Oscillations and Waves

If a particle in periodic motion moves back and *forth* (or *to* and *fro*) over the same path, then its motion is called ***oscillatory*** or ***vibratory***.

Characteristics of a Harmonic Motion

The basic quantities characterizing a periodic motion are the *amplitude*, *period* and *frequency* of vibrations.

Amplitude (*A*)

The amplitude of oscillations is the maximum displacement of a vibrating body from the position of equilibrium.

Time Period (*T*)

The time period of oscillations is defined as the time between two successive identical positions passed by the body in the same direction.

Frequency (*f*)

The frequency of oscillations is the number of cycles of vibrations of a body completed in one second. The frequency is related to the time period as

$$f = \frac{1}{T}$$

The *SI* unit of frequency is s^{-1} or Hz (hertz)

Simple Harmonic Motion

Let us consider an oscillatory particle along a straight line whose potential energy function varies as

$$U(x) = \frac{1}{2}kx^2$$

where *k* is a constant

Simple Harmonic Motion.

$x = A \sin(\omega t + \phi)$ is the *general* equation of SHM.

$$\frac{d^2x}{dt^2} = \square\square^2 x \quad or \quad \frac{d^2x}{dt^2} + \square^2 x = 0$$

Above equation is the *standard* differential equation of SHM.

The Spring-Mass System

Time period of a *spring-mass* is given by

$$T = 2\pi\sqrt{\frac{m}{k}}$$

Series and Parallel Combinations of springs

For **Series Combinations of springs** , the *equivalent* stiffness of the combination is given by

$$\frac{1}{k} = \frac{1}{k_1} + \frac{1}{k_2} \Rightarrow k = \frac{k_1 k_2}{k_1 + k_2}$$

For *parallel* **Combinations of springs**, the *equivalent* stiffness of the combination is given by

$$k = k_1 + k_2$$

ENERGY CONSERVATION IN SHM

In a *spring-mass* system, the instantaneous *potential energy* and *kinetic energy* are expressed as

$$U = \frac{1}{2}kx^2 = \frac{1}{2}kA^2 \sin^2(\omega t + \phi)$$

and $K = \frac{1}{2}mv^2 = \frac{1}{2}m\omega^2 A^2 \cos^2(\omega t + \phi)$

Since $\omega^2 = \frac{k}{m}$, therefore,

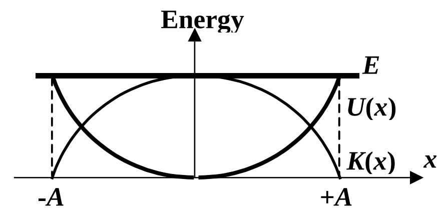
$$K = \frac{1}{2}kA^2\cos^2(\omega t+\phi)$$

The ***total mechanical energy*** is given by

$$E = K + U \quad \text{or} \quad E = \frac{1}{2}kA^2\left[\sin^2(\omega t+\phi)+\cos^2(\omega t+\phi)\right]$$

or $\quad E = \frac{1}{2}kA^2 = \text{constant}$

Thus, the total energy of SHM is constant and proportional to the square of the amplitude.



The variation of the kinetic energy , potential energy, and total energy as a function of position.

The variation of K and U as function of x is shown in figure. When $x = \pm A$, the kinetic energy is zero and the total energy is equal to the maximum potential energy.

$$E = U_{max} = \frac{1}{2}kA^2$$

There are extreme points or turning points of the SHM.

At $\quad x = 0$, $U = 0$ and the energy is purely kinetic,

i.e. $\quad E = K_{\max} = \frac{1}{2}m(\omega A)^2$

WAVE

The *wave function*

$$y = A \sin [k(x \pm vt)]$$

$$y = A \sin (kx \pm \square t)$$

The *negative* sign is used when the wave travels along the positive x – axis, and vice-versa.

Some Important Points

$$k = \frac{2\pi}{\lambda}$$

k is called the ***wave number*** and □ is called the ***wavelength***.).

$$\square = \frac{2\pi}{T} = 2\square f$$

where □ is called the ***angular frequency*** (measured in rad/s)and T is the time period and f is the frequency.

Time Period (T)

$$T = \frac{1}{f}$$

Frequency (*f*)

The number of complete vibrations of a point on the string that occur in one second or, the number of wavelengths that pass a given point in one second.

