FINAL JEE–MAIN EXAMINATION – FEBRUARY, 2021

Held On Wednesday 24th February, 2021

TIME: 9:00 AM to 12:00 NOON

SECTION-A

1. The statement among the following that is a tautology is :

(1) $A \vee (A \wedge B)$

(2) $A \wedge (A \vee B)$

(3) $B \to [A \wedge (A \to B)]$

(4) $[A \wedge (A \to B)] \to B$

Official Ans. by NTA (4)

Sol. $\left(A \wedge (A \to B)\right) \to B$

$= \left(A \wedge (\sim A \vee B)\right) \to B$

$= \left((A \wedge \sim A) \vee (A \wedge B)\right) \to B$

$= (A \wedge B) \to B$

$= \sim (A \wedge B) \vee B$

$= (\sim A \vee \sim B) \vee B$

$= T$

2. A man is walking on a straight line. The arithmetic mean of the reciprocals of the intercepts of this line on the coordinate axes is $\frac{1}{4}$. Three stones A, B and C are placed at the points (1,1), (2, 2) and (4, 4) respectively. Then which of these stones is / are on the path of the man ?

(1) A only (2) C only

(3) All the three (4) B only

Official Ans. by NTA (4)

Sol. Let the line be y = mx + c

x-intercept : $-\frac{c}{m}$

y-intercept : c

A.M of reciprocals of the intercepts :

$$\frac{-\frac{m}{c}+\frac{1}{c}}{2} = \frac{1}{4} \Rightarrow 2(1-m) = c$$

line : y = mx + 2(1 – m) = c

$\Rightarrow$ (y – 2) – m(x – 2) = 0

$\Rightarrow$ line always passes through (2, 2)

Ans. 4

3. The equation of the plane passing through the point (1, 2, –3) and perpendicular to the planes 3x + y – 2z = 5 and 2x – 5y – z = 7, is

(1) 3x – 10y – 2z + 11 = 0

(2) 6x – 5y – 2z – 2 = 0

(3) 11x + y + 17z + 38 = 0

(4) 6x – 5y + 2z + 10 = 0

Official Ans. by NTA (3)

Sol. Normal vector :

$$\begin{vmatrix} \hat{i} & \hat{j} & \hat{k} \\ 3 & 1 & -2 \\ 2 & -5 & -1 \end{vmatrix} = -11\hat{i} - \hat{j} + 17\hat{k}$$

So drs of normal to the required plane is <11, 1, 17>

plane passes through (1, 2, –3)

So eq^n of plane :

11(x – 1) + 1(y – 2) + 17(z + 3) = 0

$\Rightarrow$ 11x + y + 17z + 38 = 0

4. The population P = P(t) at time 't' of a certain species follows the differential equation $\frac{dP}{dt} = 0.5P - 450$. If P(0) = 850, then the time at which population becomes zero is :

(1) $\log_e 18$ (2) $\log_e 9$

(3) $\frac{1}{2}\log_e 18$ (4) $2\log_e 18$

Official Ans. by NTA (4)

Sol. $\frac{dP}{dt} = 0.5P - 450$

$$\Rightarrow \int_0^t \frac{dp}{P-900} = \int_0^t \frac{dt}{2}$$

$$\Rightarrow \left[\ell n|P(t)-900|\right]_0^t = \left[\frac{t}{2}\right]_0^t$$

$\Rightarrow \quad \ell n|P(t)-900|-\ell n|P(0)-900|=\frac{t}{2}$

$\Rightarrow \quad \ell n|P(t)-900|-\ell n|50|=\frac{t}{2}$

for P(t) = 0

$\Rightarrow \quad \ell n\left|\frac{900}{50}\right|=\frac{t}{2} \Rightarrow t=2\ell n18$

5. The system of linear equations

3x – 2y – kz = 10

2x – 4y – 2z = 6

x + 2y – z = 5m

is inconsistent if :

(1) $k=3, m=\frac{4}{5}$ (2) $k\neq 3, m\in R$

(3) $k\neq 3, m\neq\frac{4}{5}$ (4) $k=3, m\neq\frac{4}{5}$

Official Ans. by NTA (4)

