FINAL JEE–MAIN EXAMINATION – AUGUST, 2021

Held On Thursday 26th August, 2021

TIME: 3:00 PM to 06:00 PM

SECTION-A

1. Let [t] denote the greatest integer less than or equal to t. Let $f(x) = x - [x]$, $g(x) = 1 - x + [x]$, and $h(x) = \min\{f(x), g(x)\}$, $x \in [-2, 2]$. Then h is :

(1) continuous in [–2, 2] but not differentiable at more than four points in (–2, 2)

(2) not continuous at exactly three points in [–2, 2]

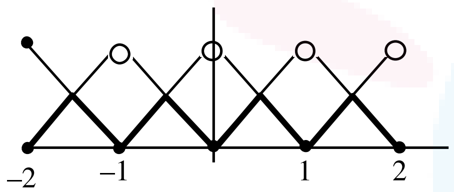
(3) continuous in [–2, 2] but not differentiable at exactly three points in (–2, 2)

(4) not continuous at exactly four points in [–2, 2]

Official Ans. by NTA (1)

Sol. $\min\{x - [x], 1 - x + [x]\}$

$h(x) = \min\{x - [x], 1 - [x - [x]]\}$



$\Rightarrow$ always continuous in [–2, 2]

but non differentiable at 7 Points

2. Let $A = \begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 1 \\ 1 & 0 & 0 \end{pmatrix}$. Then $A^{2025} - A^{2020}$ is equal to :

(1) $A^6 - A$ (2) A^5

(3) $A^5 - A$ (4) A^6

Official Ans. by NTA (1)

Sol. $A = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 1 \\ 1 & 0 & 0 \end{bmatrix} \Rightarrow A^2 = \begin{bmatrix} 1 & 0 & 0 \\ 1 & 1 & 1 \\ 1 & 0 & 0 \end{bmatrix}$

$A^3 = \begin{bmatrix} 1 & 0 & 0 \\ 2 & 1 & 1 \\ 1 & 0 & 0 \end{bmatrix} \Rightarrow A^4 = \begin{bmatrix} 1 & 0 & 0 \\ 3 & 1 & 1 \\ 1 & 0 & 0 \end{bmatrix}$

$A^n = \begin{bmatrix} 1 & 0 & 0 \\ n-1 & 1 & 1 \\ 1 & 0 & 0 \end{bmatrix}$

$A^{2025} - A^{2020} = \begin{bmatrix} 0 & 0 & 0 \\ 5 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix}$

$A^6 - A = \begin{bmatrix} 0 & 0 & 0 \\ 5 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix}$

3. The local maximum value of the function $f(x) = \left(\frac{2}{x}\right)^{x^2}$, $x > 0$, is

(1) $\left(2\sqrt{e}\right)^{\frac{1}{e}}$ (2) $\left(\frac{4}{\sqrt{e}}\right)^{\frac{e}{4}}$

(3) $(e)^{\frac{2}{e}}$ (4) 1

Official Ans. by NTA (3)

Sol. $f(x) = \left(\frac{2}{x}\right)^{x^2}$; $x > 0$

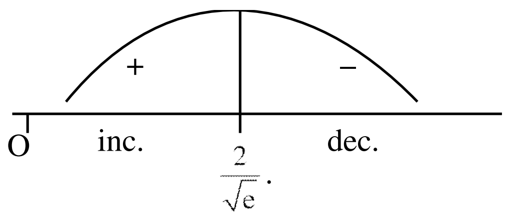
$\ell n\, f(x) = x^2 (\ell n 2 - \ell n x)$

$f'(x) = f(x) \{-x + (\ell n\, 2 - \ell n\, x)2x\}$

$f'(x) = \underbrace{f(x)}_{+} . \underbrace{x}_{+} \underbrace{(2\ell n 2 - 2\ell n x - 1)}_{g(x)}$

$g(x) = 2\ell\, n^2 - 2\ell\, n\, x - 1$

$= \ell n \frac{4}{x^2} - 1 = 0 \Rightarrow x = \frac{2}{\sqrt{e}}$



$LM = \frac{2}{\sqrt{e}}$

Local maximum value $= \left(\frac{2}{2/\sqrt{e}}\right)^{\frac{4}{e}} \Rightarrow e^{\frac{2}{e}}$

