MATHEMATICS

1st Feb Shift - 1

SECTION - A

Multiple Choice Questions: This section contains 20 multiple choice questions. Each question has 4 choices (1), (2), (3) and (4), out of which **ONLY ONE** is correct.

Choose the correct answer :

1. If 3, a, b, c are in A.P. and 3, $a-1$, $b+1$ are in G.P. Then arithmetic mean of a, b and c is

(1) 11 (2) 10

(3) 9 (4) 13

Answer (1)

Sol. 3, a, b, c are in A.P.

$a - 3 = b - a$ (common diff.)

$2a = b + 3$

and 3, $a-1$, $b+1$ are in G.P.

$$\frac{a-1}{3} = \frac{b+1}{a-1}$$

$a^2 + 1 - 2a = 3b + 3$

$a^2 - 8a + 7 = 0$ $[\because\ 2a = b+3]$

$(a-7)(a-1) = 0$

If $a = 7$, $b = 2(7) - 3 = 11$, $b = 11$

and $c - b = a - 3$

$c - 11 = 4$

$c = 15$

$\therefore$ A.M of 7, 11, 15 = $\dfrac{7+11+15}{3}$

$$= \frac{33}{3} = 11$$

2. The value of $\displaystyle\int_0^{\pi/4} \frac{x\,dx}{\sin^4(2x)+\cos^4(2x)}$ is equal to

(1) $\dfrac{\pi^2}{16\sqrt{2}}$ (2) $\dfrac{\pi^2}{64}$

(3) $\dfrac{\pi^2}{32}$ (4) $\dfrac{\pi^2}{8\sqrt{2}}$

Answer (1)

Sol. $I = \displaystyle\int_0^{\pi/4} \frac{x\,dx}{\sin^4(2x)+\cos^4(2x)}$

Let $2x = t$ then $dx = \dfrac{1}{2}dt$

$$I = \int_0^{\pi/2} \frac{\frac{t}{2}\cdot\frac{1}{2}dt}{\sin^4 t+\cos^4 t}$$

$$= \frac{1}{4}\int_0^{\pi/2} \frac{t\,dt}{\sin^4 t+\cos^4 t}dt$$

$$\therefore\quad I = \frac{1}{4}\int_0^{\pi/2} \frac{\left(\frac{\pi}{2}-t\right)dt}{\sin^4 t+\cos^4 t}dt$$

$$\therefore\quad 2I = \frac{1}{4}\int_0^{\pi/2} \frac{\frac{\pi}{2}dt}{\sin^4 t+\cos^4 t}$$

$$2I = \frac{\pi}{8}\int_0^{\pi/2} \frac{\sin^4 t\,dt}{\tan^4 t+1}$$

Let $\tan t = y$ then

$$2I = \frac{\pi}{8}\int_0^{\infty} \frac{(1+y^2)dy}{1+y^4}$$

$$= \frac{\pi}{8}\int_0^{\infty} \frac{1+\frac{1}{y^2}}{y^2+\frac{1}{y^2}-2+2}dy$$

$$= \frac{\pi}{8}\int_0^{\infty} \frac{\left(1+\frac{1}{y^2}\right)dy}{2+\left(y-\frac{1}{y}\right)^2}$$

Let $y - \dfrac{1}{y} = u$

$$2I = \frac{\pi}{8}\int_{-\infty}^{\infty} \frac{du}{2+u^2}$$

$$= \frac{\pi}{8\sqrt{2}}\left[\tan^{-1}\frac{4}{\sqrt{2}}\right]_{-\infty}^{\infty}$$

$$\therefore\quad I = \frac{\pi^2}{16\sqrt{2}}$$

3. If $A=\begin{bmatrix}\sqrt{2} & 1\\ -1 & \sqrt{2}\end{bmatrix}$, $B=\begin{bmatrix}1 & 0\\ 1 & 1\end{bmatrix}$, $C = ABA^T$ and $X = AC^2A^T$, then $|X|$ is equal to

(1) 729 (2) 283

(3) 27 (4) 23

Answer (1)

