 0$

$11\lambda + 4a + 7 = 0$ ......(2)

Solve (1) and (2) to get a = 1, $\lambda = -1$

Now equation of plane

$x + 2y + 3z - 2 = 0$

Let the point in the line $\frac{x-3}{7} = \frac{y-2}{-1} = \frac{z-3}{-4} = t$ is $(7t + 3, -t + 2, -4t + 3)$ satisfy the equation of plane A

7t + 3 – 2t + 4 + 9 -12t – 2 = 0

t = 2

So $\alpha + \beta + \gamma = 2t + 8 = 12$

4. An ellipse $E : \frac{x^2}{a^2} + \frac{y^2}{b^2} = 1$ passes through the vertices of the hyperbola $H : \frac{x^2}{49} - \frac{y^2}{64} = -1$. Let the major and minor axes of the ellipse E coincide with the transverse and conjugate axes of the hyperbola H. Let the product of the eccentricities of E and H be $\frac{1}{2}$. If l is the length of the latus rectum of the ellipse E, then the value of $113l$ *is* equal to ________.

Official Ans. by NTA (1552)

Ans. (1552)

Sol. Hyp : $\frac{y^2}{64} - \frac{x^2}{49} = 1$

An ellipse $E : \frac{x^2}{a^2} + \frac{y^2}{b^2} = 1$ passes through the vertices of the hyperbola $H : \frac{x^2}{49} - \frac{y^2}{64} = -1$.

So $b^2 = 64$

$$e_H = \sqrt{1+\frac{a^2}{b^2}} = \sqrt{1+\frac{49}{64}}$$

Ellipse $\frac{x^2}{a^2}+\frac{y^2}{b^2}=1$

$$e_E = \sqrt{1-\frac{a^2}{b^2}} = \sqrt{1-\frac{a^2}{64}}$$

$$b=8, \sqrt{\frac{1-a^2}{64}} \times \frac{\sqrt{113}}{8} = \frac{1}{2} \Rightarrow \sqrt{64-a^2} \times \sqrt{113} = 32$$

$$\left(64-a^2\right) = \frac{32^2}{113}$$

$$\Rightarrow a^2 = 64 - \frac{32^2}{113}$$

$$l = \frac{2a^2}{b} = \frac{2}{8}\left(64-\frac{32^2}{113}\right) = \frac{1552}{113}$$

$$113l = 1552$$

5. Let y = y(x) be the solution curve of the differential equation

$$\sin\left(2x^2\right)\log_e\left(\tan x^2\right)dy + \left(4xy - 4\sqrt{2}x\sin\left(x^2-\frac{\pi}{4}\right)\right)dx = 0,$$

$0 < x < \sqrt{\frac{\pi}{2}}$, which passes through the point $\left(\sqrt{\frac{\pi}{6}}, 1\right)$. Then $\left|y\left(\sqrt{\frac{\pi}{3}}\right)\right|$ is equal to ________.

Official Ans. by NTA (1)

Ans. (1)

Sol. $$\sin\left(2x^2\right)\ln\left(\tan x^2\right)dy + \left(4xy - 4\sqrt{2}x\sin\left(x^2-\frac{\pi}{4}\right)\right)dx = 0$$

$$\ln\left(\tan x^2\right)dy + \frac{4xydx}{\sin\left(2x^2\right)} - \frac{4\sqrt{2}\,x\sin\left(x^2-\frac{\pi}{4}\right)}{\sin\left(2x^2\right)}dx = 0$$

$$d\left(y.\ln\left(\tan x^2\right)\right) - 4\sqrt{2}x\frac{\left(\sin x^2 - \cos x^2\right)}{\sqrt{2} - 2\sin x^2\cos x^2}dx = 0$$

$$d\left(y\ln\left(\tan x^2\right)\right) - \frac{4x\left(\sin x^2 - \cos x^2\right)}{\left(\sin x^2 + \cos^2\right) - 1}dx = 0$$

$$\Rightarrow \int d\left(y\ln\left(\tan x^2\right)\right) + 2\int\frac{dt}{t^2-1} = \int 0$$

$$\Rightarrow y\ln\left(\tan x^2\right) + 2.\frac{1}{2}\ln\left|\frac{t-1}{t+1}\right| = c$$

$$y\ln\left(\tan x^2\right) + \ln\left(\frac{\sin x^2 + \cos x^2 - 1}{\sin x^2 + \cos x^2 + 1}\right) = c$$

Put y = 1 and $x = \sqrt{\frac{\pi}{6}}$

$$1\ln\left(\frac{1}{\sqrt{3}}\right) + \ln\frac{\left(\frac{1}{2}+\frac{\sqrt{3}}{2}-1\right)}{\left(\frac{1}{2}+\frac{\sqrt{3}}{2}+1\right)} = c$$

Now

$$x = \sqrt{\frac{\pi}{3}} \Rightarrow y\left(\ln\sqrt{3}\right) + \ln\frac{\left(\frac{1}{2}+\frac{\sqrt{3}}{2}-1\right)}{\left(\frac{1}{2}+\frac{\sqrt{3}}{2}+1\right)} = \ln\left(\frac{1}{\sqrt{3}}\right) + \ln\left(\frac{\sqrt{3}-1}{\sqrt{3}+3}\right)$$

$$y\left(\ln\sqrt{3}\right) = \ln\left(\frac{1}{\sqrt{3}}\right)$$

$\Rightarrow$ y = -1

$$|y| = 1$$

6. Let *M* and *N* be the number of points on the curve $y^5 - 9xy + 2x = 0$, where the tangents to the curve are parallel to x-axis and y-axis, respectively. Then the value of M + N equals ________.