Wave velocity (v)

$$v = \frac{\lambda}{T} = \lambda f = \frac{\omega}{k}$$

Amplitude (A)

The maximum displacement of a particle on the medium from the equilibrium position.

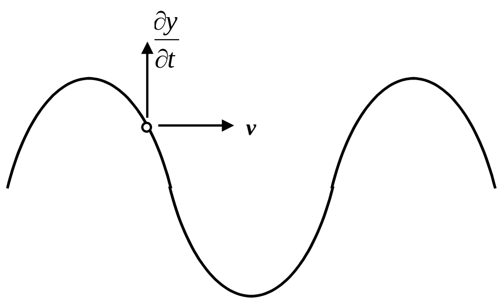
Wave Velocity and Particle Velocity

Wave velocity is the velocity of the disturbance which propagates through a medium. *It only depends on the properties of the medium and is independent of time and position.*

Particle velocity is the rate at which particle's displacement vary as a function of time, i.e.

$$\frac{\partial y}{\partial t} = \pm \omega A \cos(kx \pm \omega t + \phi)$$

The difference between the wave velocity *v* and the velocity of a particle of the medium $\left(\frac{\partial y}{\partial t}\right)$ is shown in the figure.



The velocity of a particle on a string given by $\frac{\partial y}{\partial t}$, is perpendicular to wave velocity v for a transverse wave.

VELOCITY OF A TRANSVERSE WAVE ON A STRING

$$v = \sqrt{\frac{T}{\mu}}$$

Where T is the tension in the string and μ is the mass per unit length of the string.

Note: that the velocity is measured with respect to the medium.

ENERGY TRANSPORTED BY A HARMONIC WAVE

The *average* power transmitted by the wave is

$$P_{av} = \frac{1}{2}\mu\omega^2 A^2 v$$

Where, $v = \frac{dx}{dt}$ is the wave velocity.

The mass per unit length of a wire is given by $\mu = \rho\, a$

Where, ρ is the density and a is the cross-sectional area.

The intensity of the wave is given by

$$I = \frac{P_{av}}{a} = \frac{1}{2}\rho\omega^2 A^2 v$$

Interference: Adding waves that differ in Phase only

Consider two waves travelling on the same string with the same amplitude, wavelength and frequency but with a constant phase difference as described by the following equations:

$$y_1 = a\sin(kx - \omega t)$$

$$y_2 = a\sin(kx - \omega t + \phi)$$

$$y = y_1 + y_2 = a[\sin(kx - \omega t) + \sin(kx - \omega t + \phi)]$$

Using trigonometric identity

$$y = \left(2a\cos\frac{\phi}{2}\right)\sin\left(kx - \omega t - \frac{\phi}{2}\right)$$

The term $(2a\cos\frac{\varphi}{2})$ is interpreted as the new amplitude of the wave.

$$y = A\sin\left(kx - \omega t - \frac{\phi}{2}\right)$$

where $A = 2a\cos(\frac{\phi}{2})$

Constructive & Destructive Interference

When $\phi = 2n\pi$ (where $n = 0, 1, 2, \ldots$), the positions of crests and troughs of each wave coincide with the positions of the crests and troughs of another wave.

$$A = 2a\cos\left[(2n-1)\frac{\pi}{2}\right] = 0$$

If the two waves have different amplitudes a_1 and a_2, respectively, the resultant amplitude is given by

$$A = \sqrt{a_1^2 + a_2^2 + 2a_1a_2\cos\phi}$$

Where, ϕ is the constant phase difference.

If $\phi = 2n\pi$ where $n = 0, 1, 2, \ldots\ldots\ldots$

$A_{max} = a_1 + a_2$ **(constructive)**

If $\phi = (2n-1)\pi$ where $n = 1, 2, 3, \ldots\ldots\ldots$

$A_{min} = a_1 - a_2$ **(destructive)**

Since, intensity is proportional to the square of the amplitude. Therefore,

$$I = I_1 + I_2 + 2\sqrt{I_1I_2}\cos\phi$$

If $\phi = 2n\pi$ where $n = 0, 1, 2, \ldots\ldots$

$I_{max} = \left(\sqrt{I_1} + \sqrt{I_2}\right)^2$

If $\phi = (2n-1)\pi$ where $n = 1, 2, 3, \ldots\ldots\ldots$

$I_{min} = \left(\sqrt{I_1} - \sqrt{I_2}\right)^2$

Beats:

$$f_{beat} = f_1 - f_2$$

Standing Waves: *Adding Waves That Differ In Direction Only*

When two waves identical in all respects, but travelling in opposite direction along a straight line, superimpose on each other, standing waves are produced.