Sol. $|A| = 3$

$|B| = 1$

$\Rightarrow |C| = |ABA^T| = |A||B|A^T| = |A|^2|B|$

$= 9$

$\Rightarrow |X| = |A||C|^2|A^T|$

$= 3 \times 9^2 \times 3 = 9 \times 9^2 = 729$

4. If 3, 7, 11, …., 403 = AP_1

2, 5, 8, …., 401 = AP_2

Find sum of common term of AP_1 and AP_2

(1) 3366 (2) 6699

(3) 9999 (4) 6666

Answer (2)

Sol. 3, 7, 11, 15, 19, 23, 27, … 403 = AP_1

2, 5, 8, 11, 14, 17, 20, 23, … 401 = AP_2

so common terms *A.P.*

11, 23, 35, …, 395

$\Rightarrow 395 = 11 + (n - 1)\,12$

$\Rightarrow 395 - 11 = 12\,(n - 1)$

$\frac{384}{12} = n-1$

$32 = n - 1$

$n = 33$

Sum = $\frac{33}{2}[2\times 11+(32)12]$

$=\frac{33}{2}[22+384]$

= 6699

5. $\displaystyle\int_{-\frac{\pi}{2}}^{\frac{\pi}{2}} \frac{8\sqrt{2}\cos x}{\left(1+e^{\sin x}\right)\left(1+\sin^4 x\right)}dx = a\pi + b\log\left(3+2\sqrt{2}\right)$

then find $a + b$.

(1) 4 (2) 6

(3) 8 (4) 2

Answer (1)

Sol. $\displaystyle I = \int_{-\frac{\pi}{2}}^{\frac{\pi}{2}} \frac{8\sqrt{2}\cos x}{\left(1+e^{\sin x}\right)\left(1+\sin^4 x\right)}dx$

$$= \int_0^{\frac{\pi}{2}}\left\{\frac{8\sqrt{2}\cos x}{(1+e^{\sin x})(1+\sin^4 x)}+\frac{8\sqrt{2}\cos x}{(1+e^{-\sin x})(1+\sin^4 x)}\right\}dx$$

$$= 8\sqrt{2}\int_0^{\frac{\pi}{2}}\frac{\cos x}{1+\sin^4 x}dx$$

Let $\sin x = t$

$$I = 8\sqrt{2}\int_0^1 \frac{dt}{1+t^4}$$

$$= 4\sqrt{2}\int_0^1 \frac{\left(1+\frac{1}{t^2}\right)-\left(1-\frac{1}{t^2}\right)}{t^2+\frac{1}{t^2}}dt$$

$$= 4\sqrt{2}\int_0^1 \frac{\left(1+\frac{1}{t^2}\right)dt}{\left(t-\frac{1}{t}\right)^2+2} - 4\sqrt{2}\int_0^1 \frac{\left(1-\frac{1}{t^2}\right)dt}{\left(t+\frac{1}{t}\right)^2-2}$$

$$= 4\sqrt{2}\cdot\frac{1}{\sqrt{2}}\left(\tan^{-1}\frac{t-\frac{1}{t}}{\sqrt{2}}\right)_0^1 - 4\sqrt{2}\cdot\frac{1}{2\sqrt{2}}\left[\log\left|\frac{t+\frac{1}{t}-\sqrt{2}}{t+\frac{1}{t}+\sqrt{2}}\right|\right]_0^1$$

$$= 2\pi - 2\log\left|\frac{2-\sqrt{2}}{2+\sqrt{2}}\right|$$

$$= 2\pi + 2\log\left(3+2\sqrt{2}\right)$$

$\therefore \quad a = b = 2$

6. If $(t + 1)dx = (2x + (t + 1)^3)dt$ and $x(0) = 2$, then $x(1)$ is equal to

(1) 5 (2) 12

(3) 6 (4) 8

Answer (2)

Sol. $(t + 1)dx = (2x + (t + 1)^3)dt$

$$\therefore \frac{dx}{dt}-\frac{2x}{t+1}=(t+1)^2$$

$$\therefore \text{I.F.} = e^{\int -\frac{2}{t+1}dt} = \frac{1}{(t+1)^2}$$

$\therefore$ Solution is

$$\frac{x}{(t+1)^2} = \int 1\,dt$$

$x = (t + c)\,(t + 1)^2$

$\because x(0) = 2$ then $c = 2$

$\therefore x = (t + 2)(t + 1)^2$

$\therefore x(1) = 12$

7. Five people are distributed in four identical rooms. A room can also contain zero people. Find the number of ways to distribute them.