Official Ans. by NTA (2)

Ans. (2)

Sol. $y^5 - 9xy + 2x = 0$

$$5y^4\frac{dy}{x} - 9x\frac{dy}{dx} - 9y + 2 = 0$$

$$\frac{dy}{dx}\left(5y^4 - 9x\right) = 9y - 2$$

$$\frac{dy}{dx} = \frac{9y-2}{5y^4-9x} = 0 \text{ (for horizontal tangent)}$$

$y = \frac{2}{9} \Rightarrow$ Which does not satisfy the original equation $\Rightarrow$ M = 0.

Now $5y^4 - 9x = 0$ (for vertical tangent)

$5y^4(9y-2) - 9y^5 = 0$

$y^4[45y - 10 - 9y] = 0$

$y = 0$ (Or) $36y = 10$

$$y = \frac{5}{18}$$

$$y = 0 \Rightarrow x = 0 \ \& \ y = \frac{5}{18} \Rightarrow x =$$

$$(0,0) \qquad \left(x, \frac{5}{18}\right)$$

N = 2

M + N = 0 + 2 = 2

7. Let $f(x) = 2x^2 - x - 1$ and $S = \{n \in \mathbb{Z} : |f(n)| \le 800\}$. Then, the value of $\sum_{n \in S} f(n)$ is equal to ________.

Official Ans. by NTA (10620)

Ans. (10620)

Sol. $f(x) = 2x^2 - x - 1$

$|f(x)| \le 800$

$2n^2 - n - 801 \le 0$

$$n^2 - \frac{1}{2}n - \frac{801}{2} \le 0$$

$$\left(n - \frac{1}{4}\right)^2 - \frac{801}{2} - \frac{1}{16} \le 0$$

$$\left(n - \frac{1}{4}\right)^2 - \frac{6409}{16} \le 0$$

$$\left(n - \frac{1}{4} - \frac{\sqrt{6409}}{4}\right)\left(n - \frac{1}{4} + \frac{\sqrt{6409}}{16}\right) \le 0$$

$$\frac{1 - \sqrt{6409}}{4} \le n \le \frac{1 + \sqrt{6409}}{4}$$

$$n = \{-19, -18 - 17, ...0, 1, 2, ..., 20\}$$

$$\sum_{n \in S} f(x) = \sum\left(2x^2 - x - 1\right)$$

$$= 2\left[19^2 + 18^2 + ... + 1^2 + 1^2 + 2^2 + ... + 19^2 + 20^2\right]$$

$$= 4\left[1^2 + 2^2 + ... + 19^2\right] + 2\left[20^2\right] - 20 - 40$$

$$= \frac{4 \times 19 \times 20 \times (2 \times 19 + 1)}{6} + 2 \times 400 - 60$$

$$= \frac{4 \times 19 \times 20 \times 39}{6} + 800 - 60 - 9880 + 800 - 60$$

= 10620

8. Let S be the set containing all 3 × 3 matrices with entries from {-1, 0, 1}. The total number of matrices A $\in S$ such that the sum of all the diagonal elements of $A^T A$ is 6 is __________.

Official Ans. by NTA (5376)

Ans. (5376)

Sol. $Tr\left(AA^T\right) = 6$

$$AA^T = \begin{bmatrix} a & d & g \\ b & e & h \\ c & f & i \end{bmatrix}\begin{bmatrix} a & b & c \\ d & e & f \\ g & h & i \end{bmatrix}$$

Now given $a^2 + d^2 + g^2 + b^2 + e^2 + h^2 + c^2 + f^2 + i^2 = 6$

$$= {}^9C_3 \times 2^6$$

= 5376

9. If the length of the latus rectum of the ellipse $x^2 + 4y^2 + 2x + 8y - \lambda = 0$ is 4, and l is the length of its major axis, then $\lambda + l$ is equal to _______.

Official Ans. by NTA (75)

Ans. (75)

Sol. $\lambda + \ell = 75$

$$x^2 + 4y^2 + 2x + 8y - \lambda = 0$$

$$\frac{(x+1)^2}{\lambda+5} + \frac{(y+1)^2}{\frac{\lambda+5}{4}} = 1$$

$$\because \frac{2b^2}{a} = 4$$

$$\frac{2(\lambda+5)}{4} = 4\left(\sqrt{\lambda+5}\right)$$

$$\Rightarrow \lambda = 59$$

$$\lambda \neq -5$$

$$l = 2a = 2\sqrt{\lambda+5} = 2\sqrt{65} = 16$$

$$\Rightarrow \lambda + \ell = 59 + 16 = 75$$

10. Let $S = \{z \in \mathbb{C} : z^2 + \bar{z} = 0\}$. Then $\sum_{z\in S}(\text{Re}(z) + \text{Im}(z))$ is equal to _______.

Official Ans. by NTA (0)

Ans. (0)

Sol. $S = \{z \in C : z^2 + \bar{z} = 0\}$

Let $z = x + iy$

$z^2 = x^2 - y^2 + 2ixy$

$\bar{z} = x - iy$

$$z^2 + \bar{z} = x^2 - y^2 + x + i(2xy - y) = 0$$

$$\Rightarrow x^2 + x - y^2 = 0 \;\&\; 2xy - y = 0$$

$$y = 0 \text{ or } x = \frac{1}{2}$$

If $y = 0$; $x = 0, -1$

If $x = \frac{1}{2}; y = \frac{\sqrt{3}}{2}, \frac{-\sqrt{3}}{2}$

$$\sum_{z\in S}\left(\text{Re}(z) + \text{Im}(z) = \left(0 - 1 + \frac{1}{2} + \frac{1}{2}\right) + 0 + 0 + \frac{\sqrt{3}}{2} - \frac{\sqrt{3}}{2}\right)$$