FINAL JEE–MAIN EXAMINATION – AUGUST, 2021

Held On Wednesday 01 September, 2021

TIME: 3:00 PM to 06:00 PM

SECTION-A

1. Let f : **R** → **R** be a continuous function. Then

$$\lim_{x\to\frac{\pi}{4}} \frac{\frac{\pi}{4}\int_{2}^{\sec^2 x} f(x)dx}{x^2-\frac{\pi^2}{16}}$$ is equal to :

(1) $f(2)$ (2) $2f(2)$

(3) $2f(\sqrt{2})$ (4) $4f(2)$

Official Ans. by NTA (2)

Sol. $$\lim_{x\to\frac{\pi}{4}} \frac{\frac{\pi}{4}\int_{2}^{\sec^2 x} f(x)dx}{x^2-\frac{\pi^2}{16}}$$

$$\lim_{x\to\frac{\pi}{4}} \frac{\pi}{4}.\frac{\left[f(\sec^2 x).2\sec x.\sec x\tan x\right]}{2x}$$

$$\lim_{x\to\frac{\pi}{4}} \frac{\pi}{4}f(\sec^2 x).\sec^3 x.\frac{\sin x}{x}$$

$$\frac{\pi}{4}f(2).(\sqrt{2})^3.\frac{1}{\sqrt{2}}\times\frac{4}{\pi}$$

$\Rightarrow 2f(2)$

2. $\cos^{-1}(\cos(-5)) + \sin^{-1}(\sin(6)) - \tan^{-1}(\tan(12))$ is equal to :

(The inverse trigonometric functions take the principal values)

(1) $3\pi - 11$ (2) $4\pi - 9$

(3) $4\pi - 11$ (4) $3\pi + 1$

Official Ans. by NTA (3)

Sol. $\cos^{-1}(\cos(-5)) + \sin^{-1}(\sin(6)) - \tan^{-1}(\tan(12))$

$\Rightarrow (2\pi - 5) + (6 - 2\pi) - (12 - 4\pi)$

$\Rightarrow 4\pi - 11.$

3. Consider the system of linear equations

$-x + y + 2z = 0$

$3x - ay + 5z = 1$

$2x - 2y - az = 7$

Let S_1 be the set of all a ∈ **R** for which the system is inconsistent and S_2 be the set of all a ∈ **R** for which the system has infinitely many solutions. If $n(S_1)$ and $n(S_2)$ denote the number of elements in S_1 and S_2 respectively, then

(1) $n(S_1) = 2, n(S_2) = 2$ (2) $n(S_1) = 1, n(S_2) = 0$

(3) $n(S_1) = 2, n(S_2) = 0$ (4) $n(S_1) = 0, n(S_2) = 2$

Official Ans. by NTA (3)

Sol. $$\Delta = \begin{vmatrix} -1 & 1 & 2 \\ 3 & -a & 5 \\ 2 & -2 & -a \end{vmatrix}$$

$= -1(a^2 + 10) -1(-3a-10) + 2(-6 + 2a)$

$= -a^2 - 10 + 3a + 10 - 12 + 4a$

$\Delta = -a^2 + 7a - 12$

$\Delta = -[a^2 - 7a + 12]$

$\Delta = -[(a-3)(a-4)]$

$$\Delta_1 = \begin{vmatrix} 0 & 1 & 2 \\ 1 & -a & 5 \\ 7 & -2 & -a \end{vmatrix}$$

$= 0 - 1(-a - 35) + 2(-2 + 7a)$

$\Rightarrow a + 35 - 4 + 14a$

$15a + 31$

Now $\Delta_1 = 15a + 31$

For inconsistent $\Delta = 0$ ∴ $a = 3, a = 4$

and for a = 3 and 4 $\Delta_1 \neq 0$

$n(S_1) = 2$

For infinite solution : $\Delta = 0$

and $\Delta_1 = \Delta_2 = \Delta_3 = 0$

Not possible

∴ $n(S_2) = 0$

4. Let the acute angle bisector of the two planes $x - 2y - 2z + 1 = 0$ and $2x - 3y - 6z + 1 = 0$ be the plane P. Then which of the following points lies on P?

(1) $\left(3,1,-\frac{1}{2}\right)$ (2) $\left(-2,0,-\frac{1}{2}\right)$

(3) (0, 2, –4) (4) (4, 0, – 2)

Official Ans. by NTA (2)

Sol. $P_1 : x - 2y - 2z + 1 = 0$

$P_2 : 2x - 3y - 6z + 1 = 0$

$$\left|\frac{x-2y-2z+1}{\sqrt{1+4+4}}\right| = \left|\frac{2x-3y-6z+1}{\sqrt{2^2+3^2+6^2}}\right|$$

$$\frac{x-2y-2z+1}{3} = \pm\frac{2x-3y-6z+1}{7}$$

Since $a_1a_2 + b_1b_2 + c_1c_2 = 20 > 0$

$\therefore$ Negative sign will give

acute bisector

$7x - 14y - 14z + 7 = -[6x - 9y - 18z + 3]$

$\Rightarrow 13x - 23y - 32z + 10 = 0$

$\left(-2, 0, -\frac{1}{2}\right)$ satisfy it $\therefore$ Ans (2)

5. Which of the following is equivalent to the Boolean expression $p \wedge \sim q$?

(1) $\sim (q \to p)$ (2) $\sim p \to \sim q$

(3) $\sim (p \to \sim q)$ (4) $\sim (p \to q)$

Official Ans. by NTA (4)

Sol.

