MATHEMATICS

31st Jan Shift - 2

SECTION - A

Multiple Choice Questions: This section contains 20 multiple choice questions. Each question has 4 choices (1), (2), (3) and (4), out of which **ONLY ONE** is correct.

Choose the correct answer :

1. $a = \sin^{-1}(\sin 5), b = \cos^{-1}(\cos 5)$ then $a^2 + b^2$ is equal to

(1) $8\pi^2 - 40\pi + 50$ (2) $4\pi^2 + 25$

(3) $8\pi^2 - 50$ (4) $8\pi^2 + 40\pi + 50$

Answer (1)

Sol. $a = \sin^{-1}(\sin 5) = 5 - 2\pi$

and $b = \cos^{-1}(\cos 5) = 2\pi - 5$

$\therefore\ a^2 + b^2 = (5-2\pi)^2 + (2\pi - 5)^2$

$= 8\pi^2 - 40\pi + 50$

2. A coin is biased such that head has two chances than tails, what is the probability of getting 2 heads and 1 tail?

(1) $\frac{1}{29}$ (2) $\frac{2}{29}$

(3) $\frac{1}{9}$ (4) $\frac{4}{9}$

Answer (4)

Sol. Let probability of tail is $\frac{1}{3}$

$\Rightarrow$ Probability of getting head = $\frac{2}{3}$

$\therefore$ Probability of getting 2 heads and 1 tail

$= \left(\frac{2}{3}\times\frac{2}{3}\times\frac{1}{3}\right)\times 3$

$= \frac{4}{27}\times 3$

$= \frac{4}{9}$

3. Let mean and variance of 6 observations a, b, 68, 44, 40, 60 be 55 and 194. If $a > b$ then find $a + 3b$

(1) 211.83 (2) 201.59

(3) 189.57 (4) 198.87

Answer (2)

Sol. $\frac{a+b+68+44+40+60}{6} = 55$

$212 + a + b = 330$

$\Rightarrow\ a + b = 118$

$\frac{\sum x_i^2}{n} - (\bar{x})^2 = 194$

$\frac{a^2+b^2+(68)^2+(44)^2+(40)^2+(60)^2}{6} - (55)^2 = 194$

= 3219

$11760 + a^2 + b^2 = 19314$

$\Rightarrow\ a^2 + b^2 = 19314 - 11760$

= 7554

$(a + b)^2 - 2ab = 7554$

From here $b = 41.795$

$a + b = 118$

$\Rightarrow\ a + b + 2b = 118 + 83.59$

= 201.59

4. If 2^{nd}, 8^{th}, 44^{th} terms of A.P. are 1^{st}, 2^{nd} and 3^{rd} terms respectively of G.P. and first term of A.P. is 1 then the sum of first 20 terms of A.P. is

(1) 970 (2) 916

(3) 980 (4) 990

Answer (1)

Sol. $a + d$, $a + 7d$ and $a + 43d$ are 1^{st}, 2^{nd}, 3^{rd} term of G.P.

$\frac{a+7d}{a+d} = \frac{a+43d}{a+7d}$

$\Rightarrow\ (a + 7d)^2 = (a + d)(a + 43d)$

$\Rightarrow\ a^2 + 49d^2 + 14d = a^2 + 44ad + 43d^3$

$\Rightarrow\ 6d^2 = 30ad$

$\Rightarrow\ d^2 = 5d$

$\Rightarrow\ d = 0, 5$

$a = 1, d = 5$

$S_{20} = \frac{20}{2}[2 + (19)5]$

= 10 [95 + 2]

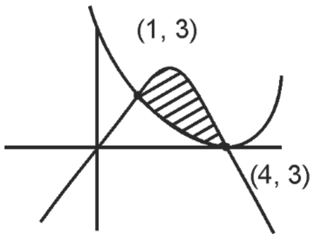
= 970

5. The area of the region enclosed by the parabolas $y = 4 - x^2$ and $3y = (x - 4)^2$ is in (sq. unit)?

(1) $\frac{14}{3}$ (2) 4

(3) $\frac{32}{3}$ (4) 6

Answer (4)



