

Class XII : MathsChapter

$$|\vec{a}| = \sqrt{1^2 + (-2)^2 + 3^2} = \sqrt{1+4+9} = \sqrt{14}$$

$$|\vec{b}| = \sqrt{3^2 + (-2)^2 + 1^2} = \sqrt{9+4+1} = \sqrt{14}$$

$$\begin{aligned} \text{Now, } \vec{a} \cdot \vec{b} &= (\hat{i} - 2\hat{j} + 3\hat{k})(3\hat{i} - 2\hat{j} + \hat{k}) \\ &= 1 \cdot 3 + (-2)(-2) + 3 \cdot 1 \\ &= 3 + 4 + 3 \\ &= 10 \end{aligned}$$

Also, we know that  $\vec{a} \cdot \vec{b} = |\vec{a}||\vec{b}|\cos\theta$ .

$$\therefore 10 = \sqrt{14}\sqrt{14}\cos\theta$$

$$\Rightarrow \cos\theta = \frac{10}{14}$$

$$\Rightarrow \theta = \cos^{-1}\left(\frac{5}{7}\right)$$

**Question 3:**

Find the projection of the vector  $\hat{i} - \hat{j}$  on the vector  $\hat{i} + \hat{j}$ .

Answer

$$\text{Let } \vec{a} = \hat{i} - \hat{j} \text{ and } \vec{b} = \hat{i} + \hat{j}.$$

Now, projection of vector  $\vec{a}$  on  $\vec{b}$  is given by,

$$\frac{1}{|\vec{b}|}(\vec{a} \cdot \vec{b}) = \frac{1}{\sqrt{1+1}}\{1 \cdot 1 + (-1)(1)\} = \frac{1}{\sqrt{2}}(1-1) = 0$$

Hence, the projection of vector  $\vec{a}$  on  $\vec{b}$  is 0.

**Question 4:**

Find the projection of the vector  $\hat{i} + 3\hat{j} + 7\hat{k}$  on the vector  $7\hat{i} - \hat{j} + 8\hat{k}$ .

Answer

$$\text{Let } \vec{a} = \hat{i} + 3\hat{j} + 7\hat{k} \text{ and } \vec{b} = 7\hat{i} - \hat{j} + 8\hat{k}.$$

Now, projection of vector  $\vec{a}$  on  $\vec{b}$  is given by,

$$\frac{1}{|\vec{b}|}(\vec{a} \cdot \vec{b}) = \frac{1}{\sqrt{7^2 + (-1)^2 + 8^2}} \{1(7) + 3(-1) + 7(8)\} = \frac{7-3+56}{\sqrt{49+1+64}} = \frac{60}{\sqrt{114}}$$

**Question 5:**

Show that each of the given three vectors is a unit vector:

$$\frac{1}{7}(2\hat{i} + 3\hat{j} + 6\hat{k}), \frac{1}{7}(3\hat{i} - 6\hat{j} + 2\hat{k}), \frac{1}{7}(6\hat{i} + 2\hat{j} - 3\hat{k})$$

Also, show that they are mutually perpendicular to each other.

Answer

$$\text{Let } \vec{a} = \frac{1}{7}(2\hat{i} + 3\hat{j} + 6\hat{k}) = \frac{2}{7}\hat{i} + \frac{3}{7}\hat{j} + \frac{6}{7}\hat{k},$$

$$\vec{b} = \frac{1}{7}(3\hat{i} - 6\hat{j} + 2\hat{k}) = \frac{3}{7}\hat{i} - \frac{6}{7}\hat{j} + \frac{2}{7}\hat{k},$$

$$\vec{c} = \frac{1}{7}(6\hat{i} + 2\hat{j} - 3\hat{k}) = \frac{6}{7}\hat{i} + \frac{2}{7}\hat{j} - \frac{3}{7}\hat{k}.$$