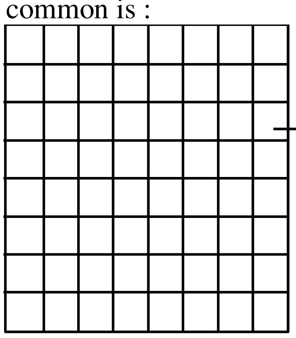
p	q	~p	~q	p – q	~(p → q)	q → p	~(q → p)
T	T	F	F	T	F	T	F
T	F	F	T	F	T	T	F
F	T	T	F	T	F	F	T
F	F	T	T	T	F	T	F

$p \wedge \sim q$	$\sim p \to \sim q$	$p \to \sim q$	$\sim(p \to \sim q)$
F	T	F	T
T	T	T	F
F	F	T	F
F	T	T	F

$p \wedge \sim q \equiv \sim (p \to q)$

Option (4)

6. Two squares are chosen at random on a chessboard (see figure). The probability that they have a side in common is :



→ 64 square

(1) $\frac{2}{7}$ (2) $\frac{1}{18}$

(3) $\frac{1}{7}$ (4) $\frac{1}{9}$

Official Ans. by NTA (2)

Sol. Total ways of choosing square = ${}^{64}C_2$

$$= \frac{64 \times 63}{2 \times 1} = 32 \times 63$$

ways of choosing two squares having common side $= 2\ (7 \times 8) = 112$

Required probability $= \frac{112}{32 \times 63} = \frac{16}{32 \times 9} = \frac{1}{18}$.

Ans. (2)

7. If $y = y(x)$ is the solution curve of the differential equation $x^2 dy + \left(y - \frac{1}{x}\right)dx = 0$; $x > 0$ and $y(1) = 1$, then $y\left(\frac{1}{2}\right)$ is equal to :

(1) $\frac{3}{2} - \frac{1}{\sqrt{e}}$ (2) $3 + \frac{1}{\sqrt{e}}$

(3) $3 + e$ (4) $3 - e$

Official Ans. by NTA (4)

Sol. $x^2 dy + \left(y - \frac{1}{x}\right)dx = 0 : x > 0, y(1) = 1$

$$x^2 dy + \frac{(xy-1)}{x}dx = 0$$

$$x^2 dy = \frac{(xy-1)}{x}dx$$

$$\frac{dy}{dx} = \frac{1-xy}{x^3}$$

$$\frac{dy}{dx} = \frac{1}{x^3} - \frac{y}{x^2}$$

$$\frac{dy}{dx} = \frac{1}{x^2}.y = \frac{1}{x^3}$$

If $e^{\int \frac{1}{x^2}dx} = e^{-\frac{1}{x}}$

$$ye^{-\frac{1}{x}} = \int \frac{1}{x^3}.e^{-\frac{1}{x}}$$

$$ye^{-\frac{1}{x}} = e^{-x}\left(1 + \frac{1}{x}\right) + C$$

$$1.e^{-1} = e^{-1}(2) + C$$

$$C = -e^{-1} = -\frac{1}{e}$$

$$ye^{-\frac{1}{x}} = e^{-\frac{1}{x}}\left(1 + \frac{1}{x}\right) - \frac{1}{e}$$

$$y\left(\frac{1}{2}\right) = 3 - \frac{1}{e} \times e^2$$

$$y\left(\frac{1}{2}\right) = 3 - e$$

8. If n is the number of solutions of the equation

$$2\cos x\left(4\sin\left(\frac{\pi}{4}+x\right)\sin\left(\frac{\pi}{4}-x\right)-1\right)=1, x\in[0,\pi]$$

and S is the sum of all these solutions, then the ordered pair (n, S) is :

(1) $(3, 13\pi/9)$ (2) $(2, 2\pi/3)$

(3) $(2, 8\pi/9)$ (4) $(3, 5\pi/3)$

Official Ans. by NTA (1)

Sol. $$2\cos x\left(4\sin\left(\frac{\pi}{4}+x\right)\sin\left(\frac{\pi}{4}-x\right)-1\right)=1$$

$$2\cos x\left(4\left(\sin^2\frac{\pi}{4}-\sin^2 x\right)-1\right)=1$$

$$2\cos x\left(4\left(\frac{1}{2}-\sin^2 x\right)-1\right)=1$$

$$2\cos x\left(2-4\sin^2 x-1\right)=1$$

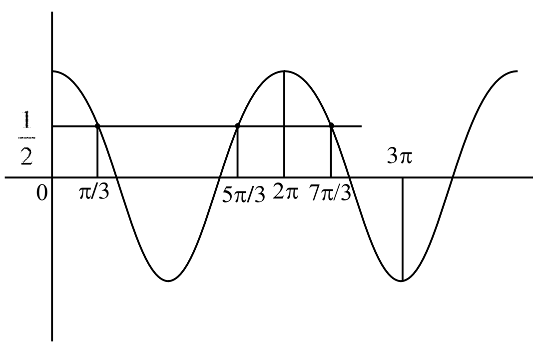
$$2\cos x\left(1-4\sin^2 x\right)=1$$

$$2\cos x\left(4\cos^2 x-3\right)=1$$

$$4\cos^3 x-3\cos x=\frac{1}{2}$$

$$\cos 3x=\frac{1}{2}$$

$x\in[0,\pi]\ \therefore\ 3x\in[0,3\pi]$



9. The function $f(x) = x^3 - 6x^2 + ax + b$ is such that $f(2) = f(4) = 0$. Consider two statements.

(S1) there exists $x_1, x_2 \in (2, 4)$, $x_1 < x_2$, such that $f'(x_1) = -1$ and $f'(x_2) = 0$.