Sol. Area = $\left|\int_1^4\left[(4-x)^2 - \frac{(x-4)^2}{3}\right]dx\right|$

Area = $\left|4x - \frac{x^3}{3} - \frac{(x-4)^3}{9}\right|_1^4$

$= \left|\left(16 - \frac{64}{3}\right) - \left(4 - \frac{1}{3} + \frac{27}{9}\right)\right|$

$= \left|16 - \frac{64}{3} - 4 + \frac{1}{3} + 3\right|$

$= |15 - 2| = 6$

6. If $A\begin{bmatrix}1\\0\\1\end{bmatrix} = 2\begin{bmatrix}1\\0\\1\end{bmatrix}$, $A\begin{bmatrix}-1\\0\\1\end{bmatrix} = 4\begin{bmatrix}-1\\0\\1\end{bmatrix}$

and $A\begin{bmatrix}0\\1\\0\end{bmatrix} = 2\begin{bmatrix}0\\1\\0\end{bmatrix}$ where, A is a 3 × 3 matrix and $(A - 3I)\begin{bmatrix}x\\y\\z\end{bmatrix} = \begin{bmatrix}-1\\2\\3\end{bmatrix}$ then the value of (x, y, z) is

(1) (1, 2, 3) (2) (1, –2, 3)

(3) (1, –2, –3) (4) (–1, –2, –3)

Answer (3)

Sol. Let $A = \begin{bmatrix}x_1 & y_1 & z_1\\x_2 & y_2 & z_2\\x_3 & y_3 & z_3\end{bmatrix}$

Given $A = \begin{bmatrix}1\\0\\1\end{bmatrix} = \begin{bmatrix}2\\0\\2\end{bmatrix}$ … (1)

$\therefore \begin{bmatrix}x_1 + z_1\\x_2 + z_2\\x_3 + z_3\end{bmatrix} = \begin{bmatrix}2\\0\\2\end{bmatrix}$

$\therefore x_1 + z_1 = 2$ … (2)

$x_2 + z_2 = 0$ … (3)

$x_3 + z_3 = 0$ … (4)

Given $A = \begin{bmatrix}-1\\0\\1\end{bmatrix} = \begin{bmatrix}-4\\0\\4\end{bmatrix}$

$\therefore \begin{bmatrix}-x_1 + z_1\\-x_2 + z_2\\-x_3 + z_3\end{bmatrix} = \begin{bmatrix}4\\0\\4\end{bmatrix}$

$\Rightarrow -x_1 + z_1 = -4$ … (5)

$-x_2 + z_2 = 0$ … (6)

$-x_3 + z_3 = 4$ … (7)

Given $A = \begin{bmatrix}0\\1\\0\end{bmatrix} = \begin{bmatrix}0\\2\\0\end{bmatrix}$

$\therefore \begin{bmatrix}y_1\\y_2\\y_3\end{bmatrix} = \begin{bmatrix}0\\2\\0\end{bmatrix}$

$\therefore y_1 = 0, y_2 = 2, y_3 = 0$

$\therefore$ from (2), (3), (4), (5), (6) and (7)

$x_1 = 3$, $x_2 = 0$, $x_3 = -1$

$y_1 = 0$, $y_2 = 2$, $y_3 = 0$

$z_1 = -1$, $z_2 = 0$, $z_3 = 3$

$\therefore A = \begin{bmatrix}3 & 0 & -1\\0 & 2 & 0\\-1 & 0 & 3\end{bmatrix}$

$\therefore$ Now $(A - 3I)\begin{bmatrix}x\\y\\z\end{bmatrix} = \begin{bmatrix}-1\\2\\3\end{bmatrix}$

$\therefore \begin{bmatrix}0 & 0 & -1\\0 & -1 & 0\\-1 & 0 & 0\end{bmatrix}\begin{bmatrix}x\\y\\z\end{bmatrix} = \begin{bmatrix}-1\\2\\3\end{bmatrix}$