4. If the value of the integral $\int_0^5 \frac{x+[x]}{e^{x-[x]}}dx = \alpha e^{-1}+\beta$, where α, β ∈ **R**, 5α + 6β = 0, and [x] denotes the greatest integer less than or equal to x; then the value of $(\alpha+\beta)^2$ is equal to :

(1) 100 (2) 25

(3) 16 (4) 36

Official Ans. by NTA (2)

Sol. $I = \int_0^5 \frac{x+[x]}{e^{x-[x]}}dx$

$$\int_0^1 \frac{x}{e^x}dx + \int_1^2 \frac{x+1}{e^{x-1}}dx + \int_2^3 \frac{x+2}{e^{x-2}}dx + ... \int_4^5 \frac{x+4}{e^{x-4}}dx$$

$\Downarrow$ $\Downarrow$ $\Downarrow$

x = t + 1 x = z + 2 x = y + 4

$$\int_0^1 \frac{t+2}{e^t}dt + \int_0^1 \frac{z+4}{e^z}dz + ... + \int_0^1 \frac{y+8}{e^y}dx$$

$$\Rightarrow \int_0^5 \frac{5x+20}{e^x}dt = 5\int_0^1 \frac{x+4}{e^x}dx$$

$$\Rightarrow 5\int_0^1 (x+4)e^{-x}dx$$

$$\Rightarrow 5e^{-x}(-x-5)|_0^1 \Rightarrow -\frac{30}{e}+25$$

α = –30

β = 25 ⇒ 5α + 6β = 0

$(\alpha+\beta)^2 = 5^2 = 25$

5. The point $P(-2\sqrt{6}, \sqrt{3})$ lies on the hyperbola $\frac{x^2}{a^2}-\frac{y^2}{b^2}=1$ having eccentricity $\frac{\sqrt{5}}{2}$. If the tangent and normal at P to the hyperbola intersect its conjugate axis at the point Q and R respectively, then QR is equal to :

(1) $4\sqrt{3}$ (2) 6

(3) $6\sqrt{3}$ (4) $3\sqrt{6}$

Official Ans. by NTA (3)

Sol. $P(-2\sqrt{6}, \sqrt{3})$ lies on hyperbola

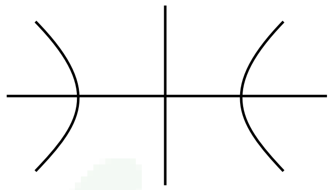
$$\Rightarrow \frac{24}{a^2}-\frac{3}{b^2}=1 \quad ....(i)$$

$$e = \frac{\sqrt{5}}{2} \Rightarrow b^2 = a^2\left(\frac{5}{4}-1\right) \Rightarrow 4b^2 = a^2$$

Put in (i) $\Rightarrow \frac{6}{b^2}-\frac{3}{b^2}=1 \Rightarrow b=\sqrt{3}$

$$\Rightarrow a = \sqrt{12}$$

$\frac{x^2}{12}-\frac{y^2}{3}=1$



Tangent at P :

$$\frac{-x}{\sqrt{6}}-\frac{y}{\sqrt{3}}=1 \Rightarrow Q\ (0, \sqrt{3})$$

Slope of $T = -\frac{1}{\sqrt{2}}$

Normal at P :

$$y-\sqrt{3}=\sqrt{2}(x+2\sqrt{6})$$

$$\Rightarrow R = (0, 5\sqrt{3})$$

$QR = 6\sqrt{3}$

6. Let y(x) be the solution of the differential equation $2x^2dy + (e^y - 2x)dx = 0, x > 0$. If y(e) = 1, then y(1) is equal to :

(1) 0 (2) 2

(3) $\log_e 2$ (4) $\log_e(2e)$

Official Ans. by NTA (3)