(S2) there exists $x_3, x_4 \in (2, 4)$, $x_3 < x_4$, such that f is decreasing in $(2, x_4)$, increasing in $(x_4, 4)$ and $2f'(x_3) = \sqrt{3}f(x_4)$.

Then

(1) both (S1) and (S2) are true

(2) (S1) is false and (S2) is true

(3) both (S1) and (S2) are false

(4) (S1) is true and (S2) is false

Official Ans. by NTA (1)

Sol. $f(x) = x^3 - 6x^2 + ax + b$

$f(2) = 8 - 24 + 2a + b = 0$

$2a + b = 16 \ldots(1)$

$f(4) = 64 - 96 + 4a + b = 0$

$4a + b = 32 \ \ldots(2)$

Solving (1) and (2)

$a = 8, b = 0$

$$\boxed{f(x) = x^3 - 6x^2 + 8x}$$

$f(x) = x^3 - 6x^2 + 8x$

$f'(x) = 3x^2 - 12x + 8$

$f''(x) = 6x - 12$

$\Rightarrow f'(x)$ is $\uparrow$ for $x > 2$, and $f'(x)$ is $\downarrow$ for $x < 2$

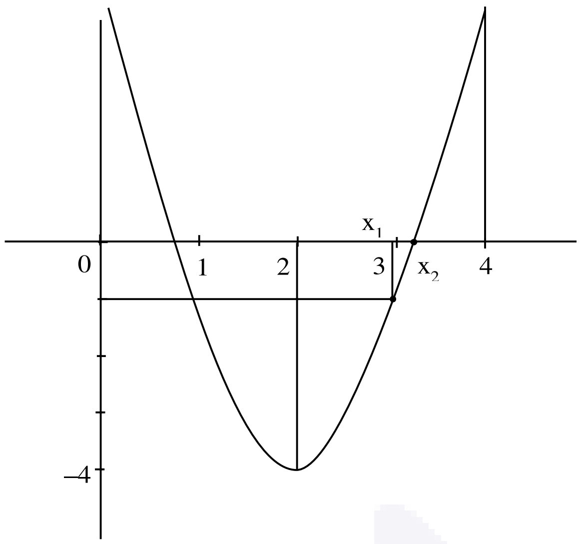
$f'(2) = 12 - 24 + 8 = -4$

$f'(4) = 48 - 48 + 8 = 8$

$f'(x) = 3x^2 - 12x + 8$

vertex $(2, -4)$

$f'(2) = -4$, $f'(4) = 8$, $f'(3) = 27 - 36 + 8$



$f'(x_1) = -1$, then $x_1 = 3$

$f'(x_2) = 0$

Again $\quad f'(x) < 0$ for $x \in (2, x_4)$

$\quad f'(x) > 0$ for $x \in (x_4, 4)$

$x_4 \in (3, 4)$

$f(x) = x^3 - 6x^2 + 8x$

$f(3) = 27 - 54 + 24 = -3$

$f(4) = 64 - 96 + 32 = 0$

For $x_4(3, 4)$

$f(x_4) < -3\sqrt{3}$

and $f'(x_3) > -4$

$2f'(x_3) > -8$

So, $2f'(x_3) = \sqrt{3}\ f(x_4)$

Correct Ans. (1)

10. Let $J_{n,m} = \int_0^{\frac{1}{2}} \frac{x^n}{x^m - 1}dx, \quad \forall\ n > m$ and $n, m \in N$.

Consider a matrix $A = [a_{ij}]_{3\times 3}$ where

$a_{ij} = \begin{cases} J_{6+i,3} - J_{i+3,3}, & i \le j \\ 0\ , & i > j \end{cases}$. Then $\left|\text{adj}A^{-1}\right|$ is :

(1) $(15)^2 \times 2^{42}$ (2) $(15)^2 \times 2^{34}$

(3) $(105)^2 \times 2^{38}$ (4) $(105)^2 \times 2^{36}$

Official Ans. by NTA (3)

Sol. $\begin{bmatrix} \overset{\surd}{a_{11}} & \overset{\surd}{a_{12}} & \overset{\surd}{a_{13}} \\ a_{21} & a_{22} & a_{23} \\ a_{31} & a_{32} & a_{33} \end{bmatrix}$

$J_{6+i,3} - J_{i+3,3}\ ; i \le j$

$\Rightarrow \int_0^{\frac{1}{2}} \frac{x^{6+i}}{x^3-1} - \int_0^{\frac{1}{2}} \frac{x^{i+3}}{x^3-1}$

$\Rightarrow \int_0^{1/2} \frac{x^{i+3}\left(x^3-1\right)}{x^3-1}$

$\Rightarrow \frac{x^{3+i+1}}{3+i+1} = \left(\frac{x^{4+i}}{4+i}\right)_0^{1/2}$

$a_{ij} = j_{6+i,3} - j_{i+3,3} = \frac{\left(\frac{1}{2}\right)^{4+i}}{4+i}$

$a_{11} = \frac{\left(\frac{1}{2}\right)^5}{5} = \frac{1}{5.2^5}$

$a_{12} = \frac{1}{5.2^5}$

$a_{13} = \frac{1}{5.2^5}$

$a_{22} = \frac{1}{6.2^6}$

$a_{23} = \frac{1}{6.2^6}$

$a_{33} = \frac{1}{7.2^7}$

$A = \begin{bmatrix} \frac{1}{5.2^5} & \frac{1}{5.2^5} & \frac{1}{5.2^5} \\ 0 & \frac{1}{6.2^6} & \frac{1}{6.2^6} \\ 0 & 0 & \frac{1}{7.2^7} \end{bmatrix}$