$$|\vec{a}| = \sqrt{\left(\frac{2}{7}\right)^2 + \left(\frac{3}{7}\right)^2 + \left(\frac{6}{7}\right)^2} = \sqrt{\frac{4}{49} + \frac{9}{49} + \frac{36}{49}} = 1$$

$$|\vec{b}| = \sqrt{\left(\frac{3}{7}\right)^2 + \left(-\frac{6}{7}\right)^2 + \left(\frac{2}{7}\right)^2} = \sqrt{\frac{9}{49} + \frac{36}{49} + \frac{4}{49}} = 1$$

$$|\vec{c}| = \sqrt{\left(\frac{6}{7}\right)^2 + \left(\frac{2}{7}\right)^2 + \left(-\frac{3}{7}\right)^2} = \sqrt{\frac{36}{49} + \frac{4}{49} + \frac{9}{49}} = 1$$

Thus, each of the given three vectors is a unit vector.

$$\vec{a} \cdot \vec{b} = \frac{2}{7} \times \frac{3}{7} + \frac{3}{7} \times \left(-\frac{6}{7}\right) + \frac{6}{7} \times \frac{2}{7} = \frac{6}{49} - \frac{18}{49} + \frac{12}{49} = 0$$

$$\vec{b} \cdot \vec{c} = \frac{3}{7} \times \frac{6}{7} + \left(-\frac{6}{7}\right) \times \frac{2}{7} + \frac{2}{7} \times \left(-\frac{3}{7}\right) = \frac{18}{49} - \frac{12}{49} - \frac{6}{49} = 0$$

$$\vec{c} \cdot \vec{a} = \frac{6}{7} \times \frac{2}{7} + \frac{2}{7} \times \frac{3}{7} + \left(-\frac{3}{7}\right) \times \frac{6}{7} = \frac{12}{49} + \frac{6}{49} - \frac{18}{49} = 0$$

Hence, the given three vectors are mutually perpendicular to each other.

**Question 6:**

Find  $|\vec{a}|$  and  $|\vec{b}|$ , if  $(\vec{a} + \vec{b}) \cdot (\vec{a} - \vec{b}) = 8$  and  $|\vec{a}| = 8|\vec{b}|$ .

Answer

$$(\vec{a} + \vec{b}) \cdot (\vec{a} - \vec{b}) = 8$$

$$\Rightarrow \vec{a} \cdot \vec{a} - \vec{a} \cdot \vec{b} + \vec{b} \cdot \vec{a} - \vec{b} \cdot \vec{b} = 8$$

$$\Rightarrow |\vec{a}|^2 - |\vec{b}|^2 = 8$$

$$\Rightarrow (8|\vec{b}|)^2 - |\vec{b}|^2 = 8 \quad [|\vec{a}| = 8|\vec{b}|]$$

$$\Rightarrow 64|\vec{b}|^2 - |\vec{b}|^2 = 8$$

$$\Rightarrow 63|\vec{b}|^2 = 8$$

$$\Rightarrow |\vec{b}|^2 = \frac{8}{63}$$

$$\Rightarrow |\vec{b}| = \sqrt{\frac{8}{63}} \quad [\text{Magnitude of a vector is non-negative}]$$

$$\Rightarrow |\vec{b}| = \frac{2\sqrt{2}}{3\sqrt{7}}$$

$$|\vec{a}| = 8|\vec{b}| = \frac{8 \times 2\sqrt{2}}{3\sqrt{7}} = \frac{16\sqrt{2}}{3\sqrt{7}}$$

**Question 7:**

Evaluate the product  $(3\vec{a} - 5\vec{b}) \cdot (2\vec{a} + 7\vec{b})$ .