(1) 47 (2) 53

(3) 43 (4) 51

Answer (4)

Sol. Total ways to partition 5 into 4 parts are:

$5\ 0\ 0\ 0 \rightarrow 1$

$4\ 1\ 0\ 0 \rightarrow \frac{5!}{4!} = 5$

$3\ 2\ 0\ 0 \rightarrow \frac{5!}{3!\cdot 2!} = 10$

$3\ 1\ 1\ 0 \rightarrow \frac{5!}{3!\cdot 2!} = 10$

$2\ 2\ 1\ 0 \rightarrow \frac{5!}{2!2!2!} = 15$

$2\ 1\ 1\ 1 \rightarrow \frac{5!}{2! \times 3!} = 10$

$51 \rightarrow$ Total way

8. $5f(x) + 4f\left(\frac{1}{x}\right) = x^2 - 4$ and $y = 9f(x)\cdot x^2$. If y is strictly increasing function, find interval of x.

(1) $\left(-\infty, \frac{-1}{\sqrt{5}}\right] \cup \left(\frac{-1}{\sqrt{5}}, 0\right)$

(2) $\left(\frac{-1}{\sqrt{5}}, 0\right) \cup \left(0, \frac{1}{\sqrt{5}}\right)$

(3) $\left(0, \frac{1}{\sqrt{5}}\right) \cup \left(\frac{1}{\sqrt{5}}, \infty\right)$

(4) $\left(-\sqrt{\frac{2}{5}}, 0\right) \cup \left(\sqrt{\frac{2}{5}}, \infty\right)$

Answer (4)

Sol. $5f(x) + 4f\left(\frac{1}{x}\right) = x^2 - 4 \quad ...(1)$

Replace x by $\frac{1}{x}$

$5f\left(\frac{1}{x}\right) + 4f(x) = \frac{1}{x^2} - 4 \quad ...(2)$

5 × equation (1) – 4 × equation (2)

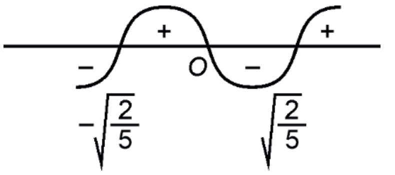
$9f(x) = 5x^2 - \frac{4}{x^2} - 4$

$y = 9f(x)\cdot x^2 = \frac{5x^4 - 4 - 4x^2}{x^2} x^2$

$y = 5x^4 - 4 - 4x^2$

$y' = 20x^3 - 8x > 0$

$4x(5x^2 - 2) > 0$



$x \in \left(-\sqrt{\frac{2}{5}}, 0\right) \cup \left(\sqrt{\frac{2}{5}}, \infty\right)$

9. If hyperbola $x^2 - y^2\operatorname{cosec}^2\theta = 5$ and ellipse $x^2\operatorname{cosec}^2\theta + y^2 = 5$ has eccentricity e_H and e_e respectively and $e_H = \sqrt{7}e_e$, then θ is equal to

(1) $\frac{\pi}{3}$

(2) $\frac{\pi}{6}$

(3) $\frac{\pi}{2}$

(4) $\frac{\pi}{4}$

Answer (1)

Sol. $x^2 - y^2\operatorname{cosec}^2\theta = 5 \quad \Rightarrow \frac{x^2}{1} - \frac{y^2}{\sin^2\theta} = 5$

$x^2\operatorname{cosec}^2\theta + y^2 = 5 \quad \Rightarrow \frac{x^2}{\sin^2\theta} + \frac{y^2}{1} = 5$