$|A| = \frac{1}{5.2^5}\left[\frac{1}{6.2^6} \times \frac{1}{7.2^7}\right]$

$|A| = \frac{1}{210.2^{18}}$

$\left|\text{adj}A^{-1}\right| = \left|A^{-1}\right|^{n-1} = \left|A^{-1}\right|^2 = \frac{1}{\left(|A|\right)^2}$

$\Rightarrow (210.2^{18})^2$

$(105)^2 \times 2^{38}$

11. The area, enclosed by the curves y = sin x + cos x and $y = |\cos x - \sin x|$ and the lines $x = 0,\ x = \frac{\pi}{2}$, is :

(1) $2\sqrt{2}\left(\sqrt{2}-1\right)$ (2) $2\left(\sqrt{2}+1\right)$

(3) $4\left(\sqrt{2}-1\right)$ (4) $2\sqrt{2}\left(\sqrt{2}+1\right)$

Official Ans. by NTA (1)

Sol. $A = \int_0^{\pi/2}\left((\sin x + \cos x) - |\cos x - \sin x|\right)dx$

$A = \int_0^{\pi/2}\left((\sin x + \cos x) - (\cos x - \sin x)\right)dx$

$+\int_{\pi/4}^{\pi/2}\left((\sin x + \cos x) - (\sin x - \cos x)\right)dx$

$A = 2\int_0^{\pi/2}\sin x dx + 2\int_{\pi/4}^{\pi/2}\cos x dx$

$A = -2\left(\frac{1}{\sqrt{2}}-1\right)+2\left(1-\frac{1}{\sqrt{2}}\right)$

$A = 4 - 2\sqrt{2} = 2\sqrt{2}\,(\sqrt{2} - 1)$

Option (1)

12. The distance of line 3y – 2z – 1 = 0 = 3x – z + 4 from the point (2, – 1, 6) is :

(1) $\sqrt{26}$ (2) $2\sqrt{5}$

(3) $2\sqrt{6}$ (4) $4\sqrt{2}$

Official Ans. by NTA (3)

Sol. $3y - 2z - 1 = 0 = 3x - z + 4$

$3y - 2z - 1 = 0$ D.R′s ⇒ (0, 3, –2)

$3x - z + 4 = 0$ D.R′s ⇒ (3, –1, 0)

Let DR′s of given line are a, b, c

Now 3b – 2c = 0 & 3a – c = 0

∴ 6a = 3b = 2c

a : b : c = 3 : 6 : 9

Any pt on line

3K – 1, 6K + 1, 9K + 1

Now 3(3K – 1) + 6(6K + 1)1 + 9(9K + 1) =0

$\Rightarrow K = \frac{1}{3}$

Point on line ⇒ (0, 3, 4)

Given point (2, –1, 6)

$\Rightarrow$ Distance = $\sqrt{4+16+4} = 2\sqrt{6}$

Option (3)

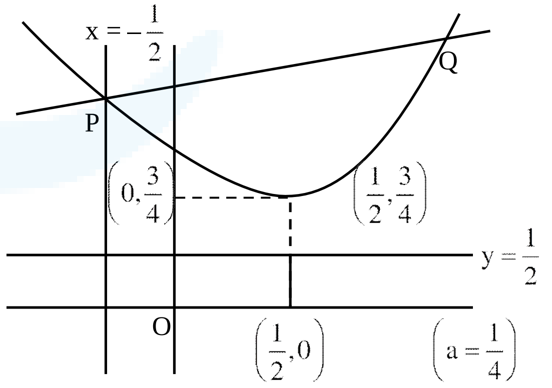
13. Consider the parabola with vertex $\left(\frac{1}{2},\frac{3}{4}\right)$ and the directrix $y = \frac{1}{2}$. Let P be the point where the parabola meets the line $x = -\frac{1}{2}$. If the normal to the parabola at P intersects the parabola again at the point Q, then $(PQ)^2$ is equal to :

(1) $\frac{75}{8}$ (2) $\frac{125}{16}$

(3) $\frac{25}{2}$ (4) $\frac{15}{2}$

Official Ans. by NTA (2)

Sol.



$\left(y-\frac{3}{4}\right)=\left(x-\frac{1}{2}\right)^2$ ... (1)

For $x = -\frac{1}{2}$

$y - \frac{3}{4} = 1 \Rightarrow y = \frac{7}{4} \Rightarrow P\left(-\frac{1}{2},\frac{7}{4}\right)$

Now $y' = 2\left(x-\frac{1}{2}\right)$ At $x = -\frac{1}{2}$

$\Rightarrow m_T = -2, m_N = \frac{1}{2}$

Equation of Normal is

$$y-\frac{7}{4}=\frac{1}{2}\left(x+\frac{1}{2}\right)$$

$$y = \frac{x}{2}+2$$

Now put y in equation (1)

$$\frac{x}{2}+2-\frac{3}{4}=\left(x-\frac{1}{2}\right)^2$$

$$\Rightarrow x = 2 \ \& \ -\frac{1}{2}$$

$\Rightarrow Q(2, 3)$

Now $(PQ)^2 = \frac{125}{16}$

Option (2)

14. The numbers of pairs (a, b) of real numbers, such that whenever α is a root of the equation $x^2 + ax + b = 0$, $\alpha^2 - 2$ is also a root of this equation, is :

(1) 6 (2) 2

(3) 4 (4) 8

Official Ans. by NTA (1)

Sol. Consider the equation $x^2 + ax + b = 0$

If has two roots (not necessarily real α & β)

Either $\alpha = \beta$ or $\alpha \neq \beta$

Case (1) If $\alpha = \beta$, then it is repeated root. Given that $\alpha^2 - 2$ is also a root