$\begin{bmatrix}-z\\-y\\-x\end{bmatrix} = \begin{bmatrix}-1\\2\\3\end{bmatrix}$

$[z = 1], [y = -2], [x = -3]$

7. Let $f:(0,\infty)\to R$ be increasing function such that $\lim\limits_{x\to\infty}\frac{f(7x)}{f(x)}=1$ then $\lim\limits_{x\to\infty}\left(\frac{f(5x)}{f(x)}-1\right)$ is equal to

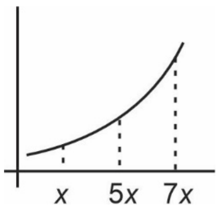
(1) Zero (2) 4

(3) 1 (4) $\frac{4}{5}$

Answer (1)

Sol. f is increasing function

$x<5x<7x$



$f(x)<f(5x)<f(7x)$

$$\frac{f(x)}{f(x)}<\frac{f(5x)}{f(x)}<\frac{f(7x)}{f(x)}$$

$$\lim_{x\to\infty}\frac{f(x)}{f(x)}<\lim_{x\to\infty}\frac{f(5x)}{f(x)}<\lim_{x\to\infty}\frac{f(7x)}{f(x)}$$

$$1<\lim_{x\to\infty}\frac{f(5x)}{f(x)}<1 \quad \Rightarrow \lim_{x\to\infty}\frac{f(5x)}{f(x)}=1$$

$$\lim_{x\to\infty}\left(\frac{f(5x)}{f(x)}-1\right)=0$$

8. Let z_1 and z_2 be two complex numbers such that $z_1 + z_2 = 5$ and $z_1^3+z_2^3=20+15i$, then the value of $\left|z_1^4+z_2^4\right|$ is equal to

(1) 75 (2) $25\sqrt{5}$

(3) $15\sqrt{15}$ (4) $30\sqrt{3}$

Answer (1)

Sol. $z_1 + z_2 = 5$

$z_1^3+z_2^3=20+15i$

$z_1^3+z_2^3=(z_1+z_2)^3-3z_1z_2(z_1+z_2)$

$z_1^3+z_2^3=125-3z_1\cdot z_2(5)$

$\Rightarrow$ $20 + 15i = 125 - 15z_1z_2$

$\Rightarrow$ $3z_1z_2 = 25 - 4 - 3i$

$3z_1z_2 = 21 - 3i$

$z_1\cdot z_2 = 7 - i$

$(z_1 + z_2)^2 = 25$

$z_1^2+z_2^2=25-2(7-i)$

$= 11 + 2i$

$\left(z_1^2+z_2^2\right)^2=121-4+44i$

$\Rightarrow$ $z_1^4+z_2^4+2(7-i)^2=117+44i$

$\Rightarrow$ $z_1^4+z_2^4=117+44i-2(49-1-14i)$

$= 21 + 72i$

$\Rightarrow$ $\left|z_1^4+z_2^4\right|=75$

9. The number of solutions of equation $e^{\sin x}-2e^{-\sin x}=2$ is

(1) More than 2 (2) 2

(3) 1 (4) 0

Answer (4)

Sol. Take $e^{\sin x} = t$ $(t > 0)$

$\Rightarrow t-\frac{2}{t}=2$

$\Rightarrow \frac{t^2-2}{t}=2$

$\Rightarrow t^2-2t-2=0$

$\Rightarrow t^2-2t+1=3$

$\Rightarrow (t-1)^2=3$

$\Rightarrow t=1\pm\sqrt{3}$

$\Rightarrow t=1\pm1.73$

$\Rightarrow t = 2.73$ or -0.73 (rejected as $t > 0$)

$\Rightarrow e^{\sin x}=2.73$

$\Rightarrow \log_e e^{\sin x}=\log_e 2.73$

$\Rightarrow \sin x=\log_e 2.73>1$

So no solution.

10. The line passes through the centre of circle $x^2 + y^2 - 16x - 4y = 0$, it interacts with the positive coordinate axis at A & B. Then find the minimum value of $OA + OB$, where O is origin.