Sol. $\Delta=\begin{vmatrix}3 & -2 & -k\\ 2 & -4 & -2\\ 1 & 2 & -1\end{vmatrix}=0$

$\Rightarrow$ 24 – 2(0) – k(8) = 0 $\Rightarrow$ k = 3

$\Delta_x=\begin{vmatrix}10 & -2 & -3\\ 6 & -4 & -2\\ 5m & 2 & -1\end{vmatrix}$

= 10(8) – 2(–10m + 6) – 3(12 + 20m)

= 8(4 – 5m)

$\Delta_y=\begin{vmatrix}3 & 10 & -3\\ 2 & 6 & -2\\ 1 & 5m & -1\end{vmatrix}$

= 3(–6 + 10m) + 10(0) – 3(10m – 6)

= 0

$\Delta_z=\begin{vmatrix}3 & -2 & 10\\ 2 & -4 & 6\\ 1 & 2 & 5m\end{vmatrix}$

= 3(–20m – 12) – 2(6 – 10m) + 10(8)

= 40m – 32 = 8(5m – 4)

for inconsistent

$k = 3$ & $m\neq\frac{4}{5}$

6. If f : R → R is a function defined by $f(x)=[x-1]\cos\left(\frac{2x-1}{2}\right)\pi$, where [.] denotes the greatest integer function, then f is :

(1) discontinuous at all integral values of x except at x = 1

(2) continuous only at x = 1

(3) continuous for every real x

(4) discontinuous only at x = 1

Official Ans. by NTA (3)

Sol. For x = n, n ∈ Z

LHL $=\lim_{x\to n^-} f(x)=\lim_{x\to n^-}[x-1]\cos\left(\frac{2x-1}{2}\right)\pi$

= 0

RHL $=\lim_{x\to n^+} f(x)=\lim_{x\to n^+}[x-1]\cos\left(\frac{2x-1}{2}\right)\pi$

= 0

$f(n)$ = 0

$\Rightarrow$ LHL = RHL = $f(n)$

$\Rightarrow$ $f(x)$ is continuous for every real x.

7. The distance of the point (1, 1, 9) from the point of intersection of the line $\frac{x-3}{1}=\frac{y-4}{2}=\frac{z-5}{2}$ and the plane x + y + z = 17 is :

(1) $2\sqrt{19}$ (2) $19\sqrt{2}$

(3) 38 (4) $\sqrt{38}$

Official Ans. by NTA (4)

Sol. Let $\frac{x-3}{1}=\frac{y-4}{2}=\frac{z-5}{2}=t$

$\Rightarrow$ x = 3 + t, y = 2t + 4, z = 2t + 5

for point of intersection with x + y + z = 17

3 + t + 2t + 4 + 2t + 5 = 17

$\Rightarrow$ 5t = 5 $\Rightarrow$ t = 1

$\Rightarrow$ point of intersection is (4, 6, 7)

distance between (1, 1, 9) and (4, 6, 7)

is $\sqrt{9+25+4}=\sqrt{38}$

8. If the tangent to the curve $y = x^3$ at the point $P(t, t^3)$ meets the curve again at Q, then the ordinate of the point which divides PQ internally in the ratio 1 : 2 is :

(1) $-2t^3$ (2) 0 (3) $-t^3$ (4) $2t^3$

Official Ans. by NTA (1)

Sol. Slope of tangent at $P(t, t^3) = \left.\frac{dy}{dx}\right]_{(t,t^3)}$

$= (3x^2)_{x=t} = 3t^2$

So equation tangent at $P(t, t^3)$:

$y - t^3 = 3t^2(x - t)$

for point of intersection with $y = x^3$

$x^3 - t^3 = 3t^2x - 3t^3$

$\Rightarrow (x - t)(x^2 + xt + t^2) = 3t^2(x - t)$

for $x \neq t$

$x^2 + xt + t^2 = 3t^2$

$\Rightarrow x^2 + xt - 2t^2 = 0 \Rightarrow (x - t)(x + 2t) = 0$

So for Q : $x = -2t$, $Q(-2t, -8t^3)$

ordinate of required point : $\frac{2t^3 - 8t^3}{2+1} = -2t^3$

9. If $\int \frac{\cos x - \sin x}{\sqrt{8 - \sin 2x}} dx = a\sin^{-1}\left(\frac{\sin x + \cos x}{b}\right) + c$, where c is a constant of integration, then the ordered pair (a, b) is equal to :