Answer

$$(3\vec{a} - 5\vec{b}) \cdot (2\vec{a} + 7\vec{b})$$

$$= 3\vec{a} \cdot 2\vec{a} + 3\vec{a} \cdot 7\vec{b} - 5\vec{b} \cdot 2\vec{a} - 5\vec{b} \cdot 7\vec{b}$$

$$= 6\vec{a} \cdot \vec{a} + 21\vec{a} \cdot \vec{b} - 10\vec{a} \cdot \vec{b} - 35\vec{b} \cdot \vec{b}$$

$$= 6|\vec{a}|^2 + 11\vec{a} \cdot \vec{b} - 35|\vec{b}|^2$$

**Question 8:**

Find the magnitude of two vectors  $\vec{a}$  and  $\vec{b}$ , having the same magnitude and such that the angle between them is  $60^\circ$  and their scalar product is  $\frac{1}{2}$ .

Answer

Let  $\theta$  be the angle between the vectors  $\vec{a}$  and  $\vec{b}$ .

It is given that  $|\vec{a}| = |\vec{b}|$ ,  $\vec{a} \cdot \vec{b} = \frac{1}{2}$ , and  $\theta = 60^\circ$ . ... (1)

We know that  $\vec{a} \cdot \vec{b} = |\vec{a}||\vec{b}|\cos\theta$ .

$$\therefore \frac{1}{2} = |\vec{a}||\vec{a}|\cos 60^\circ \quad [\text{Using (1)}]$$

$$\Rightarrow \frac{1}{2} = |\vec{a}|^2 \times \frac{1}{2}$$

$$\Rightarrow |\vec{a}|^2 = 1$$

$$\Rightarrow |\vec{a}| = |\vec{b}| = 1$$

**Question 9:**

Find  $|\vec{x}|$ , if for a unit vector  $\vec{a}$ ,  $(\vec{x} - \vec{a}) \cdot (\vec{x} + \vec{a}) = 12$ .

Answer

$$(\vec{x} - \vec{a}) \cdot (\vec{x} + \vec{a}) = 12$$

$$\Rightarrow \vec{x} \cdot \vec{x} + \vec{x} \cdot \vec{a} - \vec{a} \cdot \vec{x} - \vec{a} \cdot \vec{a} = 12$$

$$\Rightarrow |\vec{x}|^2 - |\vec{a}|^2 = 12$$

$$\Rightarrow |\vec{x}|^2 - 1 = 12 \quad [|\vec{a}| = 1 \text{ as } \vec{a} \text{ is a unit vector}]$$

$$\Rightarrow |\vec{x}|^2 = 13$$

$$\therefore |\vec{x}| = \sqrt{13}$$

**Question 10:**

If  $\vec{a} = 2\hat{i} + 2\hat{j} + 3\hat{k}$ ,  $\vec{b} = -\hat{i} + 2\hat{j} + \hat{k}$  and  $\vec{c} = 3\hat{i} + \hat{j}$  are such that  $\vec{a} + \lambda\vec{b}$  is perpendicular to  $\vec{c}$ , then find the value of  $\lambda$ .

Answer

The given vectors are  $\vec{a} = 2\hat{i} + 2\hat{j} + 3\hat{k}$ ,  $\vec{b} = -\hat{i} + 2\hat{j} + \hat{k}$ , and  $\vec{c} = 3\hat{i} + \hat{j}$ .

Now,

$$\vec{a} + \lambda\vec{b} = (2\hat{i} + 2\hat{j} + 3\hat{k}) + \lambda(-\hat{i} + 2\hat{j} + \hat{k}) = (2 - \lambda)\hat{i} + (2 + 2\lambda)\hat{j} + (3 + \lambda)\hat{k}$$

If  $(\vec{a} + \lambda\vec{b})$  is perpendicular to  $\vec{c}$ , then

$$(\vec{a} + \lambda\vec{b}) \cdot \vec{c} = 0.$$

$$\Rightarrow [(2 - \lambda)\hat{i} + (2 + 2\lambda)\hat{j} + (3 + \lambda)\hat{k}] \cdot (3\hat{i} + \hat{j}) = 0$$

$$\Rightarrow (2 - \lambda)3 + (2 + 2\lambda)1 + (3 + \lambda)0 = 0$$

$$\Rightarrow 6 - 3\lambda + 2 + 2\lambda = 0$$

$$\Rightarrow -\lambda + 8 = 0$$

$$\Rightarrow \lambda = 8$$

Hence, the required value of  $\lambda$  is 8.

