SEQUENCE AND SERIES

ARITHMETIC PROGRESSION (A.P.):

If a is the first term and d the common difference, the A.P. Then nth term a_n is given by $a_n = a + (n-1)d$. The sum S_n of the first n terms of such an A.P. is given by $S_n = \frac{n}{2}(2a + (n-1)d) = \frac{n}{2}(a + l)$ where l is the last term.

- If $a_1, a_2, a_3, \ldots\ldots a_n$ are in A.P. then $a_1 + a_n = a_2 + a_{n-1} = a_3 + a_{n-2} = \ldots\ldots$ and so on.
- If n^{th} term of any sequence is a linear expression in n, then the sequence is an AP, whose common difference is the coefficient of n.
- If sum of n terms of any sequence is a quadratic in n, whose constant term is zero, then the sequence is an AP, whose common difference is twice the coefficient of n^2. If the constant term is non-zero, then it is an A.P. from second term onwards.
- If $a_1, a_2, \ldots\ldots\ldots a_n \ldots\ldots$ is an AP then $a^{a_1}, a^{a_2}, \ldots\ldots\ldots\ldots a^{a_n}, \ldots\ldots\ldots\ldots (a > 0)$ is a G.P.
- If $\{t_n\}$ is an A.P., then the common difference, d, is given by $d = \frac{t_p - t_q}{p - q}$, $(p, q \in N)$.

Arithmetic Mean(s):

- If three terms are in A.P., then the middle term is called the arithmetic mean (A.M.) between the other two i.e. if a, b, c are in A.P. then $b = \frac{a+c}{2}$ is the A.M. of a and c.
- If $a_1, a_2, \ldots a_n$ are n numbers then the arithmetic mean (A) of these numbers is $A = \frac{1}{n}(a_1 + a_2 + a_3 + \ldots\ldots + a_n)$.

GEOMETRIC PROGRESSION (G.P.):

If 'a' is the first term and 'r' the common ratio of a G.P. can be written as a, ar, ar^2, ... the nth term 'a_n' is given by $a_n = ar^{n-1}$.

The sum S_n of the first n terms is $S_n = \frac{a(r^n - 1)}{r-1}, \quad r \neq 1$

$= na, \quad r = 1$

f $-1 < r < 1$, then the sum of the infinite G.P. $a + ar + ar^2 + \ldots\ldots = \frac{a}{1-r}$.

- If $a_1, a_2, \ldots, a_n$ are in G.P., then $a_1 a_n = a_2 a_{n-1} = a_3 a_{n-2} = \ldots$
- If $a_1, a_2, a_3, \ldots\ldots$ is a G.P. (each $a_I > 0$), then $\log a_1$, $\log a_2$, $\log a_3 \ldots$ is an A.P. The converse is also true.

Geometric Means:

- If three terms are in G.P., then the middle term is called the geometric mean (G.M.) between the two. So if a,b,c are in G.P. then $b=\sqrt{ac}$ is the geometric mean of a and c.
- If $a_1, a_2 \ldots\ldots a_n$ are positive numbers then their G.M (G) is given by

$G = (a_1 a_2 a_3 ...... a_n)^{1/n}$. If $G_1, G_2, ...G_n$ are n geometric means between a and b then $a, G_1, G_2, ..., G_n, b$ will be a G.P. Here $b = a r^{n+1}$

$\Rightarrow r = \sqrt[n+1]{\frac{b}{a}} \Rightarrow G_r = a\left(\sqrt[n+1]{\frac{b}{a}}\right)^r$ where G_r is the r^{th} mean.

HARMONIC PROGRESSION (H.P.):

The nth term a_n of the H.P. is $a_n = \frac{1}{a + (n-1)d}$, where $a = \frac{1}{a_1}$, and $d = \frac{1}{a_2} - \frac{1}{a_1}$.

Properties of H.P.:

- If a and b are two non-zero numbers, then the harmonic mean of a and b is a number H such that the numbers a, H, b are in H.P. We have $H = \frac{2ab}{a+b}$.
- If $a_1, a_2, ...a_n$ are 'n' non-zero numbers, then the harmonic mean H of these numbers is given by $\frac{1}{H} = \frac{1}{n}\left(\frac{1}{a_1} + \frac{1}{a_2} + ... + \frac{1}{a_n}\right)$.

ARITHMETICO-GEOMETRIC PROGRESSION:

Sum of n terms $S_n = ab + (a + d)br + (a + 2d)br^2 + ... + (a + (n - 2)d)br^{n-2} + (a + (n - 1)d)br^{n-1}$ is

$S_n = \frac{ab}{1-r} + \frac{dbr(1-r^{n-1})}{(1-r)^2} - \frac{(a+(n-1)d)br^n}{1-r}$. and $\lim_{n\to\infty} S_n = S = \frac{ab}{1-r} + \frac{dbr}{(1-r)^2}$.