So, $\alpha = \alpha^2 - 2 \Rightarrow (\alpha + 1)(\alpha - 2) = 0$

$\Rightarrow \alpha = -1$ or $\alpha = 2$

When $\alpha = -1$ then (a, b) = (2, 1)

$\alpha = 2$ then (a, b) = (–4, 4)

Case (2) If $\alpha \neq \beta$ Then

(I) $\alpha = \alpha^2 - 2$ and $\beta = \beta^2 - 2$

Here $(\alpha, \beta) = (2, -1)$ or $(-1, 2)$

Hence $(a, b) = (-(\alpha + \beta), \alpha\beta)$

= (–1, –2)

(II) $\alpha = \beta^2 - 2$ and $\beta = \alpha^2 - 2$

Then $\alpha - \beta = \beta^2 - \alpha^2 = (\beta - \alpha)(\beta + \alpha)$

Since $\alpha \neq \beta$ we get $\alpha + \beta = \beta^2 + \alpha^2 - 4$

$\alpha + \beta = (\alpha + \beta)^2 - 2\alpha\beta - 4$

Thus $-1 = 1 - 2\alpha\beta - 4$ which implies

$\alpha\beta = -1$ Therefore $(a, b) = (-(\alpha + \beta), \alpha\beta)$

= (1, –1)

(III) $\alpha = \alpha^2 - 2 = \beta^2 - 2$ and $\alpha \neq \beta$

$\Rightarrow \alpha = -\beta$

Thus $\alpha = 2, \beta = -2$

$\alpha = -1, \beta = 1$

Therefore (a, b) = (0, –4) & (0, –1)

(IV) $\beta = \alpha^2 - 2 = \beta^2 - 2$ and $\alpha \neq \beta$ is same as (III)

Therefore we get 6 pairs of (a, b)

Which are (2, 1), (–4, 4), (–1, –2), (1, –1) (0, –4)

Option (1)

15. Let $S_n = 1 \cdot (n-1) + 2 \cdot (n-2) + 3 \cdot (n-3) + \dots + (n-1) \cdot 1$, $n \geq 4$.

The sum $\sum_{n=4}^{\infty}\left(\frac{2S_n}{n!}-\frac{1}{(n-2)!}\right)$ is equal to :

(1) $\frac{e-1}{3}$ (2) $\frac{e-2}{6}$

(3) $\frac{e}{3}$ (4) $\frac{e}{6}$

Official Ans. by NTA (1)

Sol. Let $T_r = r(n-r)$

$T_r = nr - r^2$

$$\Rightarrow S_n = \sum_{r=1}^{n} T_r = \sum_{r=1}^{n}(nr - r^2)$$

$$S_n = \frac{n.(n)(n+1)}{2} - \frac{n(n+1)(2n+1)}{6}$$

$$S_n = \frac{n(n-1)(n+1)}{6}$$

Now $\displaystyle\sum_{r=4}^{\infty}\left(\frac{2S_n}{n!} - \frac{1}{(n-2)!}\right)$

$$= \sum_{r=4}^{\infty}\left(2.\frac{n(n-1)(n+1)}{6.n(n-1)(n-2)!} - \frac{1}{(n-2)!}\right)$$

$$= \sum_{r=4}^{\infty}\left(\frac{1}{3}\left(\frac{n-2+3}{(n-2)!}\right) - \frac{1}{(n-2)!}\right)$$

$$= \sum_{r=4}^{\infty}\frac{1}{3}\cdot\frac{1}{(n-3)!} = \frac{1}{3}(e-1)$$

Option (1)

16. Let P_1, P_2, ….., P_{15} be 15 points on a circle. The number of distinct triangles formed by points P_i, P_j, P_k such that $i + j + k \neq 15$, is :

(1) 12 (2) 419 (3) 443 (4) 455

Official Ans. by NTA (3)

Sol. Total Number of Triangles = ${}^{15}C_3$

$i + j + k = 15$ (Given)

5 Cases

i	j	k
1	2	12
1	3	11
1	4	10
1	5	9
1	6	8

4 Cases

i	j	k
2	3	10
2	4	9
2	5	8
2	6	7

3 Cases

i	j	k
3	4	8
3	5	7

1 Cases

i	j	k
4	5	6

Number of Possible triangles using the vertices P_i, P_j, P_k such that $i + j + k \neq 15$ is equal to ${}^{15}C_3 - 12 = 443$

Option (3)

17. The range of the function,

$$f(x) = \log_{\sqrt{5}}\left(3 + \cos\left(\frac{3\pi}{4} + x\right) + \cos\left(\frac{\pi}{4} + x\right) + \cos\left(\frac{\pi}{4} - x\right) - \cos\left(\frac{3\pi}{4} - x\right)\right)$$

is :

(1) $(0, \sqrt{5})$ (2) $[-2, 2]$

(3) $\left[\frac{1}{\sqrt{5}}, \sqrt{5}\right]$ (4) $[0, 2]$

Official Ans. by NTA (4)

Sol. $f(x) = \log_{\sqrt{5}}$

$$\left(3 + \cos\left(\frac{3\pi}{4} + x\right) + \cos\left(\frac{\pi}{4} + x\right) + \cos\left(\frac{\pi}{4} - x\right) - \cos\left(\frac{3\pi}{4} - x\right)\right)$$

$$f(x) = \log_{\sqrt{5}}\left[3 + 2\cos\left(\frac{\pi}{4}\right)\cos(x) - 2\sin\left(\frac{3\pi}{4}\right)\sin(x)\right]$$

$$f(x) = \log_{\sqrt{5}}\left[3 + \sqrt{2}(\cos x - \sin x)\right]$$

Since $-\sqrt{2} \leq \cos x - \sin x \leq \sqrt{2}$

$$\Rightarrow \log_{\sqrt{5}}\left[3 + \sqrt{2}(-\sqrt{2})\right] \leq f(x) \leq \log_{\sqrt{5}}\left[3 + \sqrt{2}(\sqrt{2})\right]$$

$$\Rightarrow \log_{\sqrt{5}}(1) \leq f(x) \leq \log_{\sqrt{5}}(5)$$

So Range of f(x) is [0, 2]

Option (4)

18. Let $a_1, a_2, \ldots\ldots, a_{21}$ be an AP such that $\displaystyle\sum_{n=1}^{20}\frac{1}{a_n a_{n+1}} = \frac{4}{9}$.