(1) 20 (2) 18

(3) 12 (4) 24

Answer (1)

Sol. $(y - 2) = m(x - 8)$

$\Rightarrow$ x-intercept

$\Rightarrow$ $\left(\frac{-2}{m}+8\right)$

$\Rightarrow$ y-intercept

$\Rightarrow$ $(-8m + 2)$

$\Rightarrow$ $OA + OB = \frac{-2}{m^2} + 8 - 8m + 2$

$f'(m)=\frac{2}{m^2}-8=0$

$\Rightarrow$ $m^2=\frac{1}{4}$

$\Rightarrow \quad m = \frac{-1}{2}$

$\Rightarrow \quad f\left(\frac{-1}{2}\right) = 18$

$\Rightarrow$ Minimum = 18

11. If for some m, n; ${}^6C_m + 2\left({}^6C_{m+1}\right) + {}^6C_{m+2} > {}^8C_3$ and ${}^{n-1}P_3 : {}^nP_4 = 1:8$, then ${}^nP_{m+1} + {}^{n+1}C_m$ is equal to

(1) 6756 (2) 7250
(3) 6223 (4) 6550

Answer (1)

Sol. ${}^6C_m + 2\left({}^6C_{m+1}\right) + {}^6C_{m+2} > {}^8C_3$

${}^7C_{m+1} + {}^7C_{m+2} > {}^8C_3$

${}^8C_{m+2} > {}^8C_3$

$\therefore \quad m = 2$

and ${}^{n-1}P_3 : {}^nP_4 = 1:8$

$\frac{(n-1)(n-2)(n-3)}{n(n-1)(n-2)(n-3)} = \frac{1}{8}$

$\therefore \quad n = 8$

$\therefore \quad {}^nP_{m+1} + {}^{n+1}C_m = {}^8P_5 + {}^9C_2$

$= 8 \times 7 \times 6 \times 5 \times 4 + \frac{9 \times 8}{2}$

= 6756

12. Let $f : (-\infty, -1] \to (a, b]$ be defined as $f(x) = e^{x^3 - 3x + 1}$, if f is both one and onto, then the distance from a point $P(2a + 4, b + 2)$ to curve $x + ye^{-3} - 4 = 0$ is

(1) $\sqrt{e^3 + 2}$ (2) $\frac{e^3 + 2}{\sqrt{e^3 + 1}}$

(3) $\frac{e^3 + 2}{\sqrt{e^6 + 1}}$ (4) e

Answer (3)

Sol. $f(x) = e^{x^3 - 3x + 1}$

$f'(x) = e^{x^3 - 3x + 1} \cdot (3x^2 - 3)$

$= e^{x^2 - 3x + 1} \cdot 3(x - 1)(x + 1)$

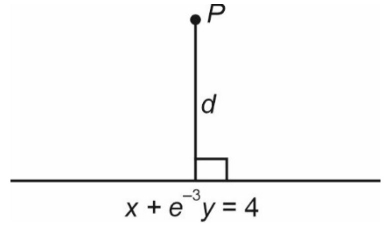
For $x \in (-\infty, -1], f'(x) \geq 0$

$\therefore f(x)$ is increasing function

$\therefore a = e^{-\infty} = 0 = f(-\infty)$

$b = e^{-1+3+1} = e^3 = f(-1)$

$\therefore P(4, e^3 + 2)$



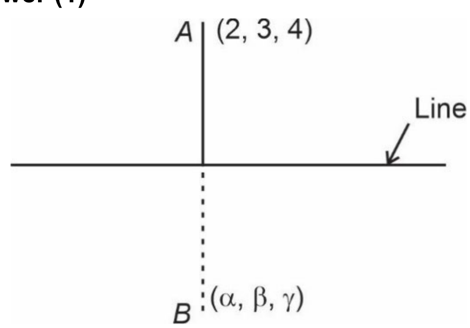
$d = \frac{(e^3 + 2)(e^{-3})}{\sqrt{1 + e^{-6}}} = \frac{1 + 2e^{-3}}{\sqrt{1 + e^{-6}}} = \frac{e^3 + 2}{\sqrt{e^6 + 1}}$

13. If (α, β, γ) is mirror image of the point (2, 3, 4) with respect to the line $\frac{x-1}{2} = \frac{y-2}{3} = \frac{z-3}{4}$. Then $2\alpha + 3\beta + 4\gamma$ is

(1) 29 (2) 30
(3) 31 (4) 32

Answer (1)

Sol.



