BINOMIALTHEOREM

BINOMIAL EXPRESSION

An expression containing two terms, is called a binomial. For example a + b/x, x + 1/y, a – y^2 etc. are binomial expressions. In general an expression containing more than two terms is called a multinomial.

STATEMENT OF BINOMIAL THEOREM

$(x + a)^n = {}^nC_0\, x^n + {}^nC_1\, x^{n-1} a + {}^nC_2\, x^{n-2} a^2 + \ldots. + {}^nC_n\, a^n$

(where $n \in N$)

(i) ${}^nC_0, {}^nC_1 {}^nC_2, \ldots., {}^nC_n$ are called binomial coefficients,

where ${}^nC_r = \frac{n!}{r!(n-r)!}$

(ii) If ${}^nC_x = {}^nC_y$, then either x = y or x + y = n.

(iii) There are (n + 1) terms in the expansion of $(x + a)^n$.

(iv) The sum of powers of a and x in each term of expansion is n.

(v) **<u>Greatest Binomial Coefficient</u>**

- If n is even : When $r = \frac{n}{2}$ i.e. ${}^nC_{n/2}$ takes the maximum value.
- If n is odd: $r = \frac{n-1}{2}$ or $\frac{n+1}{2}$ i.e. ${}^nC_{\frac{n-1}{2}} = {}^nC_{\frac{n+1}{2}}$ and take the maximum value.

(vi) General term of the expansion

$T_{r+1} = {}^nC_r\, x^{n-r}\, a^r$.

(vii) Middle term of the expansion

- If n is even $T_{\left(\frac{n}{2}+1\right)}$ is the middle term. So the middle term $T_{\left(\frac{n}{2}+1\right)} = {}^nC_{n/2}\, x^{n/2}\, y^{n/2}$

- If n is odd $T_{\left(\frac{n+1}{2}\right)}$ and $T_{\left(\frac{n+3}{2}\right)}$ are middle terms. So the middle terms are $T_{\left(\frac{n+1}{2}\right)} = {}^nC_{\left(\frac{n-1}{2}\right)}\, x^{\frac{n+1}{2}}\, y^{\frac{n-1}{2}}$ and $T_{\left(\frac{n+3}{2}\right)} = {}^nC_{\left(\frac{n+1}{2}\right)}\, x^{\frac{n-1}{2}}\, y^{\frac{n+1}{2}}$

(viii) $\frac{T_{r+1}}{T_r} = \frac{(n-r+1)}{r}\cdot\left(\frac{a}{x}\right)$

SOME RELATIONS IN BINOMIAL COEFFICIENTS

$(x + a)^n = {}^nC_0x^n + {}^nC_1x^{n-1}a + {}^nC_2x^{n-2}a^2 + \ldots.. + {}^nC_na^n$ (1)

$(1 + x)^n = {}^nC_0 + {}^nC_1x + {}^nC_2x^2 + \ldots.. + {}^nC_rx^r + \ldots\ldots + {}^nC_nx^n$ (2)

If x = 1

$2^n = {}^nC_0 + {}^nC_1 + \ldots\ldots + {}^nC_n$ (Sum of binomial coefficients)

Here (1) and (2) are identities and will hold for all values of x and a (both real and complex). On differentiating and integrating these identities with respect to x, the relations obtained after differentiation or integration are also identities and hold for all x. e.g. if we differentiating (2) we get

$n(1+x)^{n-1} = {}^nC_1 + 2\,{}^nC_2x + \ldots\ldots + n\,{}^nC_nx^{n-1}$

for x = 1 the above leads to

$n.2^{n-1} = {}^nC_1 + 2.{}^nC_2 + \ldots. + n.{}^nC_n$

Now on integrating (2)

$$\frac{(1+x)^{n+1}}{n+1} = {}^nC_0.x + {}^nC_1\frac{x^2}{2} + {}^nC_2\frac{x^3}{3} + \ldots\ldots + {}^nC_n\frac{x^{n+1}}{n+1} + k$$

(k is a constant of integration)

If we put x = 0, we get $k = \frac{1}{n+1}$

$$\frac{(1+x)^{n+1}-1}{(n+1)} = {}^nC_0x + {}^nC_1\frac{x^2}{2} + {}^nC_2\frac{x^3}{3} + \ldots\ldots. + {}^nC_n\frac{x^{n+1}}{n+1}$$

Put x = 1

$$ {}^nC_0 + \frac{{}^nC_1}{2} + \frac{{}^nC_2}{3} + \ldots.. + \frac{{}^nC_n}{n+1} = \frac{2^{n+1}-1}{(n+1)}$$

NUMERICALLY GREATEST TERM OF BINOMIAL EXPANSION

$(a+x)^n = C_0a^n + C_1a^{n-1}x + \ldots. + C_nx^n.$

$$\left|\frac{T_{r+1}}{T_r}\right| = \left|\frac{{}^nC_r}{{}^nC_{r-1}}\right|\left|\frac{x}{a}\right| \qquad = \left|\frac{n-r+1}{r}\right|\left|\frac{x}{a}\right|$$

Take $\left|\frac{n-r+1}{r}\right|\left|\frac{x}{a}\right| \geq 1$ $\qquad \{As\ |T_{r+1}| \geq |T_r|\}$

$$r \leq \frac{n+1}{1+\left|\frac{a}{x}\right|}$$

So the greatest term will be T_{r+1} where r = $\left[\dfrac{n+1}{1+\left|\frac{a}{x}\right|}\right]$

[.] denotes the greatest integer function.

Note: If $\dfrac{n+1}{1+\left|\frac{a}{x}\right|}$ itself is a natural number, then $T_r = T_{r+1}$ and both of them are the numerically greatest terms.

BINOMIAL THEOREM FOR ANY INDEX

$(1 + x)^n = 1 + nx + n(n-1)\dfrac{x^2}{2!} + \ldots + \dfrac{n(n-1)(n-2)\ldots(n-r+1)}{r!}.x^r$

$+\ldots+$ terms up to ∞.

where 'n' may be negative integer or positive or negative fraction.

- expansion is valid only when $|x| < 1$ or $-1 < x < 1$
- As the series never terminate, the number of terms in series is infinite
- ${}^nC_r$ can not be used because it is defined only for natural number.
- General term of the series $(1 + x)^{-n} = T_{r+1} = (-1)^r \dfrac{n(n+1)(n+2)\ldots(n+r-1)}{r!}.x^r$
- General term of series $(1 - x)^{-n}$
  $T_{r+1} = \dfrac{n(n+1)(n+2)\ldots(n+r-1)}{r!}.x^r$

Other important expansion

- $(1 + x)^{-1} = 1 - x + x^2 - x^3 + \ldots + (-1)^r.x^r + \ldots$
- $(1 - x)^{-1} = 1 + x + x^2 + x^3 + \ldots + x^r + \ldots$