(1) (–1, 3) (2) (3, 1)
(3) (1, 3) (4) (1, –3)

Official Ans. by NTA (3)

Sol. $\int \frac{\cos x - \sin x}{\sqrt{8 - \sin 2x}} dx$

$= \int \frac{\cos x - \sin x}{\sqrt{9 - (\sin x + \cos x)^2}} dx$

Let $\sin x + \cos x = t$

$\int \frac{dt}{\sqrt{9 - t^2}} = \sin^{-1}\frac{t}{3} + c$

$= \sin^{-1}\left(\frac{\sin x + \cos x}{3}\right) + c$

So a = 1, b = 3.

10. The value of $-{}^{15}C_1 + 2.{}^{15}C_2 - 3.{}^{15}C_3 + \ldots\ldots -15.{}^{15}C_{15} + {}^{14}C_1 + {}^{14}C_3 + {}^{14}C_5 + \ldots. + {}^{14}C_{11}$ is :

(1) $2^{16} - 1$ (2) $2^{13} - 14$
(3) 2^{14} (4) $2^{13} - 13$

Official Ans. by NTA (2)

Sol. $(-{}^{15}C_1 + 2.{}^{15}C_2 - 3.{}^{15}C_3 + \ldots\ldots -15.{}^{15}C_{15}) + ({}^{14}C_1 + {}^{14}C_3 + \ldots. + {}^{14}C_{11})$

$= \sum_{r=1}^{15}(-1)^r.r\,{}^{15}C_r + \left({}^{14}C_1 + {}^{14}C_3 + \ldots + {}^{14}C_{11} + {}^{14}C_{13}\right) - {}^{14}C_3$

$= \sum_{r=1}^{15}(-1)^r\,15.{}^{14}C_{r-1} + 2^{13} - 14$

$= 15(-{}^{14}C_0 + {}^{14}C_1 \ldots\ldots\ldots - {}^{14}C_{14}) + 2^{13} - 14$

$= 2^{13} - 14$

11. The function

$f(x) = \frac{4x^3 - 3x^2}{6} - 2\sin x + (2x - 1)\cos x$:

(1) increases in $\left[\frac{1}{2}, \infty\right)$

(2) increases in $\left(-\infty, \frac{1}{2}\right]$

(3) decreases in $\left[\frac{1}{2}, \infty\right)$

(4) decreases in $\left(-\infty, \frac{1}{2}\right]$

Official Ans. by NTA (1)

Sol. $f(x) = \frac{4x^3 - 3x^2}{6} - 2\sin x + (2x - 1)\cos x$

$f'(x) = (2x^2 - x) - 2\cos x + 2\cos x - \sin x(2x - 1)$

$= (2x - 1)(x - \sin x)$

for $x > 0$, $x - \sin x > 0$

$x < 0$, $x - \sin x < 0$

for $x \in (-\infty, 0] \cup \left[\frac{1}{2}, \infty\right)$, $f'(x) \geq 0$

for $x \in \left[0, \frac{1}{2}\right]$, $f'(x) \leq 0$

$\Rightarrow f(x)$ increases in $\left[\frac{1}{2}, \infty\right)$.

12. Let $f : R \to R$ be defined as $f(x) = 2x - 1$ and $g : R - \{1\} \to R$ be defined as $g(x) = \dfrac{x - \frac{1}{2}}{x - 1}$.

Then the composition function $f(g(x))$ is :

(1) onto but not one-one

(2) both one-one and onto

(3) one-one but not onto

(4) neither one-one nor onto

Official Ans. by NTA (3)

Sol. $f(g(x)) = 2g(x) - 1 = 2\left(\dfrac{2x-1}{2(x-1)}\right) - 1$

$= \dfrac{x}{x-1} = 1 + \dfrac{1}{x-1}$

Range of $f(g(x) = \mathbb{R} - \{1\}$

Range of $f(g(x))$ is not onto

& $f(g(x))$ is one-one

So $f(g(x))$ is one-one but not onto.