Let $y_1 = A\sin(\omega t - kx)$ and $y_2 = A\sin(\omega t + kx)$

□ $y = y_1 + y_2 = 2A\cos kx \sin\omega t$

$2A\cos kx$ represents the amplitude of particle located at 'x'.

Some Important Points:

1. There are positions along the string for which the amplitude of oscillation is always zero (called ***nodes***), and other positions where the amplitude of oscillation is *always 2a* (called ***antinodes***).
2. Distance between consecutive nodes = distance between consecutive antinodes $= \frac{\lambda}{2}$.

Positions of Nodes

Any position for which kx is a multiple of π yields a zero; all these are the positions of nodes

$$kx = 0, \pi, 2\pi, 3\pi, \ldots\ldots$$

Since $k = \frac{2\pi}{\lambda}$, therefore

$$x = 0, \frac{\lambda}{2}, \frac{2\lambda}{2}, \frac{3\lambda}{2}, \ldots\ldots$$

Note that the distance between two successive nodes is $\frac{\lambda}{2}$.

Positions for which kx is an *odd* multiple of $\frac{\pi}{2}$ yields maximum value of $2a\sin kx$; all these are the positions of *antinodes*.

$$kx = \frac{\pi}{2}, \frac{3\pi}{2}, \frac{5\pi}{2}, \ldots\ldots \qquad \text{or} \qquad x = \frac{\lambda}{4}, \frac{3\lambda}{4}, \frac{5\lambda}{4}, \ldots\ldots$$

Note that the distance between two successive antinodes is $\frac{\lambda}{2}$. Also, *nodes* and *antinodes* occur alternatively and equally spaced from each other.

The standing wave patterns possible on the string are called the ***normal modes***.

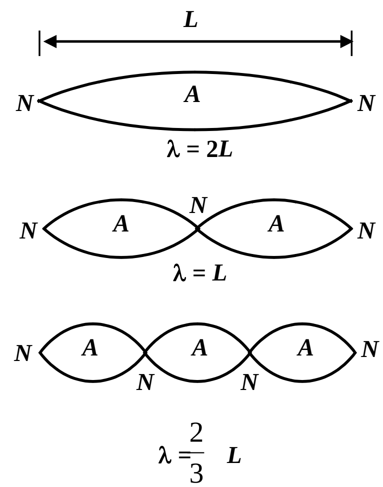
Figure shows that the wavelength of the normal modes must be

$\lambda = 2L, L, \frac{2}{3}L, \ldots\ldots\ldots$

In general

$\lambda = \frac{2L}{n}$

where $n = 1, 2, 3\ldots\ldots$



Normal modes of a string fixed at both ends.
(a) Fundamental frequency or first harmonic
(b) First overtone or second harmonic
(c) Second overtone or third harmonic

The respective frequencies are given by

Thus,

$= \frac{v}{2L}, \frac{v}{L}, \frac{3v}{2L}, \ldots\ldots$

In general,

$f = n\left(\frac{v}{2L}\right)$

where $n = 1, 2, 3, \ldots\ldots$

The *first* frequency $\left(\frac{v}{2L}\right)$ is called the ***fundamental frequency*** (or the ***first harmonic***). The *second* frequency ($n = 2$) is called the ***first overtone*** (or ***second harmonic***), the *third* frequency

(n = 3) is called the ***second overtone*** (or ***third harmonic***) and so on.

REFLECTION AND TRANSMISSION OF WAVES

Since the tensions are the same, the relative magnitudes of the wave velocities are determined by the mass densities.

$$v = \sqrt{\frac{T}{\mu}} \Rightarrow v \propto \frac{1}{\sqrt{\mu}}$$

Consider two strings of mass densities $\square_1$ and $\square_2$ joined together to form a single string. The respective wave velocities in the two strings will be v_1 and v_2 which are given by

