FINAL JEE–MAIN EXAMINATION – AUGUST, 2021
Held On Thursday 26th August, 2021
TIME: 9:00 AM to 12:00 NOON

SECTION-A

1. The sum of solutions of the equation $\frac{\cos x}{1+\sin x} = |\tan 2x|,\ x \in \left(-\frac{\pi}{2},\frac{\pi}{2}\right) - \left\{\frac{\pi}{4},-\frac{\pi}{4}\right\}$ is :

(1) $-\frac{11\pi}{30}$ (2) $\frac{\pi}{10}$

(3) $-\frac{7\pi}{30}$ (4) $-\frac{\pi}{15}$

Official Ans. by NTA (1)

Sol. $\frac{\cos x}{1+\sin x} = |\tan 2x|$

$\Rightarrow \frac{\cos^2 x/2 - \sin^2 x/2}{(\cos x/2 + \sin x/2)} = |\tan 2x|$

$\Rightarrow \tan^2\left(\frac{\pi}{4}-\frac{x}{2}\right) = \tan^2 2x$

$\Rightarrow 2x = n\pi \pm \left(\frac{\pi}{4}-\frac{x}{2}\right)$

$\Rightarrow x = \frac{-3\pi}{10}, \frac{-\pi}{6}, \frac{\pi}{10}$

or sum $= \frac{-11\pi}{6}$.

2. The mean and standard deviation of 20 observations were calculated as 10 and 2.5 respectively. It was found that by mistake one data value was taken as 25 instead of 35. If α and $\sqrt{\beta}$ are the mean and standard deviation respectively for correct data, then (α, β) is :

(1) (11, 26) (2) (10.5, 25)

(3) (11, 25) (4) (10.5, 26)

Official Ans. by NTA (4)

Sol. Given :

Mean $(\bar{x}) = \frac{\Sigma x_i}{20} = 10$

or $\Sigma x_i = 200$ (incorrect)

or $200 - 25 + 35 = 210 = \Sigma x_i$ (Correct)

Now correct $\bar{x} = \frac{210}{20} = 10.5$

again given S.D = 2.5 (σ)

$\sigma^2 = \frac{\Sigma x_i^2}{20} - (10)^2 = (2.5)^2$

or $\Sigma x_i^2 = 2125$ (incorrect)

or $\Sigma x_i^2 = 2125 - 25^2 + 35^2$

$= 2725$ (Correct)

$\therefore$ correct $\sigma^2 = \frac{2725}{20} - (10.5)^2$

$\sigma^2 = 26$

or $\sigma = 26$

$\therefore \alpha = 10.5, \beta = 26$

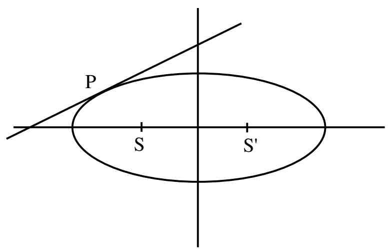
3. On the ellipse $\frac{x^2}{8} + \frac{y^2}{4} = 1$ let P be a point in the second quadrant such that the tangent at P to the ellipse is perpendicular to the line $x + 2y = 0$. Let S and S' be the foci of the ellipse and e be its eccentricity. If A is the area of the triangle SPS' then, the value of $(5 - e^2)$. A is :

(1) 6 (2) 12

(3) 14 (4) 24

Official Ans. by NTA (1)

Sol.



Equation of tangent : $y = 2x + 6$

at P

$\therefore$ P(–8/3, 2/3)

$e = \frac{1}{\sqrt{2}}$

S & S' = (–2, 0) & (2, 0)

Area of Δ SPS' = $\frac{1}{2} \times 4 \times \frac{2}{3}$

A = $\frac{4}{3}$

$\therefore (5 - e^2)A = (5 - \frac{1}{2})\frac{4}{3} = 6$

4. Let y = y(x) be a solution curve of the differential equation $(y + 1)\tan^2 x\, dx + \tan x\, dy + y\, dx = 0$, $x \in \left(0, \frac{\pi}{2}\right)$. If $\lim_{x\to 0+} x y(x) = 1$, then the value of $y\left(\frac{\pi}{4}\right)$ is :

(1) $-\frac{\pi}{4}$ (2) $\frac{\pi}{4} - 1$

(3) $\frac{\pi}{4} + 1$ (4) $\frac{\pi}{4}$

Official Ans. by NTA (4)