Sol. $2x^2dy + (e^y - 2x)dx = 0$

$$\frac{dy}{dx}+\frac{e^y-2x}{2x^2}=0 \Rightarrow \frac{dy}{dx}+\frac{e^y}{2x^2}-\frac{1}{x}=0$$

$$e^{-y}\frac{dy}{dx}-\frac{e^{-y}}{x}=-\frac{1}{2x^2} \Rightarrow \text{Put } e^{-y}=z$$

$$\frac{-dz}{dx}-\frac{z}{x}=-\frac{1}{2x^2} \Rightarrow xdz+zdx=\frac{dx}{2x}$$

$$d(xz)=\frac{dx}{2x} \Rightarrow xz=\frac{1}{2}\log_e x + c$$

$xe^{-y} = \frac{1}{2}\log_e x + c$, passes through $(e,1)$

$\Rightarrow C = \frac{1}{2}$

$xe^{-y} = \frac{\log_e ex}{2}$

$e^{-y} = \frac{1}{2} \Rightarrow y = \log_e 2$

7. Consider the two statements :

$(S1): (p \to q) \vee (\sim q \to p)$ is a tautology.

$(S2): (p \wedge \sim q) \wedge (\sim p \vee q)$ is a fallacy.

Then :

(1) only (S1) is true.

(2) both (S1) and (S2) are false.

(3) both (S1) and (S2) are true.

(4) only (S2) is true.

Official Ans. by NTA (3)

Sol. $S_1 : (\sim p \vee q) \vee (q \vee p) = (q \vee \sim p) \vee (q \vee p)$

$S_1 = q \vee (\sim p \vee p) = qvt = t =$ tautology

$S_2 : (p \wedge \sim q) \wedge (\sim p \vee q) = (p \wedge \sim q) \wedge \sim (p \wedge \sim q) = C$

$=$ fallacy

8. The domain of the function $\operatorname{cosec}^{-1}\left(\frac{1+x}{x}\right)$ is :

(1) $\left(-1,-\frac{1}{2}\right] \cup (0,\infty)$ (2) $\left[-\frac{1}{2},0\right) \cup [1,\infty)$

(3) $\left(-\frac{1}{2},\infty\right) - \{0\}$ (4) $\left[-\frac{1}{2},\infty\right) - \{0\}$

Official Ans. by NTA (4)

Sol. $\frac{1+x}{x} \in (-\infty,-1] \cup [1,\infty)$

$\frac{1}{x} \in (-\infty,-2] \cup [0,\infty)$

$x \in \left[-\frac{1}{2},0\right) \cup (0,\infty)$

$x \in \left[-\frac{1}{2},\infty\right) - \{0\}$

9. A fair die is tossed until six is obtained on it. Let X be the number of required tosses, then the conditional probability $P(X \geq 5 \mid X > 2)$ is :

(1) $\frac{125}{216}$ (2) $\frac{11}{36}$ (3) $\frac{5}{6}$ (4) $\frac{25}{36}$

Official Ans. by NTA (4)

Sol. $P(x \geq 5 \mid x > 2) = \frac{P(x \geq 5)}{P(x > 2)}$

$$\frac{\left(\frac{5}{6}\right)^4 \cdot \frac{1}{6} + \left(\frac{5}{6}\right)^5 \cdot \frac{1}{6} + \ldots\ldots + \infty}{\left(\frac{5}{6}\right)^2 \cdot \frac{1}{6} + \left(\frac{5}{6}\right)^3 \cdot \frac{1}{6} + \ldots\ldots + \infty}$$

$$\frac{\dfrac{\left(\frac{5}{6}\right)^4 \cdot \frac{1}{6}}{1-\frac{5}{6}}}{\dfrac{\left(\frac{5}{6}\right)^2 \cdot \frac{1}{6}}{1-\frac{5}{6}}} = \left(\frac{5}{6}\right)^2 = \frac{25}{36}$$

10. If $\sum_{r=1}^{50} \tan^{-1}\frac{1}{2r^2} = p$, then the value of tan p is :

(1) $\frac{101}{102}$ (2) $\frac{50}{51}$ (3) 100 (4) $\frac{51}{50}$

Official Ans. by NTA (2)

Sol. $\sum_{r=1}^{50} \tan^{-1}\left(\frac{2}{4r^2}\right) = \sum_{r=1}^{50} \tan^{-1}\left(\frac{(2r+1)-(2r-1)}{1+(2r+1)(2r-1)}\right)$