**Question 11:**

Show that  $|\vec{a}|\vec{b} + |\vec{b}|\vec{a}$  is perpendicular to  $|\vec{a}|\vec{b} - |\vec{b}|\vec{a}$ , for any two nonzero vectors  $\vec{a}$  and  $\vec{b}$

Answer

$$(|\vec{a}|\vec{b} + |\vec{b}|\vec{a}) \cdot (|\vec{a}|\vec{b} - |\vec{b}|\vec{a})$$

$$= |\vec{a}|^2 \vec{b} \cdot \vec{b} - |\vec{a}||\vec{b}|\vec{b} \cdot \vec{a} + |\vec{b}||\vec{a}|\vec{a} \cdot \vec{b} - |\vec{b}|^2 \vec{a} \cdot \vec{a}$$

$$= |\vec{a}|^2 |\vec{b}|^2 - |\vec{b}|^2 |\vec{a}|^2$$

$$= 0$$

Hence,  $|\vec{a}|\vec{b} + |\vec{b}|\vec{a}$  and  $|\vec{a}|\vec{b} - |\vec{b}|\vec{a}$  are perpendicular to each other.

Question 12:

If  $\vec{a} \cdot \vec{a} = 0$  and  $\vec{a} \cdot \vec{b} = 0$ , then what can be concluded about the vector  $\vec{b}$ ?

Answer

It is given that  $\vec{a} \cdot \vec{a} = 0$  and  $\vec{a} \cdot \vec{b} = 0$ .

Now,

$$\vec{a} \cdot \vec{a} = 0 \Rightarrow |\vec{a}|^2 = 0 \Rightarrow |\vec{a}| = 0$$

$\therefore \vec{a}$  is a zero vector.

Hence, vector  $\vec{b}$  satisfying  $\vec{a} \cdot \vec{b} = 0$  can be any vector.

Question 13:

If  $\vec{a}, \vec{b}, \vec{c}$  are unit vectors such that  $\vec{a} + \vec{b} + \vec{c} = \vec{0}$  find the value of  $\vec{a} \cdot \vec{b} + \vec{b} \cdot \vec{c} + \vec{c} \cdot \vec{a}$ .

Answer 13:

$$\begin{aligned} |\vec{a} + \vec{b} + \vec{c}|^2 &= (\vec{a} + \vec{b} + \vec{c}) \cdot (\vec{a} + \vec{b} + \vec{c}) \\ &= |\vec{a}|^2 + |\vec{b}|^2 + |\vec{c}|^2 + 2(\vec{a} \cdot \vec{b} + \vec{b} \cdot \vec{c} + \vec{c} \cdot \vec{a}) \\ &\Rightarrow 0 = 1 + 1 + 1 + 2(\vec{a} \cdot \vec{b} + \vec{b} \cdot \vec{c} + \vec{c} \cdot \vec{a}) \\ &\Rightarrow (\vec{a} \cdot \vec{b} + \vec{b} \cdot \vec{c} + \vec{c} \cdot \vec{a}) = \frac{-3}{2} \end{aligned}$$

#### Question 14:

If either vector  $\vec{a} = \vec{0}$  or  $\vec{b} = \vec{0}$ , then  $\vec{a} \cdot \vec{b} = 0$ . But the converse need not be true. Justify your answer with an example.

Answer

Consider  $\vec{a} = 2\hat{i} + 4\hat{j} + 3\hat{k}$  and  $\vec{b} = 3\hat{i} + 3\hat{j} - 6\hat{k}$ .