Sol. $(y + 1)\tan^2 x\, dx + \tan x\, dy + y\, dx = 0$

or $\frac{dy}{dx} + \frac{\sec^2 x}{\tan x} . y = -\tan x$

IF = $e^{\int \frac{\sec^2 x}{\tan x} dx} = e^{\ln \tan x} = \tan x$

$\therefore y \tan x = -\int \tan^2 x dx$

or y tanx = – tan x + x + C

or $y = -1 + \frac{x}{\tan x} + \frac{C}{\tan x}$

or $\lim_{x\to 0} xy = -x + \frac{x^2}{\tan x} + \frac{Cx}{\tan x} = 1$

or C = 1

y(x) = cot x + x cot x – 1

$y\left(\frac{\pi}{4}\right) = \frac{\pi}{4}$

5. Let A and B be independent events such that P(A) = p, P(B) = 2p. The largest value of p, for which P (exactly one of A, B occurs) = $\frac{5}{9}$, is :

(1) $\frac{1}{3}$ (2) $\frac{2}{9}$

(3) $\frac{4}{9}$ (4) $\frac{5}{12}$

Official Ans. by NTA (4)

Sol. P(Exactly one of A or B)

$= P(A \cap \bar{B}) + P(\bar{A} \cap B) = \frac{5}{9}$

$= P(A)P(\bar{B}) + P(\bar{A})P(B) = \frac{5}{9}$

$\Rightarrow P(A)(1 - P(B)) + (1 - P(A))P(B) = \frac{5}{9}$

$\Rightarrow p(1 - 2p) + (1 - p)\, 2p = \frac{5}{9}$

$\Rightarrow 36p^2 - 27p + 5 = 0$

$\Rightarrow p = \frac{1}{3}$ or $\frac{5}{12}$

$p_{max} = \frac{5}{12}$

6. Let $\theta \in \left(0, \frac{\pi}{2}\right)$. If the system of linear equations

$(1 + \cos^2\theta)x + \sin^2\theta\, y + 4\sin 3\theta\, z = 0$

$\cos^2\theta\, x + (1 + \sin^2\theta)\, y + 4\sin 3\theta\, z = 0$

$\cos^2\theta\, x + \sin^2\theta\, y + (1 + 4\sin 3\theta)\, z = 0$

has a non–trivial solution, then the value of θ is :

(1) $\frac{4\pi}{9}$ (2) $\frac{7\pi}{18}$ (3) $\frac{\pi}{18}$ (4) $\frac{5\pi}{18}$

Official Ans. by NTA (2)

Sol. Case–I

$$\begin{vmatrix} 1+\cos^2\theta & \sin^2\theta & 4\sin 3\theta \\ \cos^2\theta & 1+\sin^2\theta & 4\sin 3\theta \\ \cos^2\theta & \sin^2\theta & 1+4\sin 3\theta \end{vmatrix} = 0$$

$C_1 \to C_1 + C_2$

$$\begin{vmatrix} 2 & \sin^2\theta & 4\sin 3\theta \\ 2 & 1+\sin^2\theta & 4\sin 3\theta \\ 1 & \sin^2\theta & 1+4\sin 3\theta \end{vmatrix} = 0$$

$R_1 \to R_1 - R_2,\ R_2 \to R_2 - R_3$

$$\begin{vmatrix} 0 & -1 & 0 \\ 1 & 1 & -1 \\ 1 & \sin^2\theta & 1+4\sin^3\theta \end{vmatrix} = 0$$

or 4 sin 3θ = – 2

$\sin 3\theta = -\frac{1}{2}$

$\theta = \frac{7\pi}{18}$

7. Let $f(x) = \cos\left(2\tan^{-1}\sin\left(\cot^{-1}\sqrt{\frac{1-x}{x}}\right)\right)$,

$0 < x < 1$. Then :

(1) $(1-x)^2 f'(x) - 2(f(x))^2 = 0$

(2) $(1+x)^2 f'(x) + 2(f(x))^2 = 0$

(3) $(1-x)^2 f'(x) + 2(f(x))^2 = 0$

(4) $(1+x)^2 f'(x) - 2(f(x))^2 = 0$

Official Ans. by NTA (3)

Sol. $f(x) = \cos\left(2\tan^{-1}\sin\left(\cot^{-1}\sqrt{\frac{1-x}{x}}\right)\right)$

$\cot^{-1}\sqrt{\frac{1-x}{x}} = \sin^{-1}\sqrt{x}$

or $f(x) = \cos(2\tan^{-1}\sqrt{x})$

$= \cos\tan^{-1}\left(\frac{2\sqrt{x}}{1-x}\right)$

$f(x) = \frac{1-x}{1+x}$

Now $f'(x) = \frac{-2}{(1+x)^2}$

or $f'(x)(1-x)^2 = -2\left(\frac{1-x}{1+x}\right)^2$

or $(1-x)^2 f'(x) + 2(f(x))^2 = 0$.