Some Important Results:

- $1 + 2 + 3 + \ldots + n = \frac{n(n+1)}{2}$ (sum of the first n natural numbers)
- $1^2 + 2^2 + 3^2 + \ldots + n^2 = \frac{n(n+1)(2n+1)}{6}$ (sum of squares of the first n natural numbers)
- $1^3+2^3+3^3+\ldots+n^3 = \frac{n^2(n+1)^2}{4} = (1+2+3+\ldots+n)^2$ (sum of cubes of first n natural numbers)
- $1 + x + x^2 + x^3 + \ldots\ldots = (1 - x)^{-1}$, if $-1 < x < 1$
- $1 + 2x + 3x^2 + \ldots\ldots = (1 - x)^{-2}$, if $-1 < x < 1$

INEQUALITIES:

A.M. ≥ G. M. ≥ H. M. : Let $a_1, a_2, \ldots\ldots a_n$ be n positive real numbers, then their arithmetic mean (A), geometric mean (G) and harmonic mean (H) as $A = \frac{a_1 + a_2 + \ldots + a_n}{n}$, $G = (a_1 a_2 \ldots a_n)^{1/n}$ and $H = \frac{n}{\left(\frac{1}{a_1} + \frac{1}{a_2} + \frac{1}{a_3} + \ldots \frac{1}{a_n}\right)}$.

It can be shown that $A \geq G \geq H$. Equality holds at either place if and only if $a_1 = a_2 = \ldots = a_n$.

It can be shown that $A^* \geq G^* \geq H^*$. Equality holds at either place if and only if $a_1 = a_2 = \ldots. = a_n$.

Arithmetic Mean of mth Power: Let a_1, a_2, ... , a_n be n positive real numbers (not all equal) and let m be a real number , then

$$\frac{a_1^m + a_2^m + ... + a_n^m}{n} > \left(\frac{a_1 + a_2 + ... + a_n}{n}\right)^m \text{ if } m \in R - [0, 1].$$

$$\frac{a_1^m + a_2^m + ... + a_n^m}{n} < \left(\frac{a_1 + a_2 + ... + a_n}{n}\right)^m . \text{ if } m \in (0, 1)$$

$$\frac{a_1^m + a_2^m + ... + a_n^m}{n} = \left(\frac{a_1 + a_2 + ... + a_n}{n}\right)^m . \text{ if } m \in \{0, 1\}$$

EXPONENTIAL SERIES

$$e^x = \sum_{n=0}^{\infty}\left(\frac{x^n}{n!}\right) = 1 + \frac{x}{1!} + \frac{x^2}{2!} + \frac{x^3}{3!} + ...\infty \quad (1)$$

Put $x = 1$, $e = 1 + \frac{1}{1!} + \frac{1}{2!} + \frac{1}{3!} + ...\infty$

$$e = \lim_{n\to\infty}\left(1 + \frac{1}{n}\right)^n$$

The number 'e' lies between 2 and 3 and it is an irrational number

Put $x = -x$ in equation (1) $\quad e^{-x} = \sum_{n=0}^{\infty}(-1)^n .$

$$\frac{x^n}{n!} = 1 - \frac{x}{1!} + \frac{x^2}{2!} - \frac{x^3}{3!} + \frac{x^4}{4!} + ...\infty \quad (2)$$

From equation (1) and (2)

$$\frac{e^x + e^{-x}}{2} = \sum_{n=0}^{\infty}\frac{x^{2n}}{2n!} = 1 + \frac{x^2}{2!} + \frac{x^4}{4!} + ...\infty$$

$$\frac{e^x - e^{-x}}{2} = \sum_{n=0}^{\infty}\frac{x^{2n+1}}{(2n+1)!} = x + \frac{x^3}{3!} + \frac{x^5}{5!} + \ldots \infty$$

$e^{ax} = 1 + \frac{(ax)}{1!} + \frac{(ax)^2}{2!} + \frac{(ax)^3}{3!} + \ldots + \frac{(ax)^n}{n!} + \ldots\infty$

General Term $T_{r+1} = \frac{(ax)^r}{r!}$, coefficient of x^n in $e^{ax} = \frac{a^n}{n!}$

$a^x = e^{x \ln a} = 1 + (\log a)x + \frac{(\log a)^2}{2!}x^2 + \frac{(\log a)^3}{3!}x^3 + \ldots\infty$ for $a > 0, x \in R$.

LOGARITHMIC SERIES

If $|x| < 1$, then

$\log(1 + x) = x - \frac{x^2}{2} + \frac{x^3}{3} - \frac{x^4}{4} + \ldots\infty$ (1)

or, $\log_e(1 + x) = \sum_{n=1}^{\infty}(-1)^{n-1}\frac{x^n}{n}$

Put $x = -x$ in equation (1)

$\log(1 - x) = -x - \frac{x^2}{2} - \frac{x^3}{3} - \frac{x^4}{4} - \ldots\infty$ (2)

or

[equation (1) – (2)]

$\log(1 + x) - \log(1 - x) = \log\left(\frac{1+x}{1-x}\right) = 2\left(x + \frac{x^3}{3} + \frac{x^5}{5} + \ldots\infty\right)$