If the sum of this AP is 189, then $a_6 a_{16}$ is equal to :

(1) 57 (2) 72

(3) 48 (4) 36

Official Ans. by NTA (2)

Sol. $\displaystyle\sum_{n=1}^{20}\frac{1}{a_n a_{n+1}} = \sum_{n=1}^{20}\frac{1}{a_n(a_n + d)}$

$$= \frac{1}{d}\sum_{n=1}^{20}\left(\frac{1}{a_n} - \frac{1}{a_n + d}\right)$$

$$\Rightarrow \frac{1}{d}\left(\frac{1}{a_1} - \frac{1}{a_{21}}\right) = \frac{4}{9} \text{ (Given)}$$

$$\Rightarrow \frac{1}{d}\left(\frac{a_{21} - a_1}{a_1 a_{21}}\right) = \frac{4}{9}$$

$$\Rightarrow \frac{1}{d}\left(\frac{a_1 + 20d - a_1}{a_1 a_2}\right) = \frac{4}{9} \Rightarrow a_1 a_2 = 45 \text{ .... (1)}$$

Now sum of first 21 terms = $\frac{21}{2}(2a_1 + 20d) = 189$

$\Rightarrow a_1 + 10d = 9$ ... (2)

For equation (1) & (2) we get

$a_1 = 3$ & $d = \frac{3}{5}$

OR

$a_1 = 15$ & $d = -\frac{3}{5}$

So, $a_6.a_{16} = (a_1 + 5d)(a_1 + 15d)$

$\Rightarrow a_6 a_{16} = 72$

Option (2)

19. The function f(x), that satisfies the condition

$f(x) = x + \int_0^{\pi/2} \sin x \cdot \cos y f(y) dy$, is :

(1) $x + \frac{2}{3}(\pi - 2)\sin x$ (2) $x + (\pi + 2)\sin x$

(3) $x + \frac{\pi}{2}\sin x$ (4) $x + (\pi - 2)\sin x$

Official Ans. by NTA (4)

Sol. $f(x) = x + \int_0^{\pi/2} \sin x \cos y f(y) dy$

$f(x) = x + \sin x \underbrace{\int_0^{\pi/2} \cos y f(y) dy}_{K}$

$\Rightarrow f(x) = x + K \sin x$

$\Rightarrow f(y) = y + K \sin y$

Now $K = \int_0^{\pi/2} \cos y (y + K \sin y) dy$

$K = \underbrace{\int_0^{\pi/2} y \cos dy}_{\text{Apply IBP}} + \underbrace{\int_0^{\pi/2} \cos y \sin y dy}_{\text{Put } \sin y = t}$

$K = (y \sin y)_0^{\pi/2} - \int_0^{\pi/2} \sin dy + K\int_0^1 t\, dt$

$\Rightarrow K = \frac{\pi}{2} - 1 + K\left(\frac{1}{2}\right)$

$\Rightarrow K = \pi - 2$

So $f(x) = x + (\pi - 2)\sin x$

Option (4)

20. Let θ be the acute angle between the tangents to the ellipse $\frac{x^2}{9} + \frac{y^2}{1} = 1$ and the circle $x^2 + y^2 = 3$ at their point of intersection in the first quadrant. Then tanθ is equal to :

(1) $\frac{5}{2\sqrt{3}}$ (2) $\frac{2}{\sqrt{3}}$

(3) $\frac{4}{\sqrt{3}}$ (4) 2

Official Ans. by NTA (2)

Sol. The point of intersection of the curves $\frac{x^2}{9} + \frac{y^2}{1} = 1$ and $x^2 + y^2 = 3$ in the first quadrant is

$\left(\frac{3}{2}, \frac{\sqrt{3}}{2}\right)$

Now slope of tangent to the ellipse $\frac{x^2}{9} + \frac{y^2}{1} = 1$ at $\left(\frac{3}{2}, \frac{\sqrt{3}}{2}\right)$ is

$m_1 = -\frac{1}{3\sqrt{3}}$

And slope of tangent to the circle at $\left(\frac{3}{2}, \frac{\sqrt{3}}{2}\right)$ is m_2

$= -\sqrt{3}$

So, if angle between both curves is θ then

$$\tan\theta = \left|\frac{m_1 - m_2}{1 + m_1 m_2}\right| = \left|\frac{-\frac{1}{3\sqrt{3}} + \sqrt{3}}{1 + \left(-\frac{1}{3\sqrt{3}}(-\sqrt{3})\right)}\right|$$

$= \frac{2}{\sqrt{3}}$

Option (2)

SECTION-B

1. Let X be a random variable with distribution.

x	–2	–1	3	4	6
P(X = x)	$\frac{1}{5}$	a	$\frac{1}{3}$	$\frac{1}{5}$	b

If the mean of X is 2.3 and variance of X is σ^2, then $100\,\sigma^2$ is equal to :

Official Ans. by NTA (781)

Sol.