8. The sum of the series

$\frac{1}{x+1} + \frac{2}{x^2+1} + \frac{2^2}{x^4+1} + ... + \frac{2^{100}}{x^{2^{100}}+1}$ when $x = 2$

is :

(1) $1 + \frac{2^{101}}{4^{101}-1}$ (2) $1 + \frac{2^{100}}{4^{101}-1}$

(3) $1 - \frac{2^{100}}{4^{100}-1}$ (4) $1 - \frac{2^{101}}{4^{101}-1}$

Official Ans. by NTA (4)

Allen Ans. (BONUS)

Sol. $S = \frac{1}{x+1} + \frac{2}{x^2+1} + \frac{2^2}{x^4+1} + ... \frac{2^{100}}{x^{2^{100}}+1}$

$S + \frac{1}{1-x} = \frac{1}{1-x} + \frac{1}{x+1} + ... = \frac{2}{1-x^2} + \frac{2}{1+x^2} + ...$

$S + \frac{1}{1-x} = \frac{2^{101}}{1-x^{2^{101}}}$

Put $x = 2$

$S = 1 - \frac{2^{101}}{2^{2^{101}}-1}$

Not in option (BONUS)

9. If ${}^{20}C_r$ is the co-efficient of x^r in the expansion of $(1+x)^{20}$, then the value of $\sum_{r=0}^{20} r^2\, {}^{20}C_r$ is equal to :

(1) 420×2^{19} (2) 380×2^{19}

(3) 380×2^{18} (4) 420×2^{18}

Official Ans. by NTA (4)

Sol. $\sum_{r=0}^{20} r^2 . {}^{20}C_r$

$\sum(4(r-1)+r).{}^{20}C_r$

$\sum r(r-1).\frac{20\times19}{r(r-1)}.{}^{18}C_r + r.\frac{20}{r}.\sum {}^{19}C_{r-1}$

$\Rightarrow 20 \times 19.2^{18} + 20.2^{19}$

$\Rightarrow 420 \times 2^{18}$

10. Out of all the patients in a hospital 89% are found to be suffering from heart ailment and 98% are suffering from lungs infection. If K% of them are suffering from both ailments, then K can not belong to the set :

(1) {80, 83, 86, 89} (2) {84, 86, 88, 90}

(3) {79, 81, 83, 85} (4) {84, 87, 90, 93}

Official Ans. by NTA (3)

Sol. $n(A \cup B) \geq n(A) + n(B) - n(A \cap B)$

$100 \geq 89 + 98 - n(A \cup B)$

$n(A \cup B) \geq 87$

$87 \leq n(A \cup B) \leq 89$

Option (3)

11. The equation $\arg\left(\frac{z-1}{z+1}\right)=\frac{\pi}{4}$ represents a circle with:

(1) centre at (0, –1) and radius $\sqrt{2}$

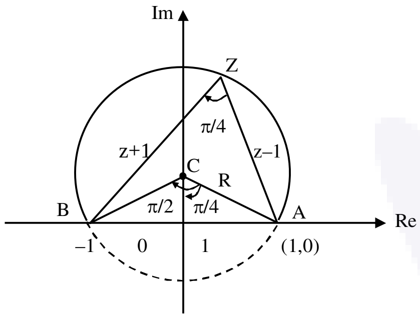
(2) centre at (0, 1) and radius $\sqrt{2}$

(3) centre at (0,0) and radius $\sqrt{2}$

(4) centre at (0,1) and radius 2

Official Ans. by NTA (2)

Sol.



In ΔOAC

$$\sin\left(\frac{\pi}{4}\right)=\frac{1}{AC}$$

$$\Rightarrow AC=\sqrt{2}$$

Also, $\tan\frac{\pi}{4}=\frac{OA}{OC}=\frac{1}{OC}$

$$\Rightarrow OC = 1$$

$\therefore$ centre (0, 1) ; Radius = $\sqrt{2}$

12. Let $\vec{a}=\hat{i}+\hat{j}+\hat{k}$ and $\vec{b}=\hat{j}-\hat{k}$. If $\vec{c}$ is a vector such that $\vec{a}\times\vec{c}=\vec{b}$ and $\vec{a}.\vec{c}=3$, then $\vec{a}.(\vec{b}\times\vec{c})$ is equal to :

(1) –2 (2) –6 (3) 6 (4) 2

Official Ans. by NTA (1)

Sol. $|\vec{a}|=\sqrt{3};\ \vec{a}.\vec{c}=3;\ \vec{a}\times\vec{b}=-2\hat{i}+\hat{j}+\hat{k},\ \vec{a}\times\vec{c}=\vec{b}$

Cross with $\vec{a}$.