13. An ordinary dice is rolled for a certain number of times. If the probability of getting an odd number 2 times is equal to the probability of getting an even number 3 times, then the probability of getting an odd number for odd number of times is :

(1) $\dfrac{1}{32}$ (2) $\dfrac{5}{16}$ (3) $\dfrac{3}{16}$ (4) $\dfrac{1}{2}$

Official Ans. by NTA (4)

Sol. ${}^nC_2\left(\dfrac{1}{2}\right)^n = {}^nC_3\left(\dfrac{1}{2}\right)^n \Rightarrow {}^nC_2 = {}^nC_3$

$\Rightarrow \quad n = 5$

Probability of getting an odd number for odd number of times is

${}^5C_1\left(\dfrac{1}{2}\right)^5 + {}^5C_3\left(\dfrac{1}{2}\right)^5 + {}^5C_5\left(\dfrac{1}{2}\right)^5 = \dfrac{1}{2^5}(5+10+1)$

$= \dfrac{1}{2}$

14. A scientific committee is to be formed from 6 Indians and 8 foreigners, which includes at least 2 Indians and double the number of foreigners as Indians. Then the number of ways, the committee can be formed, is :

(1) 1625 (2) 575

(3) 560 (4) 1050

Official Ans. by NTA (1)

Sol.

Indians	Foreigners	Number of ways
2	4	${}^6C_2 \times {}^8C_4 = 1050$
3	6	${}^6C_3 \times {}^8C_6 = 560$
4	8	${}^6C_4 \times {}^8C_8 = 15$

Total number of ways = 1625

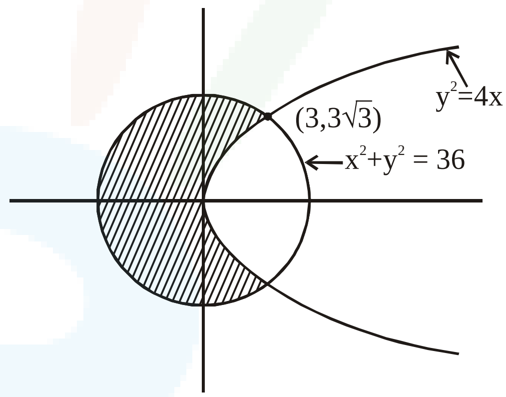
15. The area (in sq. units) of the part of the circle $x^2 + y^2 = 36$, which is outside the parabola $y^2 = 9x$, is :

(1) $24\pi + 3\sqrt{3}$ (2) $12\pi - 3\sqrt{3}$

(3) $24\pi - 3\sqrt{3}$ (4) $12\pi + 3\sqrt{3}$

Official Ans. by NTA (3)

Sol.



Required area

$= \pi \times (6)^2 - 2\int_0^3 \sqrt{9x}\,dx - \int_3^6 \sqrt{36 - x^2}\,dx$

$= 36\pi - 12\sqrt{3} - 2\left(\dfrac{x}{2}\sqrt{36-x^2} + 18\sin^{-1}\dfrac{x}{6}\right)_3^6$

$= 36\pi - 12\sqrt{3} - 2\left(9\pi - 3\pi - \dfrac{9\sqrt{3}}{2}\right)$

$= 24\pi - 3\sqrt{3}$

16. Let p and q be two positive numbers such that $p + q = 2$ and $p^4 + q^4 = 272$. Then p and q are roots of the equation :

(1) $x^2 - 2x + 2 = 0$

(2) $x^2 - 2x + 8 = 0$

(3) $x^2 - 2x + 136 = 0$

(4) $x^2 - 2x + 16 = 0$

Official Ans. by NTA (4)