Take $\frac{x-1}{2} = \frac{y-2}{3} = \frac{z-3}{4} = \lambda$

$x = 2\lambda + 1, \; y = 3\lambda + 2, \; z = 4\lambda + 3$

$\overrightarrow{AB} = (\alpha - 2)\hat{i} + (\beta - 3)\hat{j} + (\gamma - 4)\hat{k}$

Now,

$(\alpha - 2) \cdot 2 + (\beta - 3) \cdot 3 + (\gamma - 4) \cdot 4 = 0$

$2\alpha - 4 + 3\beta - 9 + 4\gamma - 16 = 0$

$\Rightarrow \; 2\alpha + 3\beta + 4\gamma = 29$

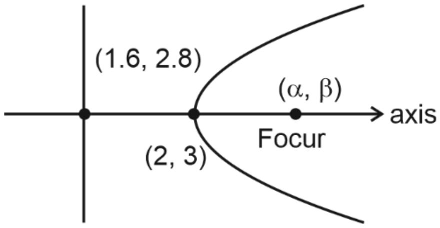
14. A parabola has vertex (2, 3), equation of directrix is $2x - y = 1$ and equation of ellipse is $\frac{x^2}{a^2} + \frac{y^2}{b^2} = 1, e = \frac{1}{\sqrt{2}}$ and ellipse passing through focur of parabola then square of length of latus rectum of ellipse is

(1) $\frac{6564}{25}$ (2) $\frac{3288}{25}$

(3) $\frac{6272}{25}$ (4) $\frac{4352}{25}$

Answer (4)

Sol.



Slope of axis $= \dfrac{1}{2}$

$$y - 3 = \frac{1}{2}(x - 2)$$

$\Rightarrow$ $2y - 6 = x - 2$

$\Rightarrow$ $2y - x - 4 = 0$

$2x + y - 6 = 0$

$4x + 2y - 12 = 0$

$\alpha + 1.6 = 4 \Rightarrow \alpha = 2.4$

$\beta + 2.8 = 6 \Rightarrow \beta = 3.2$

Ellipse passes through (2.4, 3.2)

$$\Rightarrow \frac{\left(\frac{24}{10}\right)^2}{a^2} + \frac{\left(\frac{32}{10}\right)^2}{b^2} = 1$$

Also $1 - \dfrac{a^2}{b^2} = \dfrac{1}{2}$

$$\frac{a^2}{b^2} = \frac{1}{2}$$

$$\frac{144}{25}b^2 + \frac{256}{25}a^2 = a^2b^2$$

$$\frac{144}{25} + \frac{256}{25} \times \frac{1}{2} = a^2$$

$$\Rightarrow \frac{(128 + 144)}{25} = a^2 \Rightarrow \frac{272}{25} = a^2$$

$$\Rightarrow b^2 = \frac{2 \times 272}{25}$$

Latus rectum $= \dfrac{2b^2}{a}$

(Latus rectum)^2

$$= \frac{4b^4}{a^2} = 4\left(\frac{b^2}{a^2}\right)b^2 = \frac{8 \times 272 \times 2}{25} = \frac{4352}{25}$$

15.

16.

17.

18.

19.

20.

SECTION - B

Numerical Value Type Questions: This section contains 10 Numerical based questions. The answer to each question should be rounded-off to the nearest integer.

21. The value of $\dfrac{120}{\pi^3}\left|\displaystyle\int_0^{\pi} \frac{x^2 \sin x \cdot \cos x}{(\sin x)^4 + (\cos x)^4} dx\right|$ is

Answer (15)