$$\vec{a}\times(\vec{a}\times\vec{c})=\vec{a}\times\vec{b}$$

$$\Rightarrow (\vec{a}.\vec{c})\vec{a}-a^2\vec{c}=\vec{a}\times\vec{b}$$

$$\Rightarrow 3\vec{a}-3\vec{c}=-2\hat{i}+\hat{j}+\hat{k}$$

$$\Rightarrow 3\hat{i}+3\hat{j}+3\hat{k}-3\vec{c}=-2\hat{i}+\hat{j}+\hat{k}$$

$$\Rightarrow \vec{c}=\frac{5\hat{i}}{3}+\frac{2\hat{j}}{3}+\frac{2\hat{k}}{3}$$

$$\therefore\ \vec{a}.(\vec{b}\times\vec{c})=(\vec{a}\times\vec{b}).\vec{c}=\frac{-10}{3}+\frac{2}{3}+\frac{2}{3}=-2$$

13. If a line along a chord of the circle $4x^2+4y^2+120x+675=0$, passes through the point (–30, 0) and is tangent to the parabola $y^2=30x$, then the length of this chord is :

(1) 5 (2) 7 (3) $5\sqrt{3}$ (4) $3\sqrt{5}$

Official Ans. by NTA (4)

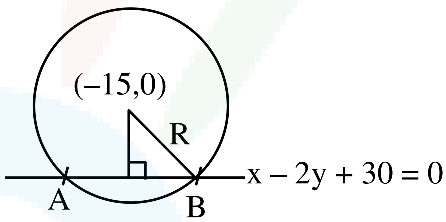
Sol. Equation of tangent to $y^2 = 30\,x$

$$y = mx + \frac{30}{4m}$$

Pass thru (–30, 0) : $a=-30m+\frac{30}{4m}\Rightarrow m^2=1/4$

$$\Rightarrow m=\frac{1}{2}\ \text{or}\ m=-\frac{1}{2}$$

At $m=\frac{1}{2}$: $y=\frac{x}{2}+15\Rightarrow x-2y+30=0$



$$P=\frac{15}{\sqrt{5}}$$

$$\ell_{AB}=2\sqrt{R^2-P^2}=2\sqrt{\frac{225}{4}-\frac{225}{5}}$$

$$\Rightarrow \ell_{AB}=30.\sqrt{\frac{1}{20}}=\frac{15}{\sqrt{5}}=3\sqrt{5}$$

14. The value of $\int_{-1/\sqrt{2}}^{1/\sqrt{2}}\left(\left(\frac{x+1}{x-1}\right)^2+\left(\frac{x-1}{x+1}\right)^2-2\right)^{1/2}dx$ is:

(1) $\log_e 4$ (2) $\log_e 16$

(3) $2\log_e 16$ (4) $4\log_e(3+2\sqrt{2})$

Official Ans. by NTA (2)

Sol. $$I=\int_{-1/\sqrt{2}}^{1/\sqrt{2}}\left(\left(\frac{x+1}{x-1}-\frac{x-1}{x+1}\right)^2\right)^{1/2}dx$$

$$I=\int_{-1/\sqrt{2}}^{1/\sqrt{2}}\left|\frac{4x}{x^2-1}\right|dx\Rightarrow I=2.4\int_{0}^{1/\sqrt{2}}\left|\frac{x}{x^2-1}\right|dx$$

$$\Rightarrow I=-4\int_{0}^{1/\sqrt{2}}\frac{2x}{x^2-1}dx\Rightarrow I=-4\ln\left|x^2-1\right|_0^{1/\sqrt{2}}$$

$$\Rightarrow I = 4\ln 2 \Rightarrow I = \ln 16$$

15. A plane P contains the line

$x + 2y + 3z + 1 = 0 = x - y - z - 6$,

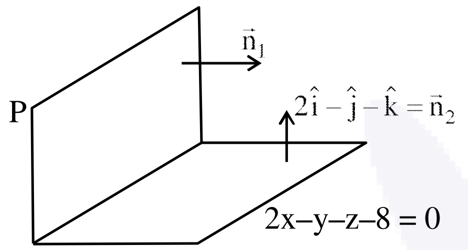
and is perpendicular to the plane $-2x + y + z + 8 = 0$. Then which of the following points lies on P ?

(1) (–1, 1, 2) (2) (0, 1, 1)

(3) (1, 0, 1) (4) (2, –1, 1)

Official Ans. by NTA (2)

Sol. Equation of plane P can be assumed as



$P : x + 2y + 3z + 1 + \lambda (x - y - z - 6) = 0$

$\Rightarrow P : (1 + \lambda)x + (2 - \lambda)y + (3 - \lambda)z + 1 - 6\lambda = 0$

$\Rightarrow \vec{n}_1 = (1+\lambda)\hat{i} + (2-\lambda)\hat{j} + (3-\lambda)\hat{k}$

$\therefore \vec{n}_1 . \vec{n}_2 = 0$

$\Rightarrow 2(1 + \lambda) - (2 - \lambda) - (3 - \lambda) = 0$

$\Rightarrow 2 + 2\lambda - 2 + \lambda - 3 + \lambda = 0 \Rightarrow \lambda = \frac{3}{4}$