Sol. Consider $(p^2 + q^2)^2 - 2p^2q^2 = 272$

$((p + q)^2 - 2pq)^2 - 2p^2q^2 = 272$

$16 - 16pq + 2p^2q^2 = 272$

$(pq)^2 - 8pq - 128 = 0$

$$(pq)^2 - 8pq - 128 = 0$$

$$pq = \frac{8 \pm 24}{2} = 16, -8$$

$\therefore$ pq = 16

$\therefore$ Required equation : $x^2 - (2)x + 16 = 0$

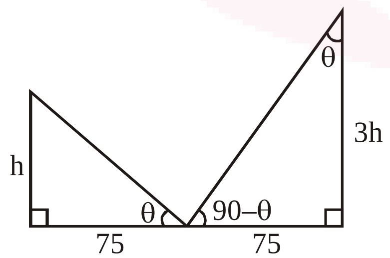
17. Two vertical poles are 150 m apart and the height of one is three times that of the other. If from the middle point of the line joining their feet, an observer finds the angles of elevation of their tops to be complementary, then the height of the shorter pole (in meters) is :

(1) $20\sqrt{3}$ (2) $25\sqrt{3}$

(3) 30 (4) 25

Official Ans. by NTA (2)

Sol.



$$\tan\theta = \frac{h}{75} = \frac{75}{3h}$$

$$\Rightarrow \quad h^2 = \frac{(75)^2}{3}$$

$$h = 25\sqrt{3}\text{ m}$$

18. $\lim\limits_{x\to 0} \dfrac{\int_0^{x^2} \left(\sin\sqrt{t}\right)dt}{x^3}$ is equal to :

(1) $\frac{2}{3}$ (2) $\frac{3}{2}$ (3) 0 (4) $\frac{1}{15}$

Official Ans. by NTA (1)

Sol. $$\lim_{x\to 0^+} \frac{\int_0^{x^2} \sin\sqrt{t}\,dt}{x^3} = \lim_{x\to 0^+} \frac{(\sin x)2x}{3x^2}$$

$$= \lim_{x\to 0^+}\left(\frac{\sin x}{x}\right) \times \frac{2}{3} = \frac{2}{3}$$

19. If $e^{(\cos^2 x + \cos^4 x + \cos^6 x + ...\infty)\log_e 2}$ satisfies the equation $t^2 - 9t + 8 = 0$, then the value of $\dfrac{2\sin x}{\sin x + \sqrt{3}\cos x}\left(0 < x < \dfrac{\pi}{2}\right)$ is

(1) $2\sqrt{3}$ (2) $\frac{3}{2}$

(3) $\sqrt{3}$ (4) $\frac{1}{2}$

Official Ans. by NTA (4)

Sol. $e^{(\cos^2\theta + \cos^4\theta + ...\infty)\ell n^2} = 2^{\cos^2\theta + \cos^4\theta + ...\infty}$

$= 2^{\cot^2\theta}$

Now $t^2 - 9t + 9 = 0 \Rightarrow t = 1, 8$

$$\Rightarrow \quad 2^{\cot^2\theta} = 1, 8 \Rightarrow \cot^2\theta = 0, 3$$

$$0 < \theta < \frac{\pi}{2} \Rightarrow \cot\theta = \sqrt{3}$$

$$\Rightarrow \quad \frac{2\sin\theta}{\sin\theta + \sqrt{3}\sin\theta} = \frac{2}{1 + \sqrt{3}\cot\theta} = \frac{2}{4} = \frac{1}{2}$$

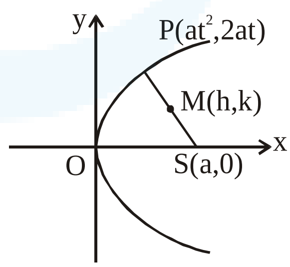
20. The locus of the mid-point of the line segment joining the focus of the parabola $y^2 = 4ax$ to a moving point of the parabola, is another parabola whose directrix is :

(1) $x = -\frac{a}{2}$ (2) $x = \frac{a}{2}$

(3) x = 0 (4) x = a

Official Ans. by NTA (3)

Sol.