x	–2	–1	3	4	6
P(X = x)	$\frac{1}{5}$	a	$\frac{1}{3}$	$\frac{1}{5}$	b

$\overline{X} = 2.3$

$-a + 6b = \frac{9}{10}$ .... (1)

$\sum P_i = \frac{1}{5} + a + \frac{1}{3} + \frac{1}{5} + b = 1$

$a + b = \frac{4}{15}$ .... (2)

From equation (1) and (2)

$a = \frac{1}{10}, \quad b = \frac{1}{6}$

$\sigma^2 = \Sigma p_i x_i^2 - (\overline{X})^2$

$\frac{1}{5}(4) + a(1) + \frac{1}{3}(9) + \frac{1}{5}(16) + b(36) - (2.3)^2$

$= \frac{4}{5} + a + 3 + \frac{16}{5} + 36b - (2.3)^2$

$= 4 + a + 3 + 36b - (2.3)^2$

$= 7 + a + 36b - (2.3)^2$

$= 7 + \frac{1}{10} + 6 - (2.3)^2$

$= 13 + \frac{1}{10} - \left(\frac{23}{10}\right)^2$

$= \frac{131}{10} - \left(\frac{23}{10}\right)^2$

$= \frac{1310 - (23)^2}{100}$

$= \frac{1310 - 529}{100}$

$\sigma^2 = \frac{781}{100}$

$100\sigma^2 = 781$

2. Let $f(x) = x^6 + 2x^4 + x^3 + 2x + 3$, $x \in \mathbf{R}$. Then the natural number n for which $\lim_{x\to 1} \frac{x^n f(1) - f(x)}{x - 1} = 44$ is________.

Official Ans. by NTA (7)

Sol. $f(n) = x^6 + 2x^4 + x^3 + 2x + 3$

$\lim_{x\to 1} \frac{x^n f(1) - f(x)}{x - 1} = 44$

$\lim_{x\to 1} \frac{9x^n - (x^6 + 2x^4 + x^3 + 2x + 3)}{x - 1} = 44$

$\lim_{x\to 1} \frac{9nx^{n-1} - (6x^5 + 8x^3 + 3x^2 + 2)}{1} = 44$

$\Rightarrow 9n - (19) = 44$

$\Rightarrow 9n = 63$

$\Rightarrow n = 7$

3. If for the complex numbers z satisfying $|z - 2 - 2i| \le 1$, the maximum value of $|3iz + 6|$ is attained at $a + ib$, then a + b is equal to ________.

Official Ans. by NTA (5)

Sol. $|z - 2 - 2i| \le 1$

$|x + iy - 2 - 2i| \le 1$

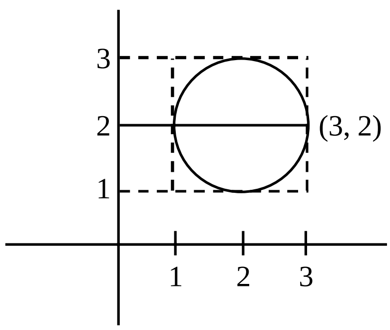
$|(x - 2) + i(y - 2)| \le 1$

$(x - 2)^2 + (y - 2)^2 \le 1$

$|3iz + 6|_{max}$ at a + ib

$|3i| \left|z + \frac{6}{3i}\right|$

$3|z - 2i|_{max}$



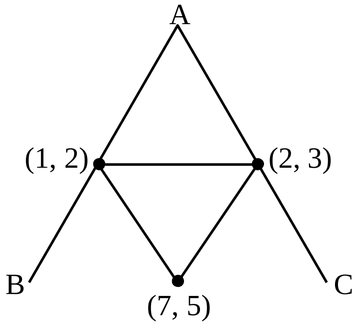
From Figure maximum distance at 3 + 2i

a + ib = 3 + 2i = a + b = 3 + 2 = 5 **Ans.**

4. Let the points of intersections of the lines $x - y + 1 = 0$, $x - 2y + 3 = 0$ and $2x - 5y + 11 = 0$ are the mid points of the sides of a triangle ABC. Then the area of the triangle ABC is ________.

Official Ans. by NTA (6)

Sol. intersection point of give lines are (1, 2), (7, 5), (2,3)



$$\Delta = \frac{1}{2}\begin{vmatrix} 1 & 2 & 1 \\ 7 & 5 & 1 \\ 2 & 3 & 1 \end{vmatrix}$$

$$= \frac{1}{2}[1(5-3) - 2(7-2) + 1(21-10)]$$

$$= \frac{1}{2}[2 - 10 + 11]$$

$$\Delta DEF = \frac{1}{2}(3) = \frac{3}{2}$$

$$\Delta ABC = 4\ \Delta DEF = 4\left(\frac{3}{2}\right) = 6$$

5. Let f(x) be a polynomial of degree 3 such that $f(k) = -\frac{2}{k}$ for k = 2, 3, 4, 5. Then the value of $52 - 10\ f(10)$ is equal to :

Official Ans. by NTA (26)

Sol. $k\ f(k) + 2 = \lambda\ (x-2)\ (x-3)\ (x-4)\ (x-5)$ …(1)

put x = 0

we get $\lambda = \frac{1}{60}$

Now put λ in equation (1)

$$\Rightarrow kf(k) + 2 = \frac{1}{60}\ (x-2)\ (x-3)\ (x-4)\ (x-5)$$

Put x = 10

$$\Rightarrow 10f(10) + 2 = \frac{1}{60}(8)\ (7)\ (6)\ (5)$$

$$\Rightarrow 52 - 10f(10) = 52 - 26 = 26$$

6. All the arrangements, with or without meaning, of the word FARMER are written excluding any word that has two R appearing together. The arrangements are listed serially in the alphabetic order as in the English dictionary. Then the serial number of the word FARMER in this list is ______.