$\Rightarrow P : \frac{7x}{4} + \frac{5}{4}y + \frac{9z}{4} - \frac{14}{4} = 0$

$\Rightarrow 7x + 5y + 9z = 14$

(0, 1, 1) lies on P

16. If $A = \begin{pmatrix} \frac{1}{\sqrt{5}} & \frac{2}{\sqrt{5}} \\ \frac{-2}{\sqrt{5}} & \frac{1}{\sqrt{5}} \end{pmatrix}$, $B = \begin{pmatrix} 1 & 0 \\ i & 1 \end{pmatrix}$, $i = \sqrt{-1}$, and

$Q = A^TBA$, then the inverse of the matrix $A\,Q^{2021}\,A^T$ is equal to :

(1) $\begin{pmatrix} \frac{1}{\sqrt{5}} & -2021 \\ 2021 & \frac{1}{\sqrt{5}} \end{pmatrix}$ (2) $\begin{pmatrix} 1 & 0 \\ -2021i & 1 \end{pmatrix}$

(3) $\begin{pmatrix} 1 & 0 \\ 2021i & 1 \end{pmatrix}$ (4) $\begin{pmatrix} 1 & -2021i \\ 0 & 1 \end{pmatrix}$

Official Ans. by NTA (2)

Sol. $AA^T = \begin{pmatrix} \frac{1}{5} & \frac{2}{\sqrt{5}} \\ \frac{-2}{\sqrt{5}} & \frac{1}{\sqrt{5}} \end{pmatrix} \begin{pmatrix} \frac{1}{\sqrt{5}} & \frac{-2}{\sqrt{5}} \\ \frac{2}{\sqrt{5}} & \frac{1}{\sqrt{5}} \end{pmatrix}$

$AA^T = \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix} = I$

$Q^2 = A^TBA\,A^TBA = A^TBIBA$

$\Rightarrow Q^2 = A^T B^2 A$

$Q^3 = A^TB^2AA^TBA \Rightarrow Q^3 = A^TB^3A$

Similarly : $Q^{2021} = A^T B^{2021}\,A$ ...... (1)

Now $B^2 = \begin{pmatrix} 1 & 0 \\ i & 1 \end{pmatrix}\begin{pmatrix} 1 & 0 \\ i & 1 \end{pmatrix} = \begin{pmatrix} 1 & 0 \\ 2i & 1 \end{pmatrix}$

$B^3 = \begin{pmatrix} 1 & 0 \\ 2i & 1 \end{pmatrix}\begin{pmatrix} 1 & 0 \\ i & 1 \end{pmatrix} \Rightarrow B^3 = \begin{pmatrix} 1 & 0 \\ 3i & 1 \end{pmatrix}$

Similarly $B^{2021} = \begin{pmatrix} 1 & 0 \\ 2021i & 1 \end{pmatrix}$

$\therefore AQ^{2021}\,A^T = AA^T\,B^{2021}\,AA^T = IB^{2021}I$

$\Rightarrow AQ^{2021}\,A^T = B^{2021} = \begin{pmatrix} 1 & 0 \\ 2021i & 1 \end{pmatrix}$

$\therefore (AQ^{2021}\,A^T)^{-1} = \begin{pmatrix} 1 & 0 \\ 2021i & 1 \end{pmatrix}^{-1} = \begin{pmatrix} 1 & 0 \\ -2021i & 1 \end{pmatrix}$

17. If the sum of an infinite GP a, ar, ar^2, ar^3,... is 15 and the sum of the squares of its each term is 150, then the sum of ar^2, ar^4, ar^6, ... is :

(1) $\frac{5}{2}$ (2) $\frac{1}{2}$

(3) $\frac{25}{2}$ (4) $\frac{9}{2}$

Official Ans. by NTA (2)

Sol. Sum of infinite terms :

$\frac{a}{1-r} = 15$ ....(i)

Series formed by square of terms:

a^2, a^2r^2, a^2r^4, a^2r^6 .....

Sum $= \frac{a^2}{1-r^2} = 150$

$\Rightarrow \frac{a}{1-r} . \frac{a}{1+r} = 150 \Rightarrow 15 . \frac{a}{1+r} = 150$

$\Rightarrow \frac{a}{1+r} = 10$ .... (ii) by (i) and (ii) a = 12 ; $r = \frac{1}{5}$

Now series : ar^2, ar^4, ar^6

$$\text{Sum} = \frac{ar^2}{1-r^2} = \frac{12.\left(\frac{1}{25}\right)}{1-\frac{1}{25}} = \frac{1}{2}$$