Then,

$$\vec{a} \cdot \vec{b} = 2.3 + 4.3 + 3(-6) = 6 + 12 - 18 = 0$$

We now observe that:

$$|\vec{a}| = \sqrt{2^2 + 4^2 + 3^2} = \sqrt{29}$$

$$\therefore \vec{a} \neq \vec{0}$$

$$|\vec{b}| = \sqrt{3^2 + 3^2 + (-6)^2} = \sqrt{54}$$

$$\therefore \vec{b} \neq \vec{0}$$

Hence, the converse of the given statement need not be true.

#### Question 15:

If the vertices A, B, C of a triangle ABC are (1, 2, 3), (-1, 0, 0), (0, 1, 2), respectively,

then find  $\sphericalangle ABC$ . [ $\sphericalangle ABC$  is the angle between the vectors  $\overrightarrow{BA}$  and  $\overrightarrow{BC}$ ]

Answer

The vertices of  $\triangle ABC$  are given as A (1, 2, 3), B (-1, 0, 0), and C (0, 1, 2).

Also, it is given that  $\sphericalangle ABC$  is the angle between the vectors  $\overrightarrow{BA}$  and  $\overrightarrow{BC}$ .



$$\overline{BA} = \{1 - (-1)\}\hat{i} + (2 - 0)\hat{j} + (3 - 0)\hat{k} = 2\hat{i} + 2\hat{j} + 3\hat{k}$$

$$\overline{BC} = \{0 - (-1)\}\hat{i} + (1 - 0)\hat{j} + (2 - 0)\hat{k} = \hat{i} + \hat{j} + 2\hat{k}$$

$$\therefore \overline{BA} \cdot \overline{BC} = (2\hat{i} + 2\hat{j} + 3\hat{k}) \cdot (\hat{i} + \hat{j} + 2\hat{k}) = 2 \times 1 + 2 \times 1 + 3 \times 2 = 2 + 2 + 6 = 10$$

$$|\overline{BA}| = \sqrt{2^2 + 2^2 + 3^2} = \sqrt{4 + 4 + 9} = \sqrt{17}$$

$$|\overline{BC}| = \sqrt{1^2 + 1^2 + 2^2} = \sqrt{6}$$

Now, it is known that:

$$\overline{BA} \cdot \overline{BC} = |\overline{BA}| |\overline{BC}| \cos(\angle ABC)$$

$$\therefore 10 = \sqrt{17} \times \sqrt{6} \cos(\angle ABC)$$

$$\Rightarrow \cos(\angle ABC) = \frac{10}{\sqrt{17} \times \sqrt{6}}$$

$$\Rightarrow \angle ABC = \cos^{-1}\left(\frac{10}{\sqrt{102}}\right)$$

#### Question 16:

Show that the points A (1, 2, 7), B (2, 6, 3) and C (3, 10, -1) are collinear.

Answer

The given points are A (1, 2, 7), B (2, 6, 3), and C (3, 10, -1).

$$\therefore \overline{AB} = (2 - 1)\hat{i} + (6 - 2)\hat{j} + (3 - 7)\hat{k} = \hat{i} + 4\hat{j} - 4\hat{k}$$

$$\overline{BC} = (3 - 2)\hat{i} + (10 - 6)\hat{j} + (-1 - 3)\hat{k} = \hat{i} + 4\hat{j} - 4\hat{k}$$

$$\overline{AC} = (3 - 1)\hat{i} + (10 - 2)\hat{j} + (-1 - 7)\hat{k} = 2\hat{i} + 8\hat{j} - 8\hat{k}$$

$$|\overline{AB}| = \sqrt{1^2 + 4^2 + (-4)^2} = \sqrt{1 + 16 + 16} = \sqrt{33}$$

$$|\overline{BC}| = \sqrt{1^2 + 4^2 + (-4)^2} = \sqrt{1 + 16 + 16} = \sqrt{33}$$

$$|\overline{AC}| = \sqrt{2^2 + 8^2 + (-8)^2} = \sqrt{4 + 64 + 64} = \sqrt{132} = 2\sqrt{33}$$

$$\therefore |\overline{AC}| = |\overline{AB}| + |\overline{BC}|$$

Hence, the given points A, B, and C are collinear.