Official Ans. by NTA (77)

Sol. FARMER (6)

A, E, F, M, R, R

A					
E					
F	A	E			
F	A	M			
F	A	R	E		
F	A	R	M	E	R

$$\frac{\lfloor 5}{\lfloor 2} - \lfloor 4 = 60 - 24 = 36$$

$$\frac{\lfloor 3}{\lfloor 2} - \lfloor 2 = 3 - 2 = 1$$

= 1

= 2

= 1

77

7. If the sum of the coefficients in the expansion of $(x + y)^n$ is 4096, then the greatest coefficient in the expansion is _______.

Official Ans. by NTA (924)

Sol. $(x + y)^n \Rightarrow 2^n = 4096$ $\quad 2^{10} = 1024 \times 2$

$\Rightarrow 2^n = 2^{12}$ $\quad 2^{11} = 2048$

$n = 12$ $\quad 2^{12} = \underline{4096}$

$$^{12}C_6 = \frac{12 \times 11 \times 10 \times 9 \times 8 \times 7}{6 \times 5 \times 4 \times 3 \times 2 \times 1}$$

$= 11 \times 3 \times 4 \times 7$

$= 924$

8. Let $\vec{a} = 2\hat{i} - \hat{j} + 2\hat{k}$ and $\vec{b} = \hat{i} + 2\hat{j} - \hat{k}$. Let a vector $\vec{v}$ be in the plane containing $\vec{a}$ and $\vec{b}$. If $\vec{v}$ is perpendicular to the vector $3\hat{i} + 2\hat{j} - \hat{k}$ and its projection on $\vec{a}$ is 19 units, then $|2\vec{v}|^2$ is equal to _____.

Official Ans. by NTA (1494)

Sol. $\vec{a} = 2\hat{i} - \hat{j} + 2\hat{k}$

$\vec{b} = \hat{i} + 2\hat{j} - \hat{k}$

$\vec{c} = 3\hat{i} + 2\hat{j} - \hat{k}$

$\vec{v} = x\vec{a} + y\vec{b}$ $\qquad \vec{v}(3\hat{i} + 2\hat{j} - k) = 0$

$\vec{v}\cdot\hat{a} = 19$

$\vec{v} = \lambda\vec{c} \times (\vec{a} \times \vec{b})$

$\vec{v} = \lambda\left[(\vec{c}.\vec{b})\vec{a} - (\vec{c}\cdot\vec{a})\vec{b}\right]$

$= \lambda[(3+4+1)(2\hat{i} - \hat{j} + 2\hat{k}) - \left(\frac{6-2-2}{2}\right)(\hat{i} + 2\hat{j} + \hat{k})$

$= \lambda[16\hat{i} - 8\hat{j} + 16\hat{k} - 2\hat{i} - 4\hat{j} + 2\hat{k}]$

$\vec{v} = \lambda\left[14\hat{i} - 12\hat{j} + 18\hat{k}\right]$

$\lambda[14\hat{i} - 12\hat{j} + 18\hat{k}] \cdot \frac{(2\hat{i} - \hat{j} + 2\hat{k})}{\sqrt{4+1+4}} = 19$

$\lambda\frac{[28+12+36]}{3} = 19$

$\lambda\left(\frac{76}{3}\right) = 19$

$4\lambda = 3 \Rightarrow \lambda = \frac{3}{4}$

$|2v^2| = \left|2 \times \frac{3}{4}(14\hat{i} - 12\hat{j} + 18\hat{k})\right|^2$

$\frac{9}{4} \times 4(7\hat{i} - 6\hat{j} + 9\hat{k})^2$

$= 9\ (49 + 36 + 81)$

$= 9\ (166)$

$= 1494$

9. Let [t] denote the greatest integer $\leq$ t. The number of points where the function

$$f(x) = [x]\left|x^2 - 1\right| + \sin\left(\frac{\pi}{[x]+3}\right) - [x+1], x \in (-2, 2)$$

is not continuous is ______.

Official Ans. by NTA (2)

Sol. $f(x) = [x]\ |x^2 - 1| + \sin\frac{\pi}{[x+3]} - [x+1]$

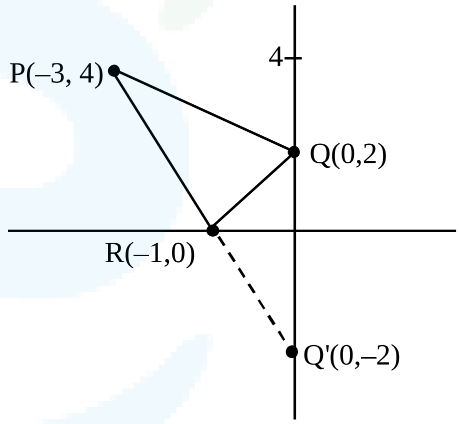
$$f(x) = \begin{cases} 3 - 2x^2, & -2 < x < -1 \\ x^2, & -1 \leq x < 0 \\ \frac{\sqrt{3}}{2} + 1 & 0 \leq x < 1 \\ x^2 + 1 + \frac{1}{\sqrt{2}}, & 1 \leq x < 2 \end{cases}$$

discontinuous at x=0 , 1

10. A man starts walking from the point P(–3,4), touches the x-axis at R, and then turns to reach at the point Q(0, 2). The man is walking at a constant speed. If the man reaches the point Q in the minimum time, then $50\left((PR)^2 + (RQ)^2\right)$ is equal to ______.

Official Ans. by NTA (1250)

Sol.



$50(PR^2 + RQ^2)$

50 (20 + 5)

50(25)

= 1250