$e_H = \sqrt{7}e_e$

$e_H = \sqrt{1 + \frac{\sin^2\theta}{1}}$

and $e_e = \sqrt{1 - \frac{\sin^2\theta}{1}}$

$\Rightarrow \sqrt{1 + \sin^2\theta} = \sqrt{7}\sqrt{1 - \sin^2\theta}$

$\Rightarrow 1 + \sin^2\theta = 7 - 7\sin^2\theta$

$\Rightarrow 8\sin^2\theta = 6$

$\Rightarrow \sin\theta = \sqrt{\frac{3}{4}} = \frac{\sqrt{3}}{2}$

$\Rightarrow \theta = \frac{\pi}{3}$

10. A bag contains 8 balls (black and white). If four balls are chosen without replacement then $2W$ and $2B$ are found then the probability that number of white and black balls are same in bag is equal to

(1) $\frac{1}{7}$ (2) $\frac{2}{7}$

(3) $\frac{3}{5}$ (4) $\frac{1}{2}$

Answer (2)

Sol. $P(2W \text{ and } 2B) = P(2B, 6W) \times P(2W \text{ and } 2B)$

$+ P(3B, 5W) \times P(2W \text{ and } 2B)$

$+ P(4B, 4W) \times P(2W \text{ and } 2B)$

$+ P(5B, 3W) \times P(2W \text{ and } 2B)$

$+ P(6B, 2W) \times P(2W \text{ and } 2B)$

$$= \frac{1}{9}\left(0+0+\frac{{}^2C_2 \times {}^6C_2}{{}^8C_4}+\frac{{}^3C_2 \cdot {}^5C_2}{{}^8C_4}+\frac{{}^4C_2 \cdot {}^4C_2}{{}^8C_2} + \frac{{}^5C_2 \cdot {}^3C_2}{{}^8C_4}+\frac{{}^6C_2 \cdot {}^2C_2}{{}^8C_4}+0+0\right)$$

$$= \frac{1}{9}\times\frac{1}{{}^8C_4}(15+30+36+30+15)$$

$$= \frac{1}{9}\times\frac{1}{{}^8C_4}\times 126$$

$$P\left(\frac{4B \text{ and } 4W}{2W \text{ and } 2B}\right) = \frac{\frac{1}{9}\times\frac{{}^4C_2 \times {}^4C_2}{{}^8C_4}}{\frac{1}{9}\times\frac{1}{{}^8C_4}\times 126}$$

$$= \frac{36}{126}$$

$$= \frac{18}{63}$$

$$= \frac{6}{21}$$

$$= \frac{2}{7}$$

11. If two circle $x^2 + y^2 = 4$ and $x^2 + y^2 - 4\lambda x + 9 = 0$ intersect at two distinct points, then find the range of λ.

(1) $\left(-\infty, -\frac{13}{2}\right)\cup\left(\frac{13}{2}, \infty\right)$

(2) $\left(-\infty, -\frac{13}{8}\right)\cup\left(\frac{13}{8}, \infty\right)$

(3) $\left[-\frac{13}{8}, \frac{13}{8}\right]$

(4) $\lambda \in \left(\frac{3}{2}, \infty\right)$

Answer (2)

Sol. $|r_1 - r_2| < c_1c_2 < r_1 + r_2$

$$\Rightarrow \left|2-\sqrt{4\lambda^2-9}\right| < |2\lambda| < 2+\sqrt{4\lambda^2-9}$$

$$\Rightarrow |2\lambda| - 2 < \sqrt{4\lambda^2-9}$$

$$\Rightarrow 4\lambda^2 + 4 - 8|\lambda| < 4\lambda^2 - 9$$

$$\lambda > \frac{13}{8}, \lambda < -\frac{13}{8}$$

$$\sqrt{4\lambda^2-9} > 0$$

$$\Rightarrow \lambda > \frac{3}{2}, \lambda < -\frac{3}{2}$$

$$\therefore \lambda \in \left(-\infty, -\frac{13}{8}\right)\cup\left(\frac{13}{8}, \infty, \right)$$

Now,

$$\left|2-\sqrt{4\lambda^2-9}\right| < |2\lambda|$$

$$\Rightarrow 4+4\lambda^2-9-4\sqrt{4\lambda^2-9} < 4\lambda^2$$

$$\Rightarrow 4\sqrt{4\lambda^2-9} > -5 \Rightarrow \lambda \in R$$

$$\therefore \lambda \in \left(-\infty, -\frac{13}{8}\right)\cup\left(\frac{13}{8}, \infty\right)$$