$\sum_{r=1}^{50} \tan^{-1}(2r+1) - \tan^{-1}(2r-1)$

$\tan^{-1}(101) - \tan^{-1}1 \Rightarrow \tan^{-1}\frac{50}{51}$

11. Two fair dice are thrown. The numbers on them are taken as λ and μ, and a system of linear equations

$x + y + z = 5$
$x + 2y + 3z = \mu$
$x + 3y + \lambda z = 1$

is constructed. If p is the probability that the system has a unique solution and q is the probability that the system has no solution, then :

(1) $p = \frac{1}{6}$ and $q = \frac{1}{36}$ (2) $p = \frac{5}{6}$ and $q = \frac{5}{36}$

(3) $p = \frac{5}{6}$ and $q = \frac{1}{36}$ (4) $p = \frac{1}{6}$ and $q = \frac{5}{36}$

Official Ans. by NTA (2)

Sol. $D \neq 0 \Rightarrow \begin{vmatrix} 1 & 1 & 1 \\ 1 & 2 & 3 \\ 1 & 3 & \lambda \end{vmatrix} \neq 0 \Rightarrow \lambda \neq 5$

For no solution $D = 0 \Rightarrow \lambda = 5$

$D_1 = \begin{vmatrix} 1 & 1 & 5 \\ 1 & 2 & \mu \\ 1 & 3 & 1 \end{vmatrix} \neq 0 \Rightarrow \mu \neq 3$

$p = \frac{5}{6}$

$q = \frac{1}{6} \times \frac{5}{6} = \frac{5}{36}$

Option (2)

12. The locus of the mid points of the chords of the hyperbola $x^2 - y^2 = 4$, which touch the parabola $y^2 = 8x$, is :

(1) $y^3(x-2) = x^2$ (2) $x^3(x-2) = y^2$

(3) $y^2(x-2) = x^3$ (4) $x^2(x-2) = y^3$

Official Ans. by NTA (3)

Sol. $T = S_1$

$xh - yk = h^2 - k^2$

$y = \frac{xh}{k} - \frac{(h^2 - k^2)}{k}$

this touches $y^2 = 8x$ then $c = \frac{a}{m}$

$\left(\frac{k^2 - h^2}{k}\right) = \frac{2k}{h}$

$2y^2 = x(y^2 - x^2)$

$y^2(x-2) = x^3$

13. The value of

$$2\sin\left(\frac{\pi}{8}\right)\sin\left(\frac{2\pi}{8}\right)\sin\left(\frac{3\pi}{8}\right)\sin\left(\frac{5\pi}{8}\right)\sin\left(\frac{6\pi}{8}\right)\sin\left(\frac{7\pi}{8}\right)$$

is :

(1) $\frac{1}{4\sqrt{2}}$ (2) $\frac{1}{4}$

(3) $\frac{1}{8}$ (4) $\frac{1}{8\sqrt{2}}$

Official Ans. by NTA (3)

Sol. $2\sin\left(\frac{\pi}{8}\right)\sin\left(\frac{2\pi}{8}\right)\sin\left(\frac{3\pi}{8}\right)\sin\left(\frac{5\pi}{8}\right)\sin\left(\frac{6\pi}{8}\right)\sin\left(\frac{7\pi}{8}\right)$

$2\sin^2\frac{\pi}{8}\sin^2\frac{2\pi}{8}\sin^2\frac{3\pi}{8}$

$\sin^2\frac{\pi}{8}\sin^2\frac{3\pi}{8}$

$\sin^2\frac{\pi}{8}\cos^2\frac{\pi}{8}$

$\frac{1}{4}\sin^2\left(\frac{\pi}{4}\right) = \frac{1}{8}$

14. If $(\sqrt{3} + i)^{100} = 2^{99}(p + iq)$, then p and q are roots of the equation :

(1) $x^2 - (\sqrt{3} - 1)x - \sqrt{3} = 0$

(2) $x^2 + (\sqrt{3} + 1)x + \sqrt{3} = 0$

(3) $x^2 + (\sqrt{3} - 1)x - \sqrt{3} = 0$

(4) $x^2 - (\sqrt{3} + 1)x + \sqrt{3} = 0$

Official Ans. by NTA (1)