$$h = \frac{at^2 + a}{2},\ k = \frac{2at + 0}{2}$$

$$\Rightarrow \quad t^2 = \frac{2h - a}{a} \text{ and } t = \frac{k}{a}$$

$$\Rightarrow \quad \frac{k^2}{a^2} = \frac{2h - a}{a}$$

$\Rightarrow$ Locus of (h, k) is $y^2 = a(2x - a)$

$$\Rightarrow \quad y^2 = 2a\left(x - \frac{a}{2}\right)$$

Its directrix is $x - \frac{a}{2} = -\frac{a}{2} \Rightarrow x = 0$

SECTION-B

1. If the least and the largest real values of α, for which the equation $z + \alpha|z - 1| + 2i = 0$ ($z \in C$ and $i = \sqrt{-1}$) has a solution, are p and q respectively; then $4(p^2 + q^2)$ is equal to ____

Official Ans. by NTA (10)

Sol. Put $z = x + iy$

$x + iy + \alpha|x + iy - 1| + 2i = 0$

$\Rightarrow \quad x+\alpha\sqrt{(x-1)^2+y^2}+i(y+2)=0+0i$

$\Rightarrow \quad y + 2 = 0$ and $x+\alpha\sqrt{(x-1)^2+y^2}=0$

$\Rightarrow \quad y = -2$ and $\alpha^2=\dfrac{x^2}{x^2-2x+5}$

Now $\dfrac{x^2}{x^2-2x+5}\in\left[0,\dfrac{5}{4}\right]$

$\therefore \quad \alpha^2\in\left[0,\dfrac{5}{4}\right] \quad \Rightarrow \quad \alpha\in\left[-\dfrac{\sqrt5}{2},\dfrac{\sqrt5}{2}\right]$

$\therefore \quad p=-\dfrac{\sqrt5}{2};\ q=\dfrac{\sqrt5}{2}$

$\Rightarrow \quad 4(p^2+q^2)=4\left(\dfrac{5}{4}+\dfrac{5}{4}\right)=10$

2. If $\int_{-a}^{a}(|x|+|x-2|)dx=22,\ (a>2)$ and $[x]$ denotes the greatest integer $\le$ x, then $\int_{a}^{-a}(x+[x])\,dx$ is equal to _______.

Official Ans. by NTA (3)

Sol. $\int_{-a}^{0}(-2x+2)dx+\int_{0}^{2}(x+2-x)dx+\int_{2}^{a}(2x-2)dx=22$

$x^2-2x\Big|_0^{-a}+2x\Big|_0^2+x^2-2x\Big|_0^a=22$

$a^2 + 2a + 4 + a^2 - 2a - (4 - 4) = 22$

$2a^2 = 18 \quad \Rightarrow \quad a = 3$

$\int_{3}^{-3}(x+[x])dx=-(-3-2-1+1+2)=3$

3. Let $A = \{n \in N : n$ is a 3-digit number$\}$

$B = \{9k + 2 : k \in N\}$

and $C = \{9k + l : k \in N\}$ for some l $(0 < l < 9)$

If the sum of all the elements of the set $A \cap (B \cup C)$ is 274×400, then l is equal to ______.

Official Ans. by NTA (5)

Sol. B and C will contain three digit numbers of the form $9k + 2$ and $9k + \ell$ respectively. We need to find sum of all elements in the set $B \cup C$ effectively.

Now, $S(B \cup C) = S(B) + S(C) - S(B \cap C)$

where S(k) denotes sum of elements of set k.

Also, $B = \{101, 109, \ldots\ldots, 992\}$

$\therefore \quad S(B)=\dfrac{100}{2}(101+992)=54650$

Case-I : If $\ell = 2$

then $B \cap C = B$

$\therefore \quad S(B \cup C) = S(B)$

which is not possible as given sum is $274 \times 400 = 109600$.