18. The value of $\lim\limits_{n\to\infty} \frac{1}{n}\sum\limits_{r=0}^{2n-1} \frac{n^2}{n^2+4r^2}$ is:

(1) $\frac{1}{2}\tan^{-1}(2)$ (2) $\frac{1}{2}\tan^{-1}(4)$

(3) $\tan^{-1}(4)$ (4) $\frac{1}{4}\tan^{-1}(4)$

Official Ans. by NTA (2)

Sol. $L = \lim\limits_{n\to\infty} \frac{1}{n}.\sum\limits_{r=0}^{2n-1} \frac{1}{1+4\left(\frac{r}{n}\right)^2}$

$$\Rightarrow L = \int_0^2 \frac{1}{1+4x^2}dx$$

$$\Rightarrow L = \frac{1}{2}\tan^{-1}(2x)\Big|_0^2 \Rightarrow L = \frac{1}{2}\tan^{-1}4$$

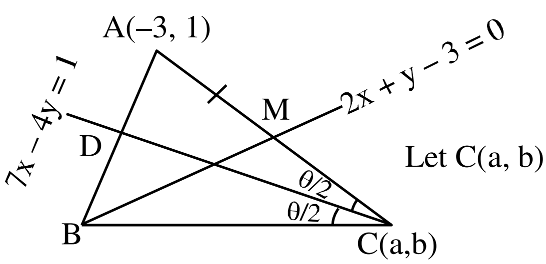
19. Let ABC be a triangle with A(–3, 1) and $\angle ACB = \theta$, $0 < \theta < \frac{\pi}{2}$. If the equation of the median through B is 2x + y – 3 = 0 and the equation of angle bisector of C is 7x – 4y – 1 = 0, then tanθ is equal to:

(1) $\frac{1}{2}$ (2) $\frac{3}{4}$

(3) $\frac{4}{3}$ (4) 2

Official Ans. by NTA (3)

Sol.



Let C(a, b)

$\therefore$ $M\left(\frac{a-3}{2}, \frac{b+1}{2}\right)$ lies on 2x + y – 3 = 0

$\Rightarrow$ 2a + b = 11 .... (i)

$\because$ C lies on 7x – 4y = 1

$\Rightarrow$ 7a – 4b = 1 ......(ii)

$\therefore$ by (i) and (ii) : a = 3, b = 5

$\Rightarrow$ C(3, 5)

$\therefore$ $m_{AC} = 2/3$

Also, $m_{CD} = 7/4$

$$\Rightarrow \tan\frac{\theta}{2} = \left|\frac{\frac{2}{3}-\frac{4}{4}}{1+\frac{14}{12}}\right| \Rightarrow \tan\frac{\theta}{2} = \frac{1}{2}$$

$$\Rightarrow \tan\theta = \frac{2.\frac{1}{2}}{1-\frac{1}{4}} = \frac{4}{3}$$

20. If the truth value of the Boolean expression $((p \vee q) \wedge (q \rightarrow r) \wedge (\sim r)) \rightarrow (p \wedge q)$ is false, then the truth values of the statements p, q, r respectively can be:

(1) T F T (2) F F T

(3) T F F (4) F T F

Official Ans. by NTA (3)

Sol.

p	q	r	$p\vee q$ (a)	$q\rightarrow r$ (b)	$a\wedge b$	$\sim r$	$a\wedge b\wedge(\sim r)$ (c)	$p\wedge q$ (d)	$c\rightarrow d$
T	F	T	T	T	T	F	F	F	T
F	F	T	F	T	F	F	F	F	T
T	F	F	T	T	T	T	T	F	F
F	T	F	T	F	F	T	F	F	T

SECTION-B

1. Let $z = \frac{1-i\sqrt{3}}{2}$, $i = \sqrt{-1}$. Then the value of

$$21+\left(z+\frac{1}{z}\right)^3+\left(z^2+\frac{1}{z^2}\right)^3+\left(z^3+\frac{1}{z^3}\right)^3+...+\left(z^{21}+\frac{1}{z^{21}}\right)^3$$

is ________.

Official Ans. by NTA (13)

Sol. $Z = \dfrac{1-\sqrt{3}i}{2} = e^{-i\frac{\pi}{3}}$

$$z^r + \frac{1}{z^r} = 2\cos\left(-\frac{\pi}{3}\right)r = 2\cos\frac{r\pi}{3}$$

$$\Rightarrow 21 + \sum_{r=1}^{21}\left(z^r + \frac{1}{z^r}\right)^3 = 8\left(\cos^3\frac{r\pi}{3}\right) = 2\left(\cos r\pi + 3\cos\frac{r\pi}{3}\right)$$

$$\Rightarrow 21 + \left(z + \frac{1}{2}\right)^3 + \left(z^2 + \frac{1}{z^2}\right)^3 + ...\left(z^{21} + \frac{1}{z^{21}}\right)^3$$

$$= 21 + \sum_{r=1}^{21}\left(z^r + \frac{1}{z^r}\right)^3$$

$$= 21 + \sum_{r=1}^{21}\left(2\cos r\pi + 6\cos\frac{r\pi}{3}\right)$$

$= 21 - 2 - 6$

$= 13$

2. The sum of all integral values of k ($k \neq 0$) for which the equation $\dfrac{2}{x-1} - \dfrac{1}{x-2} = \dfrac{2}{k}$ in x has no real roots, is ________.