Sol. $(2e^{i\pi/6})^{100} = 2^{99}(p + iq)$

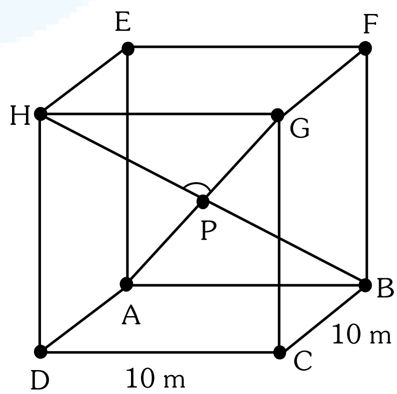
$2^{100}\left(\cos\frac{50\pi}{3} + i\sin\frac{50\pi}{3}\right) = 2^{99}(p + iq)$

$p + iq = 2\left(\cos\frac{2\pi}{3} + i\sin\frac{2\pi}{3}\right)$

$p = -1, q = \sqrt{3}$

$x^2 - (\sqrt{3} - 1)x - \sqrt{3} = 0.$

15. A hall has a square floor of dimension 10m × 10m (see the figure) and vertical walls. If the angle GPH between the diagonals AG and BH is $\cos^{-1}\frac{1}{5}$, then the height of the hall (in meters) is :



(1) 5 (2) $2\sqrt{10}$

(3) $5\sqrt{3}$ (4) $5\sqrt{2}$

Official Ans. by NTA (4)

Sol. $A(\hat{j})$. $B(10\hat{i})$

H $(h\hat{j} + 10\hat{k})$

G $(10\hat{i} + h\hat{j} + 10\hat{k})$

$\overrightarrow{AG} = 10\hat{i} + h\hat{j} + 10\hat{k}$

$\overrightarrow{BH} = -10\hat{i} + h\hat{j} + 10\hat{k}$

$\cos\theta = \dfrac{\overrightarrow{AG}\,\overrightarrow{BH}}{|\overrightarrow{AG}||\overrightarrow{BH}|}$

$\dfrac{1}{5} = \dfrac{h^2}{h^2+200}$

$4h^2 = 200 \Rightarrow h = 5\sqrt{2}$

16. Let P be the plane passing through the point (1,2,3) and the line of intersection of the planes $\vec{r}\cdot(\hat{i}+\hat{j}+4\hat{k}) = 16$ and $\vec{r}\cdot(-\hat{i}+\hat{j}+\hat{k}) = 6$. Then which of the following points does **NOT** lie on P ?

(1) (3, 3, 2) (2) (6, –6, 2)

(3) (4, 2, 2) (4) (–8, 8, 6)

Official Ans. by NTA (3)

Sol. $(x+y+4z-16)+\lambda(-x+y+z-6)=0$

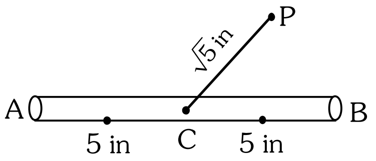
Passes through (1,2,3)

$-1+\lambda(-2) \Rightarrow \lambda = -\dfrac{1}{2}$

$2(x+y+4z-16)-(-x+y+z-6)=0$

$3x + y + 7z - 26 = 0$

17. A 10 inches long pencil AB with mid point C and a small eraser P are placed on the horizontal top of a table such that $PC = \sqrt{5}$ inches and $\angle PCB = \tan^{-1}(2)$. The acute angle through which the pencil must be rotated about C so that the perpendicular distance between eraser and pencil becomes exactly 1 inch is :

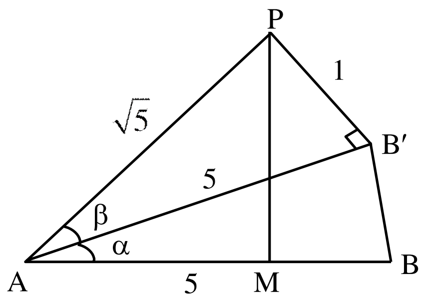


(1) $\tan^{-1}\left(\dfrac{3}{4}\right)$ (2) $\tan^{-1}(1)$

(3) $\tan^{-1}\left(\dfrac{4}{3}\right)$ (4) $\tan^{-1}\left(\dfrac{1}{2}\right)$

Official Ans. by NTA (1)

Sol.



From figure.