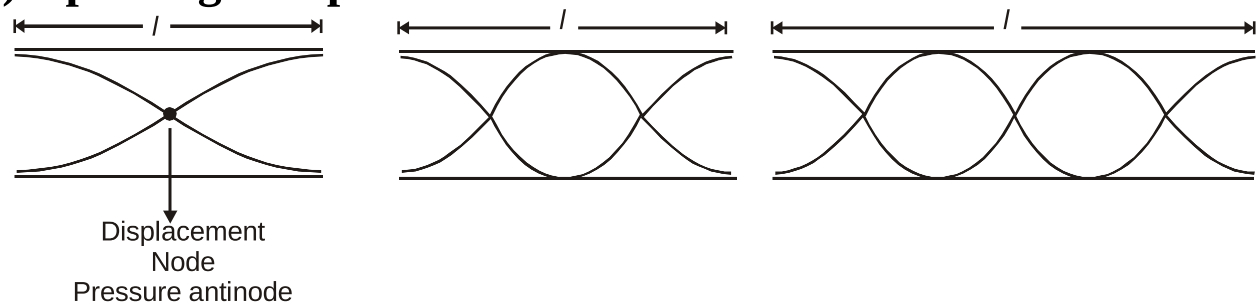
$$\frac{v_1}{v_2} = \sqrt{\frac{\mu_2}{\mu_1}}$$

Standing Sound Waves

Standing waves can be produced in air columns, for example, in organ pipes, flutes, and other wind instruments because sound waves are reflected both at a closed end and at an *open* end of a pipe. In an ***open pipe*** both the ends are open, whereas in a ***closed pipe*** one end is closed.

In this case, longitudinal stationary waves are formed

(a) Open organ Pipe :



$l = \frac{\lambda}{2}$ or □ $= 2l$ $\quad$ $l =$ □ $\quad$ $l = \frac{3\lambda}{2}$

$\nu_1 = \frac{V}{\lambda} = \frac{V}{2l}$ $\quad$ $\nu = \frac{V}{l} = 2\nu_1$ $\quad$ $\lambda = \frac{2l}{3}$

1st harmonic or $\quad$ 2nd $\quad$ harmonic

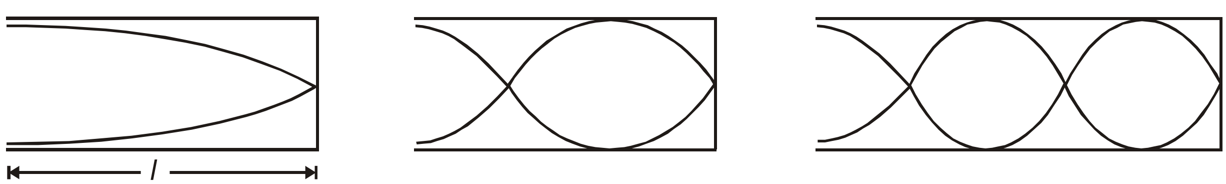
$\nu_0 = \frac{3V}{2l} = 3\nu_1$

Fundamental mode $\quad$ Ist overtone $\quad$ 3rd

harmonic

2nd overtone

(b) Closed organ pipe :



$l = \frac{\lambda}{4} \Rightarrow \lambda = 4l$ $\quad$ $l = \frac{3\lambda}{4} \Rightarrow \lambda = \frac{4l}{3}$ $\quad$ $l = \frac{5\lambda}{4}, \lambda = \frac{4l}{5}$

$\nu = \frac{V}{4l}$ $\quad$ $\nu = \frac{3V}{4l}$ $\quad$ $\nu = \frac{5V}{4l}$

Fundamental mode I^{st} overtone 2^{nd} overtone

st harmonic 3^{rd} harmonic

5^{th} harmonic

Intensity Level: *The Decibel Scale*

The intensity level β in terms of the ***decibel*** (***dB***), which is defined as

$$\beta = 10\log\frac{1}{I_0}$$

Where, I is the measured *intensity* and $I_0$is some reference value.

DOPPLER EFFECT

*The change in frequency due to the motion of the source, observer, or both is called the **Doppler Effect**.*

Table: Doppler Frequencies (f') for Different Situations

	Source Stationary	Source Towards Observer	Source Away from Observer
Observer Stationary	f	$f\left(\frac{v}{v-v_s}\right)$	$f\left(\frac{v}{v+v_s}\right)$
Observer towards source	$f\left(\frac{v+v_o}{v}\right)$	$f\left(\frac{v+v_o}{v-v_s}\right)$	$f\left(\frac{v+v_o}{v+v_s}\right)$
Observer away from source	$f\left(\frac{v-v_o}{v}\right)$	$f\left(\frac{v-v_o}{v-v_s}\right)$	$f\left(\frac{v-v_o}{v+v_s}\right)$