Sol. $\displaystyle\int_0^{\pi} \frac{x^2 \sin x \cdot \cos x}{\sin^4 x + \cos^4 x} dx$

$$= \int_0^{\frac{\pi}{2}} \frac{\sin x \cos x}{\sin^4 x + \cos^4 x}\left(x^2 - (\pi - x)^2\right)dx$$

$$= \int_0^{\frac{\pi}{2}} \frac{\sin x \cdot \cos x(2\pi x - \pi^2)}{\sin^4 x + \cos^4 x} x$$

$$= 2\pi\int_0^{\frac{\pi}{2}} \frac{x \sin x \cos x}{\sin^4 x + \cos^4 x} dx - \pi^2 \int_0^{\frac{\pi}{2}} \frac{\sin x \cos x}{\sin^4 x + \cos^4 x} dx$$

$$= 2\pi \cdot \frac{\pi}{4}\int_0^{\frac{\pi}{2}} \frac{\sin x \cos x}{\sin^4 x + \cos^4 x} dx - \pi^2 \int_0^{\frac{\pi}{2}} \frac{\sin x \cos x}{\sin^4 x + \cos^4 x} dx$$

$$= -\frac{\pi^2}{2}\int_0^{\frac{\pi}{2}} \frac{\sin x \cos x}{\sin^4 x + \cos^4 x} dx$$

$$= -\frac{\pi^2}{2}\int_0^{\frac{\pi}{2}} \frac{\sin x \cos x dx}{1 - 2\sin^2 x + \cos^2 x}$$

$$= -\frac{\pi^2}{2}\int_0^{\frac{\pi}{2}} \frac{\frac{1}{2}\sin 2x}{1 - \frac{1}{2}\sin^2 2x} dx$$

$$= -\frac{\pi^2}{2}\int_0^{\frac{\pi}{2}} \frac{\sin 2x}{2 - \sin^2 2x} dx$$

$$= -\frac{\pi^2}{2}\int_0^{\frac{\pi}{2}} \frac{\sin 2x}{1 + \cos^2 2x} dx$$

Let $\cos 2x = t$

$$= -\frac{\pi^2}{2}\int_{1}^{-1}\frac{-\frac{1}{2}dt}{1+t^2}$$

$$= -\frac{\pi^2}{4}\int_{-1}^{1}\frac{dt}{1+t^2}$$

$$= -\frac{\pi^2}{4}\cdot\frac{\pi}{2} = -\frac{\pi^3}{8}$$

$$\therefore \quad \frac{120}{\pi^3}\left|-\frac{\pi^3}{8}\right| = 15$$

22. The number of ways to distribute the 21 identical apples to three children's so that each child gets at least 2 apples.

Answer (136)

Sol. After giving 2 apples to each child 15 apples left now 15 apples can be distributed in ${}^{15+3-1}C_2 = {}^{17}C_2$ ways

$$= \frac{17\times 16}{2} = 136$$

23. If $A = \{1, 2, 3, \ldots 100\}$, $R = \{(x, y) \mid 2x = 3y, x, y \in A\}$ is symmetric relation on A and the number of elements in R is n, the smallest integer value of n is

Answer (0)

Sol. $\because$ R is symmetric relation

$\Rightarrow$ $(y, x) \in R \;\forall\; (x, y) \in R$

$(x, y) \in R \Rightarrow 2x = 3y$ and $(y, x) \in R \Rightarrow 3x = 2y$

Which holds only for (0, 0)

Which does not belongs to R.

$\therefore$ Value of $n = 0$

24. Matrix A of order 3 × 3 is such that $|A| = 2$ if $n = \underbrace{\left|\text{adj}(\text{adj}(\text{adj}\ldots(a)))\right|}_{2024 \text{ times}}$ then remainder when n is divided by 9 is

Answer (7)

Sol. $|A| = 2$

$$\underbrace{\text{adj}(\text{adj}(\text{adj}\ldots(a)))}_{2024 \text{ times}} = |A|^{(n-1)^{2024}}$$

$$= |A|^{2^{2024}}$$

$$= 2^{2^{2024}}$$

$$2^{2024} = (2^2)2^{2022} = 4(8)^{674} = 4(9-1)^{674}$$

$$\Rightarrow \quad 2^{2024} \equiv 4 \pmod 9$$

$$\Rightarrow \quad 2^{2024} \equiv 9m + 4, \quad m \leftarrow \text{even}$$

$$2^{9m+4} \equiv 16\cdot(2^3)^{3m} \equiv 16 \pmod 9$$

$$\equiv 7$$

25.

26.

27.

28.

29.

30.