12. If $S = \left\{x \in R : 3\left(\sqrt{3}+\sqrt{2}\right)^x + \left(\sqrt{3}-\sqrt{2}\right)^x = \frac{10}{3}\right\}$

then number of elements in set S is

(1) Zero (2) 1

(3) 2 (4) 3

Answer (3)

Sol. $\sqrt{3}-\sqrt{2} = \frac{\left(\sqrt{3}+\sqrt{2}\right)\left(\sqrt{3}-\sqrt{2}\right)}{\left(\sqrt{3}+\sqrt{2}\right)} = \frac{1}{\sqrt{3}+\sqrt{2}}$

Let $\sqrt{3}+\sqrt{2} = t$

$$\Rightarrow t^x + \frac{1}{t^x} = \frac{10}{3}$$

Let $t^x = y \quad \Rightarrow y + \frac{1}{y} = \frac{10}{3}$

$$\Rightarrow y = 3 \text{ or } \frac{1}{3}$$

$$\Rightarrow \left(\sqrt{3}+\sqrt{2}\right)^x = 3 \text{ or } \frac{1}{3}$$

$$x\log\left(\sqrt{3}+\sqrt{2}\right) = \ln 3 \text{ or } -\ln 3$$

$$\Rightarrow x = \frac{\ln 3}{\ln\left(\sqrt{3}+\sqrt{3}\right)} \text{ or } \frac{-\ln 3}{\sqrt{3}+\sqrt{2}}$$

$\Rightarrow$ two real values of x

13. $f(x)=\begin{cases} e^{-x} & , x<0 \\ \ln x & , x>0 \end{cases}$

$g(x)=\begin{cases} e^{x} & , x<0 \\ x & , x>0 \end{cases}$

The $gof : A \to R$ is

(1) Onto but not one-one

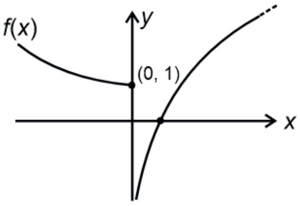
(2) Into and many one

(3) Onto and one-one

(4) Into and one-one

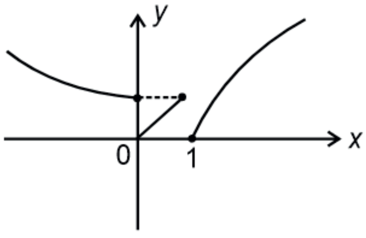
Answer (2)

Sol.



$gof(x)=\begin{cases} f(x) & , f(x)<0 \\ f(x) & , f(x)>0 \end{cases}$

$=\begin{cases} e^{\ln x}=x & (0, 1) \\ e^{-x} & (-\infty, 0) \\ \ln x & (1, \infty) \end{cases}$



$\therefore$ $gof(x)$ is many one and into

14. If $\tan A=\frac{1}{\sqrt{x^2+x+1}}$, $\tan B=\frac{\sqrt{x}}{\sqrt{x^2+x+1}}$ and $\tan C=\frac{1}{\sqrt{x(x^2+x+1)}}$, then $A + B =$

(1) 0 (2) $\pi - C$

(3) $\frac{\pi}{2}-C$ (4) None

Answer (3)

Sol. $\tan B\times \tan C=\frac{\sqrt{x}}{\sqrt{x^2+x+1}}\times\frac{1}{\sqrt{x(x^2+x+1)}}$

$=\frac{1}{x^2+x+1}=\tan^2 A$

$\tan^2 A = \tan B \tan C$

It is only possible when $A = B = C$ at $x = 1$

$\Rightarrow$ $A = 30°$, $B = 30°$, $C = 30°$

$\left[\tan A=\tan B=\tan C=\frac{1}{\sqrt{3}}\right]$

$\therefore$ $A+B=\frac{\pi}{2}-C$

15. $\lim_{x\to 0}\frac{\cos^{-1}\left(1-\{x\}^2\right)\sin^{-1}\left(1-\{x\}\right)}{\{x\}-\{x\}^3}$, where { } is fractional part function.