Case-II : If $\ell \ne 2$

then $B \cap C = \phi$

$\therefore \quad S(B \cup C) = S(B) + S(C) = 400 \times 274$

$\Rightarrow \quad 54650+\sum_{k=11}^{110}9k+\ell=109600$

$\Rightarrow \quad 9\sum_{k=11}^{110}k+\sum_{k=11}^{110}\ell=54950$

$\Rightarrow \quad 9\left(\dfrac{100}{2}(11+110)\right)+\ell(100)=54950$

$\Rightarrow \quad 54450 + 100\ell = 54950$

$\Rightarrow \quad \ell = 5$

4. Let M be any 3×3 matrix with entries from the set $\{0, 1, 2\}$. The maximum number of such matrices, for which the sum of diagonal elements of M^TM is seven, is _______.

Official Ans. by NTA (540)

Sol. $\begin{bmatrix} a & b & c \\ d & e & f \\ g & h & i \end{bmatrix}\begin{bmatrix} a & d & g \\ b & e & h \\ c & f & i \end{bmatrix}$

$a^2 + b^2 + c^2 + d^2 + e^2 + f^2 + g^2 + h^2 + i^2 = 7$

Case-I : Seven (1's) and two (0's)

$^9C_2 = 36$

Case-II : One (2) and three (1's) and five (0's)

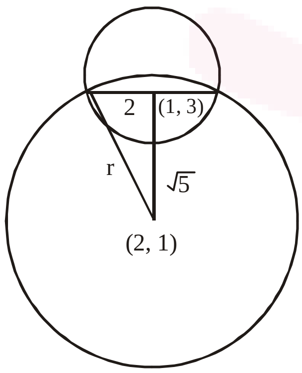
$\frac{9!}{5!3!} = 504$

$\therefore$ Total = 540

5. If one of the diameters of the circle $x^2 + y^2 - 2x - 6y + 6 = 0$ is a chord of another circle 'C', whose center is at (2, 1), then its radius is _______ .

Official Ans. by NTA (3)

Sol.



$x^2 + y^2 + 2x - 6y + 6 = 0$

center (1, 3)

radius = 2

distance between (1, 3) and (2, 1) is $\sqrt{5}$

$\therefore \left(\sqrt{5}\right)^2 + (2)^2 = r^2$

$\Rightarrow r = 3$

6. The minimum value of α for which the equation $\frac{4}{\sin x} + \frac{1}{1-\sin x} = \alpha$ has at least one solution in $\left(0, \frac{\pi}{2}\right)$ is _______.

Official Ans. by NTA (9)

Sol. Let $f(x) = \frac{4}{\sin x} + \frac{1}{1-\sin x}$

$\Rightarrow f'(x) = 0 \Rightarrow \sin x = 2/3$

$\therefore f(x)_{min} = \frac{4}{2/3} + \frac{1}{1-2/3} = 9$

$f(x)max \rightarrow \infty$

$f(x)$ is continuous function

$\therefore \alpha_{min} = 9$

7. $\lim_{n\to\infty} \tan\left\{\sum_{r=1}^{n} \tan^{-1}\left(\frac{1}{1+r+r^2}\right)\right\}$ is equal to ____.

Official Ans. by NTA (1)

Sol. $\lim_{n\to\alpha} \tan\left(\sum_{r=1}^{n} \tan^{-1}\left(\frac{1}{1+r(r+1)}\right)\right)$

$= \lim_{n\to\alpha} \tan\left(\sum_{r=1}^{n} \tan^{-1}\left(\frac{r+1-r}{1+r(r+1)}\right)\right)$

$= \tan\left(\lim_{n\to\alpha} \sum_{r=1}^{n}\left[\tan^{-1}(r+1) - \tan^{-1}(r)\right]\right)$

$= \tan\left(\lim_{n\to\infty}\left(\tan^{-1}(n+1) - \frac{\pi}{4}\right)\right)$

$= \tan\left(\frac{\pi}{4}\right) = 1$

8. Let three vectors $\vec{a}, \vec{b}$ and $\vec{c}$ be such that $\vec{c}$ is coplanar with $\vec{a}$ and $\vec{b}, \vec{a}.\vec{c} = 7$ and $\vec{b}$ is perpendicular to $\vec{c}$, where $\vec{a} = -\hat{i} + \hat{j} + \hat{k}$ and $\vec{b} = 2\hat{i} + \hat{k}$, then the value of $2\left|\vec{a} + \vec{b} + \vec{c}\right|^2$ is ______.