Official Ans. by NTA (66)

Sol. $\dfrac{2}{x-1} - \dfrac{1}{x-2} = \dfrac{2}{k}$

$x \in R - \{1, 2\}$

$\Rightarrow k(2x - 4 - x + 1) = 2(x^2 - 3x + 2)$

$\Rightarrow k(x - 3) = 2(x^2 - 3x + 2)$

for $x \neq 3$, $\quad k = 2\left(x - 3 + \dfrac{2}{x-3} + 3\right)$

$$x - 3 + \frac{2}{x-3} \geq 2\sqrt{2}, \ \forall x > 3$$

$$\& \ x - 3 + \frac{2}{x-3} \leq -2\sqrt{2}, \ \forall x < -3$$

$$\Rightarrow 2\left(x - 3 + \frac{2}{x-3} + 3\right) \in \left(-\infty, 6 - 4\sqrt{2}\right] \cup \left[6 + 4\sqrt{2}, \infty\right)$$

for no real roots

$k \in (6 - 4\sqrt{2}, 6 + 4\sqrt{2}) - \{0\}$

Integral $\ k \in \{1, 2..... 11\}$

Sum of k = 66

3. Let the line L be the projection of the line

$$\frac{x-1}{2} = \frac{y-3}{1} = \frac{z-4}{2}$$

in the plane $x - 2y - z = 3$. If d is the distance of the point (0, 0, 6) from L, then d^2 is equal to ________.

Official Ans. by NTA (26)

Sol. $L_1 : \dfrac{x-1}{2} = \dfrac{y-3}{1} = \dfrac{z-4}{2}$

for foot of $\perp r$ of (1, 3, 4) on $x - 2y - z - 3 = 0$

$(1 + t) - 2(3 - 2t) - (4 - t) - 3 = 0$

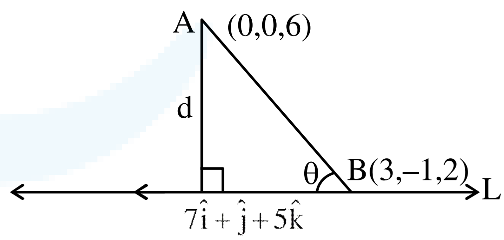
$\Rightarrow t = 2$

So foot of $\perp r \triangleq (3, -1, 2)$

& point of intersection of L_1 with plane is (–11, –3, –8)

dr's of L is <14, 2, 10>

$\cong$ <7, 1, 5>



$$d = AB\sin\theta = \left|\frac{\begin{vmatrix} \hat{i} & \hat{j} & \hat{k} \\ 3 & -1 & -4 \\ 7 & 1 & 5 \end{vmatrix}}{\sqrt{7^2 + 1^2 + 5^2}}\right|$$

$$\Rightarrow d^2 = \frac{1^2 + (43)^2 + (10)^2}{49 + 1 + 25} = 26$$

4. If ${}^1P_1 + 2\cdot{}^2P_2 + 3\cdot{}^3P_3 + ... + 15\cdot{}^{15}P_{15} = {}^qP_r - s, 0 \leq s \leq 1$, then ${}^{q+s}C_{r-s}$ is equal to ________.

Official Ans. by NTA (136)

Sol. ${}^{1}P_1 + 2\cdot{}^{2}P_2 + 3\cdot{}^{3}P_3 + \ldots + 15\cdot{}^{15}P_{15}$

$= 1! + 2.2! + 3.3! + \ldots.15 \times 15!$

$= \sum_{r=1}^{15}(r+1-1)r!$

$= \sum_{r=1}^{15}(r+1)! - (r)!$

$= 16! - 1$

$= {}^{16}P_{16} - 1$

$\Rightarrow q = r = 16,\ s = 1$

${}^{q+s}C_{r-s} = {}^{17}C_{15} = 136$

5. A wire of length 36 m is cut into two pieces, one of the pieces is bent to form a square and the other is bent to form a circle. If the sum of the areas of the two figures is minimum, and the circumference of the circle is k (meter), then $\left(\frac{4}{\pi}+1\right)k$ is equal to _________.