If L.H.L = L and R.H.L = R, then the correct relation between L and R is

(1) $\sqrt{2}R=4L$ (2) $\sqrt{2}L=4R$

(3) $R = L$ (4) $R = 2L$

Answer (1)

Sol. $RHL \Rightarrow \lim_{x\to 0^+}\frac{\cos^{-1}(1-x^2)\sin^{-1}(1-x)}{x-x^3}$

$\Rightarrow \lim_{x\to 0^+}\frac{\pi}{2}\cdot\frac{\cos^{-1}(1-x^2)}{x}$

$\frac{\pi}{2}\lim_{x\to 0^+}\frac{-1}{\sqrt{\left(1-(1-x^2)^2\right)}}(-2x)$

$=\frac{\pi}{2}\lim_{x\to 0^+}\frac{2x}{\sqrt{2x^2-x^4}}=\pi\lim_{x\to 0^+}\frac{x}{x\sqrt{2-x^2}}$

$=\frac{\pi}{\sqrt{2}}$

$LHL \Rightarrow \lim_{x\to 0^-}\frac{\cos^{-1}\left(1-(1+x)^2\right)\sin^{-1}(1-(1+x))}{1\cdot\left(1-(1+x)^2\right)}$

$=\lim_{x\to 0^-}\frac{\cos^{-1}(-x^2-2x).\sin^{-1}(-x)}{-x^2-2x}$

$=\frac{\pi}{2}\lim_{x\to 0^-}\frac{-\sin^{-1}x}{-x(x+2)}=\frac{\pi}{2}\times\frac{1}{2}=\frac{\pi}{4}$

16.

17.

18.

19.

20.

SECTION - B

Numerical Value Type Questions: This section contains 10 Numerical based questions. The answer to each question should be rounded-off to the nearest integer.

21. Let $S = \{1, 2, 3, \ldots., 20\}$

$R_1 = \{(a, b) : a \text{ divide } b\}$

$R_2 = \{(a, b) : a \text{ is integral multiple of } b\}$ $a, b \in s$

$n(R_1 - R_2) = ?$

Answer (46)

Sol. R_1 = {(1, 1), (1, 2), (1, 3)…., (1, 20), (2, 2), (2, 4)….(2, 20), (3, 3), (3, 6), ….(3, 18), (4, 4), (4, 8), … (4, 20), (5, 5), (5, 10), (5, 15), (5, 20), (6, 6), (6, 12), (6, 18), (7, 7), (7, 14), (8, 8), (8, 16), (9, 9), (9, 18), (10, 10), (10, 20), (11, 11), (12, 12), ….(20, 20)}

$n(R_1) = 66$

$R_2 = \{a$ is integral multiple of $b\}$

So $n(R_1 - R_2) = 66 - 20 = 46$

as $R_1 \cap R_2 = \{(a, a) : a \in s\} = \{(1, 1), (2, 2)\ldots, (20, 20)\}$

22. The number of solution of equation $x + 2y + 3z = 42$ and $x, y, z \in z$ and $x, y, z \geq 0$ is

Answer (168)

Sol. $x + 2y + 3z = 42$

0 $x + 2y = 42 \Rightarrow$ 22 cases

1 $x + 2y = 39 \Rightarrow$ 19 cases

2 $x + 2y = 36 \Rightarrow$ 19 cases

3 $x + 2y = 33 \Rightarrow$ 17 cases

4 $x + 2y = 30 \Rightarrow$ 16 cases

5 $x + 2y = 27 \Rightarrow$ 14 cases

6 $x + 2y = 24 \Rightarrow$ 13 cases

7 $x + 2y = 21 \Rightarrow$ 11 cases

8 $x + 2y = 18 \Rightarrow$ 10 cases

9 $x + 2y = 15 \Rightarrow$ 8 cases

10 $x + 2y = 12 \Rightarrow$ 7 cases

11 $x + 2y = 9 \Rightarrow$ 5 cases

12 $x + 2y = 6 \Rightarrow$ 4 cases

13 $x + 2y = 3 \Rightarrow$ 2 cases

14 $x + 2y = 0 \Rightarrow$ 1 cases

23.

24.

25.

26.

27.

28.

29.

30.