$\sin\beta = \dfrac{1}{\sqrt{5}}$

$\therefore \tan\beta = \dfrac{1}{2}$

$\tan(\alpha+\beta) = 2$

$\dfrac{\tan\alpha + \tan\beta}{1-\tan\alpha\cdot\tan\beta} = 2$

$\dfrac{\tan\alpha + \frac{1}{2}}{1-\tan\alpha\left(\frac{1}{2}\right)} = 2$

$\tan\alpha = \dfrac{3}{4}$

$\boxed{\alpha = \tan^{1}\left(\dfrac{3}{4}\right)}$

18. The value of $\displaystyle\int_{-\frac{\pi}{2}}^{\frac{\pi}{2}}\left(\frac{1+\sin^2 x}{1+\pi^{\sin x}}\right)dx$ is

(1) $\dfrac{\pi}{2}$ (2) $\dfrac{5\pi}{4}$

(3) $\dfrac{3\pi}{4}$ (4) $\dfrac{3\pi}{2}$

Official Ans. by NTA (3)

Sol. $I = \displaystyle\int_0^{\pi/2}\frac{(1+\sin^2 x)}{(1+\pi^{\sin x})} + \frac{\pi^{\sin x}(1+\sin^2 x)}{(1+\pi^{\sin x})}dx$

$I = \displaystyle\int_0^{\pi/2}(1+\sin^2 x)dx$

$I = \dfrac{\pi}{2} + \dfrac{\pi}{2}\cdot\dfrac{1}{2} = \dfrac{3\pi}{4}$

19. A circle C touches the line x = 2y at the point (2,1) and intersects the circle $C_1 : x^2 + y^2 + 2y - 5 = 0$ at two points P and Q such that PQ is a diameter of C_1. Then the diameter of C is :

(1) $7\sqrt{5}$ (2) 15

(3) $\sqrt{285}$ (4) $4\sqrt{15}$

Official Ans. by NTA (1)

Sol. $(x-2)$

$^2 + (y-1)^2 + \lambda(x-2y) = 0$

$C : x^2 + y^2 + x(\lambda - 4) + y(-2 - 2\lambda) + 5 = 0$

$C_1 : x^2 + y^2 + 2y - 5 = 0$

$S_1 - S_2 = 0$ (Equation of PQ)

$(\lambda - 4)x - (2\lambda + 4)y + 10 = 0$ Passes through $(0,-1)$

$\Rightarrow \lambda = -7$

$C : x^2 + y^2 - 11x + 12y + 5 = 0$

$= \dfrac{\sqrt{245}}{4}$

Diometer $= 7\sqrt{5}$

20. $\lim\limits_{x\to 2}\left(\sum\limits_{n=1}^{9}\dfrac{x}{n(n+1)x^2 + 2(2n+1)x + 4}\right)$ is equal to :

(1) $\dfrac{9}{44}$ (2) $\dfrac{5}{24}$

(3) $\dfrac{1}{5}$ (4) $\dfrac{7}{36}$

Official Ans. by NTA (1)

Sol. $S = \lim\limits_{x\to 2}\sum\limits_{n=1}^{9}\dfrac{x}{n(n+1)x^2 + 2(2n+1)x + 4}$

$S = \sum\limits_{n=1}^{9}\dfrac{2}{4(n^2+3n+2)} = \dfrac{1}{2}\sum\limits_{n=1}^{9}\left(\dfrac{1}{n+1} - \dfrac{1}{n+2}\right)$

$S = \dfrac{1}{2}\left(\dfrac{1}{2} - \dfrac{1}{11}\right) = \dfrac{9}{44}$

SECTION-B

1. The sum of all 3-digit numbers less than or equal to 500, that are formed without using the digit "1" and they all are multiple of 11, is ________.

Official Ans. by NTA (7744)

So1. 209, 220, 231, ..., 495

Sum $= \dfrac{27}{2}(209 + 495) = 9504$

Number containing 1 at unit place
2 3 1
3 4 1
4 5 1

Number containing 1 at 10^{th} place
3 1 9
4 1 8

Required = 9501 – (231 + 341 + 451 + 319 + 418)

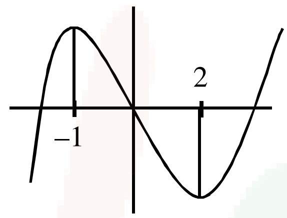
7744

2. Let a and b respectively be the points of local maximum and local minimum of the function $f(x) = 2x^3 - 3x^2 - 12x$. If A is the total area of the region bounded by $y = f(x)$, the x-axis and the lines $x = a$ and $x = b$, then 4A is equal to ________.