Official Ans. by NTA (75)

Sol. Let $\vec{c}=\lambda\left(\vec{b}\times\left(\vec{a}\times\vec{b}\right)\right)$

$=\lambda\left(\left(\vec{b}.\vec{b}\right)\vec{a}-\left(\vec{b}.\vec{a}\right)\vec{b}\right)$

$=\lambda\left(5\left(-\hat{i}+\hat{j}+\hat{k}\right)+2\hat{i}+\hat{k}\right)$

$=\lambda\left(-3\hat{i}+5\hat{j}+6\hat{k}\right)$

$\vec{c}.\vec{a}=7\Rightarrow 3\lambda+5\lambda+6\lambda=7$

$\lambda=\frac{1}{2}$

$\therefore\ 2\left|\left(\frac{-3}{2}-1+2\right)\hat{i}+\left(\frac{5}{2}+1\right)\hat{j}+(3+1+1)\hat{k}\right|^2$

$=2\left(\frac{1}{4}+\frac{49}{4}+25\right)=25+50=75$

9. Let B_i (i = 1, 2, 3) be three independent events in a sample space. The probability that only B_1 occur is α, only B_2 occurs is β and only B_3 occurs is γ. Let p be the probability that none of the events B_i occurs and these 4 probabilities satisfy the equations $(\alpha-2\beta)p=\alpha\beta$ and $(\beta-3\gamma)p=2\beta\gamma$ (All the probabilities are assumed to lie in the interval (0,1)). Then

$\frac{P(B_1)}{P(B_3)}$ is equal to______.

Official Ans. by NTA (6)

Sol. Let $P(B_1)=p_1$, $P(B_2)=p_2$, $P(B_3)=p_3$

given that $p_1(1-p_2)(1-p_3)=\alpha$ .....(i)

$p_2(1-p_1)(1-p_3)=\beta$ .....(ii)

$p_3(1-p_1)(1-p_2)=\gamma$ .....(iii)

and $(1-p_1)(1-p_2)(1-p_3)=p$ .....(iv)

$\Rightarrow\ \frac{p_1}{1-p_1}=\frac{\alpha}{p},\ \frac{p_2}{1-p_2}=\frac{\beta}{p}\ \ \&\ \ \frac{p_3}{1-p_3}=\frac{\gamma}{p}$

Also $\beta=\frac{\alpha p}{\alpha+2p}=\frac{3\gamma p}{p-2\gamma}$

$\Rightarrow\ \alpha p-2\alpha\gamma=3\alpha\gamma+6p\gamma$

$\Rightarrow\ \alpha p-6p\gamma=5\alpha\gamma$

$\Rightarrow\ \frac{p_1}{1-p_1}-\frac{6p_3}{1-p_3}=\frac{5p_1p_3}{(1-p_1)(1-p_3)}$

$\Rightarrow\ p_1-6p_3=0$

$\Rightarrow\ \frac{p_1}{p_3}=6$

10. Let $P=\begin{bmatrix}3 & -1 & -2\\ 2 & 0 & \alpha\\ 3 & -5 & 0\end{bmatrix}$, where $\alpha\in R$. Suppose $Q=[q_{ij}]$ is a matrix satisfying $PQ=kI_3$ for some non-zero $k\in R$. If $q_{23}=-\frac{k}{8}$ and $|Q|=\frac{k^2}{2}$, then α^2+k^2 is equal to _______.

Official Ans. by NTA (17)

Sol. PQ = kI

$|P|.|Q|=k^3$

$\Rightarrow$ $|P|=2k\neq 0\Rightarrow$ P is an invertible matrix

$\because$ PQ = kI

$\therefore$ $Q=kP^{-1}I$

$\therefore\ Q=\frac{adj.P}{2}$

$\because\ q_{23}=-\frac{k}{8}$

$\therefore\ \frac{-(3\alpha+4)}{2}=-\frac{k}{8}\Rightarrow k=4$

$\therefore$ $|P|=2k\Rightarrow k=10+6\alpha$ ...(i)

Put value of k in (i).. we get $\alpha=-1$