**Question 17:**

Show that the vectors  $2\hat{i} - \hat{j} + \hat{k}$ ,  $\hat{i} - 3\hat{j} - 5\hat{k}$  and  $3\hat{i} - 4\hat{j} - 4\hat{k}$  form the vertices of a right angled triangle.

Answer

Let vectors  $2\hat{i} - \hat{j} + \hat{k}$ ,  $\hat{i} - 3\hat{j} - 5\hat{k}$  and  $3\hat{i} - 4\hat{j} - 4\hat{k}$  be position vectors of points A, B, and C respectively.

$$\text{i.e., } \overline{OA} = 2\hat{i} - \hat{j} + \hat{k}, \overline{OB} = \hat{i} - 3\hat{j} - 5\hat{k} \text{ and } \overline{OC} = 3\hat{i} - 4\hat{j} - 4\hat{k}$$

Now, vectors  $\overline{AB}$ ,  $\overline{BC}$ , and  $\overline{AC}$  represent the sides of  $\Delta ABC$ .

$$\text{i.e., } \overline{OA} = 2\hat{i} - \hat{j} + \hat{k}, \overline{OB} = \hat{i} - 3\hat{j} - 5\hat{k}, \text{ and } \overline{OC} = 3\hat{i} - 4\hat{j} - 4\hat{k}$$

$$\therefore \overline{AB} = (1-2)\hat{i} + (-3+1)\hat{j} + (-5-1)\hat{k} = -\hat{i} - 2\hat{j} - 6\hat{k}$$

$$\overline{BC} = (3-1)\hat{i} + (-4+3)\hat{j} + (-4+5)\hat{k} = 2\hat{i} - \hat{j} + \hat{k}$$

$$\overline{AC} = (2-3)\hat{i} + (-1+4)\hat{j} + (1+4)\hat{k} = -\hat{i} + 3\hat{j} + 5\hat{k}$$

$$|\overline{AB}| = \sqrt{(-1)^2 + (-2)^2 + (-6)^2} = \sqrt{1+4+36} = \sqrt{41}$$

$$|\overline{BC}| = \sqrt{2^2 + (-1)^2 + 1^2} = \sqrt{4+1+1} = \sqrt{6}$$

$$|\overline{AC}| = \sqrt{(-1)^2 + 3^2 + 5^2} = \sqrt{1+9+25} = \sqrt{35}$$

$$\therefore |\overline{BC}|^2 + |\overline{AC}|^2 = 6 + 35 = 41 = |\overline{AB}|^2$$

Hence,  $\Delta ABC$  is a right-angled triangle.

**Question 18:**

If  $\vec{a}$  is a nonzero vector of magnitude 'a' and  $\lambda$  a nonzero scalar, then  $\lambda \vec{a}$  is unit vector if

(A)  $\lambda = 1$  (B)  $\lambda = -1$  (C)  $a = |\lambda|$

(D)  $a = \frac{1}{|\lambda|}$

Answer

Vector  $\lambda \vec{a}$  is a unit vector if  $|\lambda \vec{a}| = 1$ .

Now,

$$|\lambda \vec{a}| = 1$$

$$\Rightarrow |\lambda| |\vec{a}| = 1$$

$$\Rightarrow |\vec{a}| = \frac{1}{|\lambda|} \quad [\lambda \neq 0]$$

$$\Rightarrow a = \frac{1}{|\lambda|} \quad [|\vec{a}| = a]$$

Hence, vector  $\lambda \vec{a}$  is a unit vector if  $a = \frac{1}{|\lambda|}$ .

The correct answer is D.