Official Ans. by NTA (114)

Sol. $f'(x) = 6x^2 - 6x - 12 = 6(x-2)(x+1)$

Point = (2,–20) & (–1,7)



$A = \int_{-1}^{0}(2x^3 - 3x^2 - 12x)dx + \int_{0}^{2}(12x + 3x^2 - 2x^3)dx$

$A = \left(\dfrac{x^4}{2} - x^3 - 6x^2\right)_{-1}^{0} + \left(6x^2 + x^3 - \dfrac{x^4}{2}\right)_{0}^{2}$

4A = 114

3. If the projection of the vector $\hat{i} + 2\hat{j} + \hat{k}$ on the sum of the two vectors $2\hat{i} + 4\hat{j} - 5\hat{k}$ and $-\lambda\hat{i} + 2\hat{j} + 3\hat{k}$ is 1, then λ is equal to ______.

Official Ans. by NTA (5)

Sol. $\vec{a} = \hat{i} + 2\hat{j} + \hat{k}$

$\vec{b} = (2-\lambda)\hat{i} + 6\hat{j} - 2\hat{k}$

$\dfrac{\vec{a}\cdot\vec{b}}{|\vec{b}|} = 1,\ \vec{a}\cdot\vec{b} = 12 - \lambda$

$(\vec{a}\cdot\vec{b}) = |\vec{b}|^2$

$\lambda^2 - 24\lambda + 144 = \lambda^2 - 4\lambda + 4 + 40$

$20\lambda = 100 \Rightarrow \lambda = 5.$

4. Let $a_1, a_2, \ldots, a_{10}$ be an AP with common difference –3 and $b_1, b_2, \ldots, b_{10}$ be a GP with common ratio 2. Let $c_k = a_k + b_k$, $k = 1, 2, \ldots, 10$. If $c_2 = 12$ and $c_3 = 13$, then $\sum\limits_{k=1}^{10} c_k$ is equal to _____.

Official Ans. by NTA (2021)

Sol. $c_2 = a_2 + b_2 = a_1 - 3 + 2b_1 = 12$

$a_1 + 2b_1 = 15$ ____(1)

$c_3 = a_3 + b_3 = a_1 - 6 + 4b_1 = 13$

$a_1 + 4b_1 = 19$ ____(2)

from (1) & (2) $b_1 = 2, a_1 = 11$

$$\sum_{k=1}^{10} c_k = \sum_{k=1}^{10}(a_k + b_k) = \sum_{k=1}^{10} a_k + \sum_{k=1}^{10} b_k$$

$$= \frac{10}{2}(2\times 11 + 9\times(-3)) + \frac{2(2^{10}-1)}{2-1}$$

$= 5(22 - 27) + 2(1023)$
$= 2046 - 25 = 2021$

5. Let Q be the foot of the perpendicular from the point P(7,–2,13) on the plane containing the lines $\frac{x+1}{6} = \frac{y-1}{7} = \frac{z-3}{8}$ and $\frac{x-1}{3} = \frac{y-2}{5} = \frac{z-3}{7}$. Then $(PQ)^2$, is equal to ________.

Official Ans. by NTA (96)

Sol. Containing the line $\begin{vmatrix} x+1 & y-1 & z-3 \\ 6 & 7 & 8 \\ 3 & 5 & 7 \end{vmatrix} = 0$

$9(x + 1) - 18(y - 1) + 9(z - 3) = 0$

$x - 2y + z = 0$

$PQ = \left|\frac{7+4+13}{\sqrt{6}}\right| = 4\sqrt{6}$

$PQ^2 = 96$

6. Let $\binom{n}{k}$ denotes ${}^nC_k$ and $\begin{bmatrix} n \\ k \end{bmatrix} = \begin{cases} \binom{n}{k}, & \text{if } 0 \le k \le n \\ 0, & \text{otherwise} \end{cases}$