Official Ans. by NTA (36)

Sol. Let x + y = 36

x is perimeter of square and y is perimeter of circle

side of square = x/4

radius of circle = $\frac{y}{2\pi}$

Sum Areas = $\left(\frac{x}{4}\right)^2 + \pi\left(\frac{y}{2\pi}\right)^2$

$= \frac{x^2}{16} + \frac{(36-x)^2}{4\pi}$

For min Area :

$x = \frac{144}{\pi+4}$

$\Rightarrow$ Radius = $y = 36 - \frac{144}{\pi+4}$

$\Rightarrow k = \frac{36\pi}{\pi+4}$

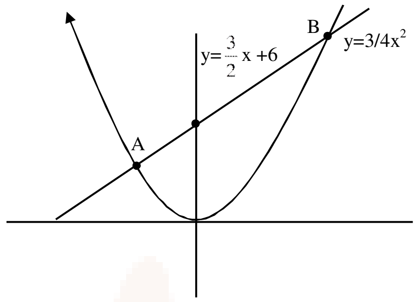
$\left(\frac{4}{\pi}+1\right)k = 36$

6. The area of the region

$S = \{(x, y) : 3x^2 \le 4y \le 6x + 24\}$ is ___________.

Official Ans. by NTA (27)

Sol.



For A & B

$3x^2 = 6x + 24 \Rightarrow x^2 - 2x - 8 = 0$

$\Rightarrow x = -2, 4$

Area = $\int_{-2}^{4}\left(\frac{3}{2}x + 6 - \frac{3}{4}x^2\right)dx$

$= \left[\frac{3x^2}{4} + 6x - \frac{x^3}{4}\right]_{-2}^{4} = 27$

7. The locus of a point, which moves such that the sum of squares of its distances from the points (0, 0), (1, 0), (0, 1) (1, 1) is 18 units, is a circle of diameter d. Then d^2 is equal to _________.

Official Ans. by NTA (16)

Sol. Let P(x, y)

$x^2 + y^2 + x^2 + (y-1)^2 + (x-1)^2 + y^2 + (x-1)^2 + (y-1)^2$;

$\Rightarrow 4(x^2 + y^2) - 4y - 4x = 14$

$\Rightarrow x^2 + y^2 - x - y - \frac{7}{2} = 0$

$d = 2\sqrt{\frac{1}{4} + \frac{1}{4} + \frac{7}{2}}$

$\Rightarrow d^2 = 16$

8. If y = y(x) is an implicit function of x such that $\log_e(x + y) = 4xy$, then $\frac{d^2y}{dx^2}$ at x = 0 is equal to ________.

Official Ans. by NTA (40)

Sol. $\ln(x + y) = 4xy$ (At $x = 0$, $y = 1$)

$x + y = e^{4xy}$

$\Rightarrow 1+\frac{dy}{dx}=e^{4xy}\left(4x\frac{dy}{dx}+4y\right)$

At $x = 0$ $\boxed{\frac{dy}{dx}=3}$

$\frac{d^2y}{dx^2}=e^{4xy}\left(4x\frac{dy}{dx}+4y\right)^2+e^{4xy}\left(4x\frac{d^2y}{dx^2}+4y\right)$

At $x = 0$, $\frac{d^2y}{dx^2}=e^0(4)^2+e^0(24)$

$\Rightarrow \frac{d^2y}{dx^2}=40$

9. The number of three-digit even numbers, formed by the digits 0, 1, 3, 4, 6, 7 if the repetition of digits is not allowed, is _________.

Official Ans. by NTA (52)

Sol. (i) When '0' is at unit place

↑	↑	0
5	4	

Number of numbers = 20

(ii) When 4 or 6 are at unit place

OX		4,6
4	× 4	↑ 2

Number of numbers = 32

So number of numbers = 52

10. Let $a, b \in \mathbf{R}$, $b \neq 0$, Define a function

$$f(x)=\begin{cases} a\sin\frac{\pi}{2}(x-1), & \text{for } x\le 0 \\ \frac{\tan 2x-\sin 2x}{bx^3}, & \text{for } x>0. \end{cases}$$

If f is continuous at $x = 0$, then $10 - ab$ is equal to _________.

Official Ans. by NTA (14)

Sol. $$f(x)=\begin{cases} a\sin\frac{\pi}{2}(x-1), & x\le 0 \\ \frac{\tan 2x-\sin 2x}{bx^3}, & x>0 \end{cases}$$

For continuity at '0'

$\lim_{x\to 0^+} f(x)=f(0)$

$\Rightarrow \lim_{x\to 0^+}\frac{\tan 2x-\sin 2x}{bx^3}=-a$

$\Rightarrow \lim_{x\to 0^+}\frac{\frac{8x^3}{3}+\frac{8x^3}{3!}}{bx^3}=-a$

$\Rightarrow 8\left(\frac{1}{3}+\frac{1}{3!}\right)=-ab$

$\Rightarrow 4 = -ab$

$\Rightarrow 10 - ab = 14$