If $A_k = \sum_{i=0}^{9}\binom{9}{i}\begin{bmatrix} 12 \\ 12-k+i \end{bmatrix} + \sum_{i=0}^{8}\binom{8}{i}\begin{bmatrix} 13 \\ 13-k+i \end{bmatrix}$

and $A_4 - A_3 = 190$ p, then p is equal to :

Official Ans. by NTA (49)

Sol. $A_k = \sum_{i=0}^{9} {}^9C_i\ {}^{12}C_{k-i} + \sum_{i=0}^{8} {}^8C_i\ {}^{13}C_{k-i}$

$A_k = {}^{21}C_k + {}^{21}C_k = 2\cdot {}^{21}C_k$

$A_4 - A_3 = 2\left({}^{21}C_4 - {}^{21}C_3\right) = 2(5985 - 1330)$

$190\text{ p} = 2(5985 - 1330) \Rightarrow p = 49.$

7. Let $\lambda \neq 0$ be in **R**. If α and β are the roots of the equation $x^2 - x + 2\lambda = 0$, and α and γ are the roots of equation $3x^2 - 10x + 27\lambda = 0$, then $\frac{\beta\gamma}{\lambda}$ is equal to _________.

Official Ans. by NTA (18)

Sol. $3\alpha^2 - 10\alpha + 27\lambda = 0$ _____(1)

$\alpha^2 - \alpha + 2\lambda = 0$ _____(2)

(1) – 3(2) gives

$-7\alpha + 21\lambda = 0 \Rightarrow \alpha = 3\lambda$

Put $\alpha = 3\lambda$ in equation (1) we get

$9\lambda^2 - 3\lambda + 2\lambda - 0$

$9\lambda^2 = \lambda \Rightarrow \lambda = \frac{1}{9}$ as $\lambda \neq 0$

Now $\alpha = 3\lambda \Rightarrow \lambda = \frac{1}{3}$

$\alpha + \beta = 1 \Rightarrow \beta = 2/3$

$\alpha + \gamma = \frac{10}{3} \Rightarrow \gamma = 3$

$$\frac{\beta\gamma}{\lambda} = \frac{\frac{2}{3}\times 3}{\frac{1}{9}} = 18$$

8. Let the mean and variance of four numbers 3, 7, x and y(x > y) be 5 and 10 respectively. Then the mean of four numbers 3 + 2x, 7 + 2y, x + y and x – y is ________.

Official Ans. by NTA (12)

Sol. $5 = \frac{3+7+x+y}{4} \Rightarrow x + y = 10$

$\text{Var}(x) = 10 = \frac{3^2+7^2+x^2+y^2}{4} - 25$

$140 = 49 + 9 + x^2 + y^2$

$x^2 + y^2 = 82$

$x + y = 10$

$\Rightarrow (x,y) = (9,1)$

Four numbers are 21,9,10,8

$\text{Mean} = \frac{48}{4} = 12$

9. Let A be a 3 × 3 real matrix.

If $\det(2\text{Adj}(2\ \text{Adj}(\text{Adj}(2A)))) = 2^{41}$, then the value of $\det(A^2)$ equal _____.

Official Ans. by NTA (4)

Sol. adj (2A) = 2^2 adjA

$\Rightarrow$ adj(adj (2A)) = adj(4 adjA) = 16 adj (adj A)

= 16 |A| A

$\Rightarrow$ adj (32 |A| A) = $(32\ |A|)^2$ adj A

$12(32|A|)^2$ |adj A| = $2^3\ (32|A|)^6$ |adj A|

$2^3.2^{30}\ |A|^6\ .\ |A|^2 = 2^{41}$

$|A|^8 = 2^8 \Rightarrow |A| = \pm 2$

$|A|^2 = |A|^2 = 4$

10. The least positive integer n such that

$\frac{(2i)^n}{(1-i)^{n-2}}, i = \sqrt{-1}$ is a positive integer, is ______.

Official Ans. by NTA (6)

Sol. $\frac{(2i)^n}{(1-i)^{n-2}} = \frac{(2i)^n}{(-2i)^{\frac{n-2}{2}}}$

$= \frac{(2i)^{\frac{n+2}{2}}}{(-1)^{\frac{n-2}{2}}} = \frac{2^{\frac{n+2}{2}} i^{\frac{n+2}{2}}}{(-1)^{\frac{n-2}{2}}}$

This is positive integer for n = 6